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LEFT IDEALS PRESERVING LINEAR MAPS BETWEEN C^*-ALGEBRAS

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Abstract: In this paper we show that if A is a unital C^*-algebra and B is a purely infinite C^*-algebra such that has a non-zero commutative maximal ideal and $\phi : A \to B$ is a unital surjective linear map which preserves the maximal left ideals in both direction. Then ϕ is a Jordan isomorphism.

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1. Introduction

Let A and B be two C^*-algebras. We always denote by e the unit both A and B. For any $a \in A$ the sets $\sigma(a), \sigma_l(a)$ and $r(a)$ will denote the spectrum, the left spectrum and the spectral radius of a respectively. A linear map $\phi : A \to B$

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is said to be spectrum preserving (or left spectrum preserving, respectively) if $\sigma(\phi(a)) = \sigma(a)$ (or $\sigma_l(\phi(a)) = \sigma_l(a)$, respectively) for all $a \in A$ and invertibility (or left invertibility, respectively) preserving if $\phi(a)$ is invertible (or left invertibe, respectively) in B for every invertible (or left invertibe, respectively) element $a \in A$. Also ϕ is called unital if $\phi(e) = e$. A linear mapping $\phi : A \to B$ is called a spectral isometry if $r(\phi(a)) = r(a)$ for every $a \in A$. The objective of this note is to study the behaviour of surjective spectral isometries between unital C^*-algebras. Combining these results with theorems on the structure of spectral isometries we shall obtain that every unital surjective spectral isometry has to be a Jordan isomorphism for some unital C^*-algebras. It is an open question whether this holds for all C^*-algebras.

A C^*-algebra A is said to be infinite if A contains an infinite projection p, that is, p is Murray-von Neumann equivalent to its subprojection, and purely infinite if every hereditary C^*-algebra B of A is infinite. For a simple C^*-algebra A, A is purely infinite if and only if A has a purely infinite hereditary C^*-algebra B. A C^*-subalgebra B of A is hereditary if $0 \leq a \leq b, a \in A, b \in B$ implies $a \in B$. A C^*-algebra is purely infinite if every hereditary subalgebra contains an infinite projection.

Recall that a map ϕ from a C^*-algebra A into another preserves the maximal left ideals if $\phi(I)$ is a maximal left ideal whenever I is; ϕ preserves the maximal left ideals in both directions if $\phi(I)$ is a maximal left ideal if and only if I is. The following conjecture seems to be still open:

any spectrum-preserving linear map from a unital Banach algebra onto a unital semi-simple (non-commutative) Banach algebra that preserves the unit is a Jordan homomorphism, (Kaplansky's conjecture).

The G-K-Z Theorem ([11], [10]) asserts that a unital linear functional defined on a Banach algebra is multiplicative if it is invertibility preserving and the theorem has inspired a number of papers on more general preserver problems. It is a straightforward conclusion of the G-K-Z Theorem that a unital and invertibility preserving linear map from a Banach algebra into a semi-simple commutative Banach algebra is a homomorphism. This conjecture is still unsolved. The most important partial results obtained is this direction are [1], [3], [7], [8], [19], [20].

Recently Aupetit [1] showed that a spectrum preserving surjective linear map from a von Neumann algebra onto another is a Jordan isomorphism. In [9] C. Jianlian, H. Jinchuan have showed that every unital linear bijection which preserves the maximal left ideals from a semi-simple Banach algebra onto a C^*-algebra of real rank zero is a Jordan isomorphism.

In this paper we show that if A is a unital C^*-algebra and B is a purely infinite C^*-algebra such that has a non-zero commutative maximal ideal, then every unital linear surjective map which preserves the maximal left ideals from A onto B is a Jordan isomorphism. And generalize the results in [9].

There are the most important C^*-algebras for what follows, for example, Let A be the C^*-algebra $B(H)\oplus C(X)$, where $B(H)$ is the algebra of all bounded linear operators on a Hilbert space H and $C(X)$ is the algebra of all continuous functions on a compact Hausdorff space X. It is clear that, $C(X)$ is a unique maximal commutative ideal in A.

