NOTES ON $\mathcal{H}(\delta)$ -OPEN SETS INDUCED BY HEREDITARY CLASSES

Won Keun Min^{1 §}, Young Key Kim²

¹Department of Mathematics
Kangwon National University
Chuncheon, 200-701, KOREA

²Department of Mathematics
MyongJi University
Yongin, 449-728, KOREA

Abstract: We introduce $\mathcal{H}(\)$ -sets defined by a given hereditary class, and investigate some properties and characterizations by using $\mathcal{H}(r)$ -open sets. We also study relationships between $\mathcal{H}(\)$ -open sets and $\mathcal{H}(\)$ -open sets.

AMS Subject Classification: 54A05

Key Words: hereditary class, ideal, $\mathcal{H}(\)$ -open sets, $\mathcal{H}(r)$ -open sets, $\mathcal{H}(\)$ -open sets.

1. Introduction

An *ideal* I [5] on X is a nonempty family $I \subseteq \exp X$ satisfying the following conditions:

- (i) $A \subseteq B$, $B \in I$ implies $A \in I$;
- (ii) $A, B \in I$ implies $A \cup B \in I$.

In [4], Jankovic and Hamlett have introduced another topology $I\text{-}\tau^*$ by using a given ideal I for a given topology τ .

Received: June 20, 2016

Revised: July 20, 2016

Published: October 21, 2016

© 2016 Academic Publications, Ltd.

url: www.acadpubl.eu

[§]Correspondence author

The idea of hereditary classes was introduced by Császár [4]. A nonempty family $\mathcal{H} \subseteq \exp X$ is called a *hereditary class* if it satisfies the above condition (i) only (i.e. $A \subseteq B$, $B \in \mathcal{H}$ implies $A \in \mathcal{H}$).

In the paper [3], for a hereditary class $\mathcal{H}$, the operator $()^* : \exp X \rightarrow \exp X$ was introduced. An operator $c^* : \exp X \rightarrow \exp X$ was defined by using the operator $()^*$ (i.e. for $A \subseteq X$, $c^*A = A \cup A^*$), which is monotonic, enlarging and idempotent. Some properties of operators $()^*$ and c^* were investigated in [3].

In [7], we introduced $\mathcal{H}(\theta)$ -open sets and $\mathcal{H}(r)$ -open sets defined by a given hereditary class, and investigated some properties and characterizations by using $\mathcal{H}(r)$ -open sets.

In this paper, we introduce the new notion of $\mathcal{H}(\delta)$ -open sets defined by a given hereditary class, which are the extended $\mathcal{H}(\theta)$ -open sets. We also investigate some properties by using $\mathcal{H}(r)$ -open sets.

2. Preliminaries

Let X be a nonempty set and $\exp X$ be the power set of X . Then $\mu \subseteq \exp X$ is called a *generalized topology* (briefly GT) [1,2] on X iff $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in \mu$. We call the pair (X, μ) a *generalized topological space* (briefly GTS) on X . Let X be a nonempty set and μ be a GT. The elements of μ are called μ -open [2] sets and the complements are called μ -closed sets. The generalized-closure of a subset S of X , denoted by cS , is the intersection of generalized closed sets including S . And the interior of S , denoted by iS , the union of generalized open sets included in S . We will denote $\mathcal{M}_\mu$ the union of all μ -open sets in a GTS (X, μ) . An operator $\gamma : \exp X \rightarrow \exp X$ is said to be *monotonic* if for $A \subseteq B \subseteq X$, $\gamma(A) \subseteq \gamma(B)$; (ii) *enlarging* if for $A \subseteq X$, $A \subseteq \gamma(A)$.

For a hereditary class $\mathcal{H}$ on X and $A \subseteq X$, we define the set $A^* \subseteq X$ [3] by $x \in A^*$ iff $x \in M \in \mu$ implies $M \cap A \notin \mathcal{H}$.

If $x \notin M_\mu$, then by definition $x \in A^*$. And $x \notin A^*$ iff there exists $x \in M \in \mu$ such that $M \cap A \in \mathcal{H}$.

Theorem 1 ([3]). *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X .*

- (1) $A \subseteq B \subseteq X$ implies $A^* \subseteq B^*$.
- (2) $A^* \subseteq cA$.
- (3) If $H \in \mathcal{H}$, then $H^* = X - \mathcal{M}_\mu$.
- (4) A^* is μ -closed.
- (5) If F is μ -closed, then $F^* \subseteq F$.

