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HYBRID CUBIC B-SPLINE METHOD FOR SOLVING NON-LINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract: A method based on hybrid cubic B-spline method (HCBSM) is developed, analyzed and applied to solve second-order non-linear two-point boundary value problems. In this method, a free parameter, γ, plays an important role in producing accurate results. Tests on four examples and a comparison of the results with the results obtained using cubic B-spline, extended cubic B-spline and shooting methods indicated that HCBSM is a feasible and accurate method.

Key Words: two-point boundary value problems, cubic b-spline, trigonometric cubic b-spline, hybrid cubic b-spline, quasilinearization

1. Introduction

Consider the non-linear second-order two-point boundary value problem:

$$y''(x) = f(x, y, y'), \quad x \in [a, b], \tag{1}$$

$$y(a) = \alpha, \quad y(b) = \beta, \tag{2}$$

where f is continuous on the set $D = \{(x, y, y')|a \leq x \leq b, y, y' \in R\}$ and a, b, α, and β are real numbers.

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Boundary value problems arise in many real life applications in science and engineering such as theoretical physics, chemistry and control theory. In general, analytical methods which yield exact analytical solutions are not available for problem (1) and (2) and the only recourse is to use numerical or approximate analytical methods. Nonlinear two-point boundary value problems have been solved approximately with shooting, finite difference, variational approach, Adomian decomposition, homotopy perturbation being amongst the more frequently used methods. Details on these methods as well as others can be found in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

Table 1 summarize some salient points (obtained from [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]) regarding some approximate methods.

Linear and nonlinear two-point boundary value problems had already been treated by using cubic spline polynomial, cubic B-spline, cubic trigonometric B-spline, extended cubic B-spline, quartic B-spline and B-spline collocation methods [14, 15, 16, 17, 18, 19]. From the literature extended cubic B-spline method seems to be more suitable than other spline methods. B-spline interpolation is the best to interpolate any smooth functions compared with finite difference, finite element, finite volume methods [15]. Cubic trigonometric B-spline produces more accurate results compared to cubic B-spline if the problems have trigonometric functions [16]. The advantage of using extended B-spline is that it possesses a free parameter, λ, that can be optimized to give more accurate results [17]. The hybrid cubic B-spline also contains a free parameter, γ. The main purpose of this study is to apply a hybrid cubic B-spline function in solving equation (1). The hybrid cubic B-spline method used is basically that of [20] for solving a generalized nonlinear Klien-Gordon equation for which the results are more accurate compared with some other methods. Therefore we seek to apply HCBSM for solving equation (1).

2. Quasilinearization

The quasilinearization technique is used to reduce the given nonlinear problem (1)-(2) to a sequence of linear problems. The approach used is adopted from Goh [18]. An initial approximation for the function $y(x)$ in $f(x, y)$ is chosen and $f(x, y)$ expanded about this function, denoted by $y^{(0)}(x)$, to obtain

$$f(x, y^{(1)}) = f(x, y^{(0)}) + (y^{(1)} - y^{(0)}) \left(\frac{\partial f}{\partial y}\right)_{(x,y^{(0)})} + ...$$

Methods	Features
Shooting method	Quite general, applicable to wide variety of differential equations but has limitations and is sensitive to initial conditions.
Finite difference method	Gives highly accurate results, but these results can be found at the chosen knots only. Conversely, for some other methods, the results can be obtained at any point in the range.
Variational approach	The results which are obtained by variational approach are remarkable simple with high accuracy. Moreover, this method can be easily applied to many nonlinear problems.
Adomian decomposition and reproducing kernel	Using this method can avoid additional calculation in finding unknown parameters; and it can solve singular BVPs easily.
Homotopy perturbation method	HPM is better than most of the other methods in the literature and gives more accurate results with rapid convergence than variety of the other methods.
Shooting reproducing kernel Hilbert space	Simplicity and applicability.
Reproducing kernel and least squares	Feasible, effective and gives more accuracy.
Reproducing kernel space method	This method satisfies all boundary conditions, it has important properties for computation and gives more accurate results.
Uniform haar wavelets method	The essential advantage of the Haar wavelet is its efficiency and applicability for a many boundary conditions.
Newton type method	Successful for resolving of the nonlinear two-point BVPs for ODEs with mixed linear boundary conditions.
Variable mesh method	Successful to solve linear and nonlinear BVPs and gives more accurate solutions when compared with a family of variable mesh method, a non-uniform mesh cubic spline, and TAGE iterative algorithms.
Piece-wise continuous method	Gives high accurate results and can apply to linear and nonlinear problems without any additional calculations.
New shooting method	By using a new type of shooting method, we can convert initial value problems to boundary value problems. Also this method gives a good results compared with other methods.