Theorem 1.1. *[1] Let A be a unital complex Banach algebra, B be a unital semi-simple commutative complex Banach algebra and $\phi : A \to B$ is a unital invertibility preserving linear map. Then ϕ is a continuous multiplicative.*

Theorem 1.2. *[14] Let $\phi : A \to B$ be a unital surjective spectrally bounded operator from the unital C^*-algebra A onto the unital semisimple Banach algebra B. If A is a purely infinite C^*-algebra of real rank zero. Then ϕ is a Jordan homomorphism.*

2. Main Results

Lemma 2.1. *Let I be a closed ideal of C^*-algebra A. If B/I be a hereditary C^*-subalgebra of A/I, then B is a hereditary C^*-subalgebra of A.*

Proof. Let $b \in B$ be positive and $a \in A$ such that $0 \leq a \leq b$. We have $a+I \leq b+I$ and $a+I \in A/I, b+I \in B/I$ are positive, it follows that $a+I \in B/I$ so that $a \in B$. Thus B is a hereditary. □

Lemma 2.2. *Let I be a closed ideal of C^*-algebra A and B be a C^*-subalgebra of A such that $I \subseteq B$. If B is infinite then B/I is infinite.*

Proof. Evidently, if $p \in B$ be a infinite projection then $p + I$ is a infinite projection in B/I.

Corollary 2.3. *Let I be a closed ideal of C^*-algebra A. If A is purely infinite, then A/I is purely infinite.*

Proof. It is clear. □

Remark 2.4. We recall that if A, B and D are C^*-algebras, and if homomorphisms $\varphi : A \longrightarrow D$ and $\psi : B \longrightarrow D$ are given, then the C^*-algebra $A \oplus_D B$ is defined as

$$A \oplus_D B = \{(a, b) \in A \oplus_D B : \varphi(a) = \psi(b)\}.$$

Let A be a C^*-algebra, by [18, Lemmas 10 and 11] A has a unique maximal commutative ideal I and a closed ideal J such that $I \cap J = \{0\}$ and A/J is commutative, furthermore, $A \cong A/J \oplus_{A/(I+J)} A/I$ by $*$-isomorphism $\varphi : A \longrightarrow A/J \oplus_{A/(I+J)} A/I$ such that $\varphi(a) = (a + J, a + I)$.

Theorem 2.5. *Let A be a unital complex Banach algebra and B be a unital C^*-algebra such that has a non-zero commutative ideal. Suppose $\phi : A \to B$ is a unital invertibility preserving linear map. Then there exists a direct sum $B \cong B/J \oplus_{B/(I+J)} B/I$ such that $\phi(a) = \phi_1(a)+\phi_2(a)$ for all $a \in A$, where $\phi_1 : A \longrightarrow B/J$ is a continuous homomorphism and $\phi_2 : A \longrightarrow B/I$ preserves idempotent elements and $\phi_2(e) = e_{B/I}$.*

Proof. By [18, Lemmas 10 and 11] B has a unique maximal commutative ideal I and a closed ideal J with the properties $I \cap J = 0$ and B/J is commutative and also $B \cong B/J \oplus_{B/(I+J)} B/I$. Since I contains every commutative ideal by the hypothesis $I \neq 0$ [see proof of Lemma 10 in 18]. Define $\phi_1 : A \longrightarrow B/J$ and $\phi_2 : A \longrightarrow B/I$ by $\phi_1(a) = b_1$ and $\phi_2(a) = b_2$ for every $a \in A$, where $\phi(a) = b_1 + b_2$, $b_1 \in B/J$, $b_2 \in B/I$. In fact $b_1 = \phi(a) + J$ and $b_2 = \phi(a) + I$.

We can show that ϕ_1 and ϕ_2 are well-defined and non-zero unital linear maps. For any $a \in A$ we have $\phi_1(a) = b_1 = \phi(a)+J$ and $\phi_2(a) = b_2 = \phi(a)+I$, it follows that if $\phi(a)$ is invertible then $\phi_1(a)$ and $\phi_2(a)$ are invertible. Hence ϕ_1 and ϕ_2 preserve invertibility. Therefore, ϕ_1 is continuous homomorphism by Theorem 1.1. (Note that B/J is commutative C^*-algebra).