Let μ be a GT in X and $\mathcal{H}$ a hereditary class. Császár [3] introduced the operator $c^* : \exp X \rightarrow \exp X$ defined as $c^*(A) = A \cup A^*$, for $A \subseteq X$.

Theorem 2 ([3]). *Let μ be a GT in X and $\mathcal{H}$ a hereditary class. For $A \subseteq X$,*

$$A^* \subseteq c^*(A) \subseteq cA$$

The author [6] introduced the operator $i^* : \exp X \rightarrow \exp X$ defined as $i^*(A) = X - c^*(X - A)$, for $A \subseteq X$.

Theorem 3 ([6]). *Let μ be a GT in X and $\mathcal{H}$ a hereditary class. For $A \subseteq X$,*

- (1) $c^*(A) = X - i^*(X - A)$.
- (2) $iA \subseteq i^*(A) \subseteq A$.

3. Main Results

Let μ be a GT in X and $\mathcal{H}$ a hereditary class. We will call the triple $(X, \mu, \mathcal{H})$ a $\mathcal{H}$ -generalized topological space, denoted by $\mathcal{H}$ -GTS.

Definition 4. Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$ -GTS in X . Let us define $\mathcal{H}(\delta) \subseteq \exp X$ by $A \in \mathcal{H}(\delta)$ iff $A \subseteq X$ and, if $x \in A$, then there is $Q \subseteq X$ such that $c^*(Q) = Q$ and $x \in i_\mu(Q) \subseteq A$.

We call an element A in $\mathcal{H}(\delta)$ a $\mathcal{H}(\delta)$ -open set. A is called a $\mathcal{H}(\delta)$ -closed set if the complement of A is $\mathcal{H}(\delta)$ -open.

Theorem 5. *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$ -GTS in X . Then every $\mathcal{H}(\delta)$ -open set is μ -open.*

Proof. Obvious. □

Theorem 6. *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$ -GTS in X . Then $\mathcal{H}(\delta)$ also is a GT included in μ .*

Proof. Let $A = \{A_i \subseteq X : A_i \in \mathcal{H}(\delta) \text{ for } i \in J\}$. Then for each $x \in \cup A$, there exists a $j \in J$ such that $x \in A_j \in \mathcal{H}(\delta)$. Moreover, there exists $Q_j \subseteq X$ such that $c^*Q_j = Q_j$ and $x \in i_\mu Q_j \subseteq A_j \subseteq A$. Therefore, $\cup A \in \mathcal{H}(\delta)$. □

Definition 7 ([7]). Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$ -GTS. For $A \subseteq X$, A is said to be $\mathcal{H}(r)$ -open (resp., $\mathcal{H}(r)$ -closed) if $A = i_\mu(c^*(A))$ (resp., $A = c_\mu(i^*(A))$).

Theorem 8. *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$ -GTS in X . Then the nonempty element of $\mathcal{H}(\delta)$ coincides with the unions of $\mathcal{H}(r)$ -open sets.*

Proof. Let A be an $\mathcal{H}(r)$ -open set. Then $A = i_\mu(c^*(A))$. Put $Q = c^*(A)$; then $c^*(Q) = c^*(c^*(A)) = c^*(A) = Q$ since $c^*(c^*(A)) = c^*(A)$. For each $x \in A$, we can say $x \in i_\mu(Q) = i_\mu(c^*(A)) \subseteq A$ where $c^*(Q) = Q$. So $A \in \mathcal{H}(\delta)$ and since $\mathcal{H}(\delta)$ is a GT, any union of $\mathcal{H}(r)$ -open sets is in $\mathcal{H}(\delta)$.

For the converse, let $G \in \mathcal{H}(\delta)$. Then for each $x \in G$, there exists $Q \subseteq X$ such that $c^*(Q) = Q$ and $x \in i_\mu(Q) \subseteq G$. Moreover, from $i_\mu(Q) \subseteq c^*(i_\mu(Q)) \subseteq c^*(Q) = Q$, we have $i_\mu(Q) = i_\mu(c^*(i_\mu(Q)))$, i.e, $i_\mu(Q)$ is $\mathcal{H}(r)$ -open. Finally, we can say G is the union of all $\mathcal{H}(r)$ -open sets. □

Theorem 9. *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$ -GTS in X . Then for $M \in \mu$, $i_\mu(c^*(M))$ is $\mathcal{H}(\delta)$ -open.*

Proof. For $M \in \mu$, $i_\mu(c^*(M)) \subseteq c^*(i_\mu(c^*(M))) \subseteq c^*(M)$ implies $i_\mu(c^*(M)) \subseteq i_\mu(c^*(i_\mu(c^*(M)))) \subseteq i_\mu(c^*(M))$.