Generally for $k = 0, 1, 2, ...,$ we can write

$$f(x, y^{(k+1)}) = f(x, y^{(k)}) + (y^{(k+1)} - y^{(k)}) \left(\frac{\partial f}{\partial y}\right)_{(x, y^{(k)})} + ...$$

Then, equation (1) can be written as:

$$y_{xx}^{(k+1)}(x)+p(x)y_x^{(k+1)}(x)+q^{(k)}(x)y^{(k+1)}(x)=r^{(k)}(x), \quad k=0,1,2,..., \tag{3}$$

where

$$q^{(k)}(x)=-\left(\frac{\partial f}{\partial y}\right)_{(x,y^{(k)})}, \quad r^{(k)}(x)=f(x,y^{(k)})-y^{(k)}\left(\frac{\partial f}{\partial y}\right)_{(x,y^{(k)})}$$

with boundary conditions

$$y^{(k+1)}(a)=\alpha, \quad y^{(k+1)}(b)=\beta.$$

The iterations can be stopped by setting the prescribed absolute error criterion $|y_i^{(k+1)}(x)-y_i^{(k)}(x)|\leq 10^{-7}$ for all i that appears.

3. Hybrid Cubic B-Spline Method

Hybrid cubic B-spline is a combination of B-spline and trigonometric cubic B-spline with one free parameter, γ. The value of γ giving the least error for the numerical solution is specified.

3.1. Hybrid Cubic B-Spline

Consider the following linear two-point boundary value problem:

$$Ly\equiv y''(x)+p(x)y'(x)+q(x)y(x)=r(x), \tag{4}$$

with boundary conditions

$$y(a)=\alpha, \quad y(b)=\beta, \tag{5}$$

where $p(x)$, $q(x)$, and $r(x)$ are smooth functions. For a finite interval [a,b], suppose that $\{x_i\}_{i=0}^n$ is a uniform partition of a finite interval [a,b] with $n\in\mathbb{Z}^+$ such that $a=x_0<x_1<...<x_n=b$. The partition can be extended by using

$$h=\frac{b-a}{n}, \quad x_0=a, \quad x_i=x_0+ih, \quad i\in\mathbb{Z}.$$

Hybrid cubic B-spline basis function is established by using a linear combination of the cubic B-spline basis function and trigonometric cubic B-spline basis [20].

Here, blending function of degree four, H_i^4 is used and the resulting function is displayed in (6).

$$H_i^4(x) = \begin{cases} \frac{\gamma}{6h^3}(x-x_i)^3 + \frac{1-\gamma}{\theta}\sin^3(\frac{x-x_i}{2}), & x \in [x_i, x_{i+1}], \\ \frac{\gamma}{6h^3}(h^3 + 3h^2(x-x_{i+1}) + 3h(x-x_{i+1})^2 - 3(x-x_{i+1})^3) & \\ \quad +\frac{1-\gamma}{\theta}(\sin^2(\frac{x-x_{i+1}+h}{2})\sin(\frac{x_{i+1}-x+h}{2}) & \\ \quad +\sin(\frac{x-x_{i+1}+h}{2})\sin(\frac{x-x_{i+1}}{2})\sin(\frac{x_{i+1}-x+2h}{2}) & \\ \quad +\sin^2(\frac{x-x_{i+1}}{2})\sin(\frac{x_i-x+3h}{2})), & x \in [x_{i+1}, x_{i+2}], \\ \frac{\gamma}{6h^3}(h^3 + 3h^2(x_{i+3}-x) + 3h(x_{i+3}-x)^2 - 3(x_{i+3}-x)^3) & \\ \quad +\frac{1-\gamma}{\theta}(\sin^2(\frac{x_{i+3}-x}{2})\sin(\frac{x-x_{i+3}+2h}{2}) & \\ \quad +\sin(\frac{x-x_{i+3}+2h}{2})\sin(\frac{x_{i+1}-x+h}{2})\sin(\frac{x_{i+3}-x}{2}) & \\ \quad +\sin^2(\frac{x_{i+3}-x+h}{2})\sin(\frac{x-x_{i+3}+h}{2})), & x \in [x_{i+2}, x_{i+3}], \\ \frac{\gamma}{6h^3}(x_{i+4}-x)^3 + \frac{1-\gamma}{\theta}\sin^3(\frac{x_{i+4}-x}{2}), & x \in [x_{i+3}, x_{i+4}], \end{cases} \tag{6}$$

where $\theta = \sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})$ and $\gamma \in \mathbb{R}$. The values of γ have an essential role and is influential in the hybrid cubic basis function. It is known that if $\gamma = 0$, the basis function is the same as cubic trigonometric B-spline basis function and if $\gamma = 1$, the basis function is the same as B-spline basis function. Consequently, this work focuses on the value of $\gamma > 1$. A plot of a family of $H_i^4(x)$ together with cubic B-spline and trigonometric cubic B-spline is shown in Figure 1.