Suppose that $p^2 = p$ in A. Since ϕ preserves idempotent elements and also $\phi(e) = e$ (see [1]), we have $\phi_2(p)^2 = \phi(p)^2 + I = \phi(p^2) + I = \phi_2(p^2)$ and $\phi_2(e) = e_{B/I}$. This completes the proof. □

Theorem 2.6. *Let A be a unital C^*-algebra and B be a purely infinite C^*-algebra such that has a non-zero commutative maximal ideal. Suppose that $\phi : A \to B$ is a unital surjective linear map which preserves the maximal left ideals in both direction. Then ϕ is a Jordan isomorphism.*

Proof. Let J, I and ϕ_1 be as in Theorem 2.5 and B be a purely infinite C^*-algebra. By Corollary 2.3, B/J and B/I are purely infinite C^*-algebras. Since I contains every commutative ideal by the hypothesis I is a commutative maximal ideal in B [see proof of Lemma 10 in 15]. Thus B/I has real rank zero by [5, Theorem V.7.4].

Note that in a Banach algebra an element is not left invertible if and only if it belongs to some maximal left ideal. Evidently, ϕ is injective and continuous (see [2]). Since ϕ is bijective and unital, we see that ϕ^{-1} is unital and preserves left invertibility. It now follows that ϕ^{-1} is left spectrum preserving. Hence ϕ is spectral isometry.

Since $I \subset Z(B)$ and by [16, Corollary 4.4] ϕ maps the center $Z(A)$ of A onto the center $Z(B)$ of B. Hence, in view of continuity ϕ, $I^{'} = \phi^{-1}(I) \subset Z(A)$ is a closed two-sided ideal in A.

Define $\phi_1 : A \longrightarrow B/J$ by $\phi_1(a) = \phi(a) + J$ for every $a \in A$. Evidently, ϕ_1 preserves left invertibility. Since B/J is commutative C^*-algebra it follows that ϕ_1 is invertibility preserving linear map. Hence ϕ_1 is continuous homomorphism by Theorem 1.1. Define $\psi : A/I' \longrightarrow B/I$ by $\psi(a + I^{'}) = \phi(a) + I$ for every $a \in A$, where $I' = \phi^{-1}(I)$. ψ is a non-zero unital linear map. We have $r(\psi(a + I')) = r(a + I^{'})$, [see the proof of Proposition 9 in 17]. Also, ψ is injective and ψ^{-1}, as well as, ψ is a Jordan isomorphisms by Theorem 1.2.

Denote $\pi_2 : A \to A/I'$ the natural quotient homomorphism, and put $\phi_2 = \psi \circ \pi_2$. Hence $\phi_2 : A \longrightarrow B/I$ by $\phi_2(a) = \phi(a) + I$, for every $a \in A$, is a Jordan homomorphism.

Now, we show that ϕ is a Jordan isomorphism. We have $\phi_1(a) = \phi(a) + J$ and $\phi_2(a) = \phi(a) + I$ for all $a \in A$. Hence for every $a \in A$

$$\phi_1(a)^2 = \phi(a)^2 + J, \quad \phi_2(a)^2 = \phi(a)^2 + I. \tag{1}$$

And also we have

$$\phi_1(a^2) = \phi(a^2) + J, \quad \phi_2(a^2) = \phi(a^2) + I. \tag{2}$$

Since ϕ_1 and ϕ_2 are Jordan homomorphism. (1) and (2) imply $\phi(a)^2 - \phi(a^2) \in J$ and $\phi(a)^2 - \phi(a^2) \in I$. But $I \cap J = 0$. Therefore, $\phi(a^2) = \phi(a)^2$ for all $a, b \in A$, that is, ϕ is a Jordan isomorphism. This completes the proof. □

Corollary 2.7. *Let A be a unital C^*-algebra and B be a purely infinite C^*-algebra such that has a non-zero commutative maximal ideal. Suppose that $\phi : A \to B$ is a surjective linear map which preserves the maximal left ideals in both direction and $\phi(e)$ is invertible in B. Then ϕ is a Jordan isomorphism multiplied by an invertible element.*

Proof. For $b \in B$, denote L_b the liner map from B into itself defined by multiplying by b from the left hand, that is, $L_b(x) = bx$ for every $x \in B$. Let $\psi = L_{\phi(e)^{-1}} \circ \phi$, then $\psi(e) = e$. As a preserver, ψ has the same property as ϕ has. Now by Theorem 2.6, ψ is a Jordan isomorphism and $\phi = L_{\phi(e)} \circ \psi$. This completes the proof. □

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