Consequently, $i_\mu(c^*(i_\mu(c^*(M)))) = i_\mu(c^*(M))$, and so $i_\mu(c^*(M))$ is $\mathcal{H}(r)$ -open. By the above theorem, $i_\mu(c^*(M))$ is $\mathcal{H}(\delta)$ -open. □

Definition 10. Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$ -GTS in X . The $\mathcal{H}(\delta)$ -closure of a subset A of X , denoted by $c_{\mathcal{H}(\delta)}A$, is the intersection of $\mathcal{H}(\delta)$ -closed sets including A . And the $\mathcal{H}(\delta)$ -interior of A , denoted by $i_{\mathcal{H}(\delta)}A$, the union of $\mathcal{H}(\delta)$ -open sets included in A .

It is obviously that the operator $c_{\mathcal{H}(\delta)}$ is enlarging but the operator $i_{\mathcal{H}(\delta)}$ is not.

Theorem 11. *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$ -GTS in X . Then the operators $c_{\mathcal{H}(\delta)}, i_{\mathcal{H}(\delta)} : \exp X \rightarrow \exp X$ are monotonic.*

Proof. Obvious. □

Theorem 12. *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$ -GTS in X . Then*

- (1) $x \in c_{\mathcal{H}(\delta)}A$ iff $x \in M$ and $M \in \mu$ implies $i_\mu(c^*(M)) \cap A \neq \emptyset$.
- (2) $x \in c_{\mathcal{H}(\delta)}A$ iff $A \cap R \neq \emptyset$ for every $\mathcal{H}(r)$ -open set R containing x .

Proof. (1) Suppose that there exists a μ -open set M of x such that $i_\mu(c^*(M)) \cap A = \emptyset$. Then $A \subseteq (X - i_\mu(c^*(M)))$, and by Theorem 3.5, $X - i_\mu(c^*(M))$ is $\mathcal{H}(\delta)$ -closed. But since the $\mathcal{H}(\delta)$ -closed set $(X - i_\mu(c^*(M)))$ does not contain x , $x \notin c_{\mathcal{H}(\delta)}A$.

For the converse, let $x \notin c_{\mathcal{H}(\delta)}A$. Then there exists a $\mathcal{H}(\delta)$ -closed set F such that $x \notin F$ and $A \subseteq F$. Since $X - F$ is a $\mathcal{H}(\delta)$ -open set containing x ,

from the definition of $\mathcal{H}(\delta)$ -open set, there exists a μ -open M of x such that $i_\mu(c^*(M)) \subseteq X - F$. It implies $A \cap i_\mu(c^*(M)) = \emptyset$.

(2) It follows from (1) and Theorem 3.5.

□

Theorem 13. Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$ -GTS in X .

(1) $x \in i_{\mathcal{H}(\delta)}A$ iff there exists a μ -open set M containing x such that $i_\mu(c^*(M)) \subseteq A$.

(2) $x \in i_{\mathcal{H}(\delta)}A$ iff there exists a $\mathcal{H}(r)$ -open set R containing x such that $R \subseteq A$.

Proof. (1) Let $x \in i_{\mathcal{H}(\delta)}A$. Then there is a $\mathcal{H}(\delta)$ -open set Q containing x such that $x \in Q \subseteq A$. Furthermore, from the definition of $\mathcal{H}(\delta)$ -open set, there is a μ -open set M such that $x \in i_\mu(c^*(M)) \subseteq Q \subseteq A$. So the condition is satisfied.

The converse is obviously obtained since $i_\mu(c^*(M))$ is $\mathcal{H}(\delta)$ -open.

(2) It follows from (1) and Theorem 3.5.

□

Definition 14 ([7]). Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$ -GTS in X . Let us define the collection $\mathcal{H}(\theta) \subseteq \exp X$ by $A \in \mathcal{H}(\theta)$ iff for each $x \in A$, there is $M \in \mu$ such that $x \in M \subseteq c^*M \subseteq A$. The elements of $\mathcal{H}(\theta)$ are called $\mathcal{H}(\theta)$ -open sets and the complements are called $\mathcal{H}(\theta)$ -closed sets.