3.2. Hybrid Cubic B-Spline for a Nonlinear Two-Point BVPs

Suppose the estimate solution $S(x)$ to the exact solution $y(x)$ at point x_i respectively, be defined as:

$$S(x) = \sum_{i=-3}^{n-1} C_i H_i^4, \quad x \in [x_0, x_n], \quad C_i \in \mathbb{R}, \tag{7}$$

where C_i are unknown real coefficients and $H_i^4(x)$ are hybrid cubic B-spline basis functions. The values of H_i and its derivatives $H_i^{'}$, $H_i^{''}$ at the nodal points are tabulated in Table 1.

where

$$a_1 = \frac{\gamma}{6} + \frac{(1-\gamma)\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}, \qquad a_2 = \frac{4\gamma}{6} + \frac{2(1-\gamma)\sin(\frac{h}{2})}{\sin(\frac{3h}{2})},$$

$$a_3 = \frac{\gamma}{2h} + \frac{3(1-\gamma)}{4\sin(\frac{3h}{2})}, \qquad a_4 = \frac{-\gamma}{2h} - \frac{3(1-\gamma)}{4\sin(\frac{3h}{2})},$$

$$a_5 = \frac{\gamma}{h^2} + \frac{3(1-\gamma)[\sin(\frac{h}{2}) - 2\sin^3(\frac{h}{2}) + \sin(\frac{3h}{2})]}{8\sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})},$$

$$a_6 = \frac{-2\gamma}{h^2} - \frac{3(1-\gamma)[\sin(2h) + 2\sin^2(\frac{h}{2})\sin(h)]}{4\sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})}.$$

From equations (6) and (7), the values of $S(x)$, $S^{'}(x)$, and $S^{''}(x)$, at the knots x_i are specified in the conditions of C_i as

$$\begin{cases} S(x_i) = C_{i-3}[\frac{\gamma}{6} + \frac{(1-\gamma)\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}] \\ \qquad +C_{i-2}[\frac{4\gamma}{6} + \frac{2(1-\gamma)\sin(\frac{h}{2})}{\sin(\frac{3h}{2})}] + C_{i-1}[\frac{\gamma}{6} + \frac{(1-\gamma)\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}] \\ S^{'}(x_i) = C_{i-3}[\frac{\gamma}{2h} + \frac{3(1-\gamma)}{4\sin(\frac{3h}{2})}] + C_{i-1}[\frac{-\gamma}{2h} - \frac{3(1-\gamma)}{4\sin(\frac{3h}{2})}] \\ S^{''}(x_i) = C_{i-3}[\frac{\gamma}{h^2} + \frac{3(1-\gamma)(\sin(\frac{h}{2}) - 2\sin^3(\frac{h}{2}) + \sin(\frac{3h}{2}))}{8\sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})}] \\ \qquad +C_{i-2}[\frac{-2\gamma}{h^2} - \frac{3(1-\gamma)(\sin(2h)+2\sin^2(\frac{h}{2})\sin(h))}{4\sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})}] \\ \qquad +C_{i-1}[\frac{\gamma}{h^2} + \frac{3(1-\gamma)(\sin(\frac{h}{2})-2\sin^3(\frac{h}{2})+\sin(\frac{3h}{2}))}{8\sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})}] \end{cases} \tag{8}$$

4. Solution of Nonlinear Two-Point BVPs

In this part, a collocation approach based on hybrid cubic B-spline basis functions is used to obtain a numerical solution of the nonlinear two-point BVPs

(1)-(2). It is required that the approximate solution (5) satisfy the differential equation at the points $x = x_i$ by putting (7) in (4), as a follows:

$$\sum_{i=-3}^{n-1} C_i H_i^{''}(x_i) + p(x_i) \sum_{i=-3}^{n-1} C_i H_i^{'}(x_i) + q(x_i) \sum_{i=-3}^{n-1} C_i H_i(x_i)$$

$$= r(x_i), \;\; i = 0, 1, 2, ..., n. \tag{9}$$

with boundary conditions

$$\sum_{i=-3}^{n-1} C_i H_i^4(x) = \alpha, \quad \text{and} \quad \sum_{i=-3}^{n-1} C_i H_i^4(x) = \beta. \tag{10}$$