Theorem 15. Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$ -GTS in X . Then $\mathcal{H}(\theta) \subseteq \mathcal{H}(\delta) \subseteq \mu$.

Proof. Let $A \in \mathcal{H}(\theta)$. Then for each $x \in A$, there exists $M \in \mu$ such that $x \in M \subseteq c^*M \subseteq A$. Put $Q = c^*M$; then $c^*Q = c^*c^*M = c^*M = Q$ and $x \in M \subseteq i_\mu Q = i_\mu c^*M \subseteq c^*M \subseteq A$. Hence $A \in \mathcal{H}(\delta)$. □

Remark 16. The following implications are obtained:

$$\begin{array}{ccccc} \mathcal{H}(\theta)\text{-open} & \Rightarrow & \mathcal{H}(\delta)\text{-open} & \Rightarrow & \text{generalized open} \\ & & \uparrow & & \\ & & \mathcal{H}(r)\text{-open} & & \end{array}$$

The converses may not be true in general as shown in the next examples:

Example 17. Let $Y = \{a, b, c, d\}$, a generalized topology $\nu = \{\emptyset, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and a hereditary class $\mathcal{H} = \{\emptyset, \{b\}\}$.

(1) For a set $A = \{a, b\}$, note that $c^*(\{a, b\}) = \{a, b\} \cup \{a, b\}^* = \{a, b, d\}$ and $ic^*(\{a, b\}) = \{a, b\}$. So we can say that A is $\mathcal{H}(\delta)$ -open but A is not $\mathcal{H}(\theta)$ -open.

(2) Consider a ν -open set $A = \{a, c\}$. Note that $c^*(\{a, c\}) = \{a, c\} \cup \{a, c\}^* = \{a, c\} \cup Y = Y$ and $ic^*(\{a, c\}) = iY = \{a, b, c\}$. Finally, A is not $\mathcal{H}(\delta)$ -open.

Example 18. Let $X = \{a, b, c\}$, a generalized topology $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and a hereditary class $\mathcal{H}' = \{\emptyset, \{a\}, \{b\}\}$.

Note that $\{a\}^* = \emptyset$; $\{b\}^* = \emptyset$; $\{a, b\}^* = \{c, d\}$; $X^* = \emptyset$.

$c^*\{a\} = \{a\}$; $c^*\{b\} = \{b\}$; $c^*\{a, b\} = X$; $c^*X = X$;

$ic^*\{a\} = \{a\}$; $ic^*\{b\} = \{b\}$; $ic^*\{a, b\} = X$; $ic^*X = X$.

So $\{a, b\}$ is $\mathcal{H}(\delta)$ -open but not $\mathcal{H}(r)$ -open.

Definition 19 ([7]). Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$ -GTS in X . (X, μ) is $\mathcal{H}$ -regular iff for each μ -open set U containing $x \in X$, there exists a μ -open set V such that $x \in V \subseteq c^*V \subseteq U$.

Theorem 20 ([7]). Let (X, μ) be a GTS. If X is $\mathcal{H}$ -regular, then $\mu = \mathcal{H}(\theta)$.

Theorem 21. Let (X, μ) be a generalized topological space. If X is $\mathcal{H}$ -regular, then $\mu = \mathcal{H}(\theta) = \mathcal{H}(\delta)$.

Proof. It follows from Theorem 3.11 and Theorem 3.16. □

References

- [1] Á. Császár; Generalized open sets, *Acta Math. Hungar.*, **75**, (1997), 65-87.
- [2] Á. Császár; Generalized topology, Generalized Continuity, *Acta Math. Hungar.*, **96**, (2002), 351-357.
- [3] Á. Császár; Modificatons of generalized topologies via hereditary classes, *Acta Math. Hungar.*, **120**(3) (2008), 275-279.
- [4] D. Jankovic and T. R. Hamlett; New topologies from old via ideals, *Amer. Math. Monthly*, **97**, (1990), 295-310.
- [5] K. Kuratowski; *Topology*, Vol. 1, Academic Press(New York, 1966).
- [6] Y. K. Kim and W. K. Min; On Operations Induced by Hereditary Classes on Generalized Topological Spaces, *Acta Math. Hungar.*, **137** (1-2) (2012), 130-138.
- [7] Y. K. Kim and W. K. Min; $\mathcal{H}(\)$ -open sets induced by hereditary classes on generalized topological spaces, *International Journal of Pure and Applied Mathematics*, **93** (3) (2014), 307 - 315.