The values in Table 1 are substituted into equations (9)-(10). Then a system of $(n+3)$ equations with $(n+3)$ unknowns C_{-3} , $C_{-2}, \dots, C_{n-1}$ are obtained. This system can be written in the matrix-vector form

$$MY = Z, \tag{11}$$

where
$Y = [C_{-3}, C_{-2}, ..., C_{n-1}]^T$, $Z = [0, f(x_0), f(x_1), ..., f(x_N), 0]^T$, and M is a $(n+3) \times (n+3)$ matrix given by

$$M = \begin{pmatrix} A & B & A & 0 & \cdots & 0 & 0 \\ \rho(x_0) & \tau(x_0) & \phi(x_0) & 0 & \cdots & 0 & 0 \\ 0 & \rho(x_1) & \tau(x_1) & \phi(x_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \rho(x_n) & \tau(x_n) & \phi(x_n) \\ . & . & . & . & A & B & A \end{pmatrix}_{(n+3)\times(n+3)}$$

The elements of the matrix M are specified below for $i = 0, 1, ..., n$.

$$A = \frac{\gamma}{6} + \frac{(1-\gamma)\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}, \quad B = \frac{4\gamma}{6} + \frac{2(1-\gamma)\sin(\frac{h}{2})}{\sin(\frac{3h}{2})},$$

$$\rho(x_i) = [-\frac{\gamma}{h^2} - \frac{3(1-\gamma)[\sin(\frac{h}{2}) - 2\sin^3(\frac{h}{2}) + \sin(\frac{3h}{2})]}{8\sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})}]$$

$$+ p(x_i)[\frac{-\gamma}{2h} - \frac{3(1-\gamma)}{4\sin(\frac{3h}{2})}] + q(x_i)[\frac{\gamma}{6} + \frac{(1-\gamma)\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}],$$

$$\tau(x_i) = [\frac{2\gamma}{h^2} + \frac{3(1-\gamma)[\sin(2h) + 2\sin^2(\frac{h}{2})\sin(h)]}{4\sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})}] + p(x_i)[\frac{4\gamma}{6} + \frac{2(1-\gamma)\sin(\frac{h}{2})}{\sin(\frac{3h}{2})}],$$

$$\phi(x_i) = [-\frac{\gamma}{h^2} - \frac{3(1-\gamma)[\sin(\frac{h}{2}) - 2\sin^3(\frac{h}{2}) + \sin(\frac{3h}{2})]}{8\sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})}] + p(x_i)[\frac{\gamma}{2h} + \frac{3(1-\gamma)}{4\sin(\frac{3h}{2})}] + q(x_i)[\frac{\gamma}{6} + \frac{(1-\gamma)\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}].$$

To find the values of $C_{-3}, C_{-2}, , C_{n-1}$ in vector Y, we can solve the equation (11) as follows:

$$Y = M^{-1}Z. \tag{12}$$

Finally, substituting the values of C_i , for $i = -3, -2, ..., n-1$ in equation (7) we obtain the approximated analytical solution of equation (11). The numerical solution can be calculated after obtaining the values of γ by trial and error [20].

5. Stability

The system (8) obtained by using HCBSM cannot be solved exactly. However we can calculate the solutions that are close to the exact solution when small errors are introduced in the input functions. Specifically, we need to prove that the difference between these solutions always depends on the coefficients of the linear system; that means, the method is stable [21].

Let ζ_N and η_N be small perturbations in M and Z. Let α be the solution to the system

$$(M + \zeta_N)\alpha = Z + \eta_N. \tag{13}$$

Suppose that M is nonsingular and

$$\|\zeta_N\| < \frac{1}{2 \parallel M^{-1} \parallel}.$$

Then, $M+\zeta_N$ is nonsingular and

$$\| (M+\zeta_N)^{-1} \| \leq 2 \| M^{-1} \| .$$

From (11) and (13), we obtain

$$Y - \alpha = M^{-1}Z - (X+\zeta_N)^{-1}(Z+\eta_N) \tag{14}$$

Multiplying (14) by $(M+\zeta_N)$, we obtain

$$\begin{aligned}(M+\zeta_N)(Y-\alpha) &= (M+\zeta_N)M^{-1}Z - (Z+\eta_N)\\ &= (MY+\zeta_N Y) - (Z+\eta_N) \end{aligned}\tag{15}$$

Substituting $Z = MY$ in (15), we get

$$\begin{aligned}(M+\zeta_N)(Y-\alpha) &= (MY+\zeta_N Y) - (MY+\eta_N)\\ &= (\zeta_N Y - \eta_N) \end{aligned}\tag{16}$$

Multiplying both side of (16) by $(M+\zeta_N)^{-1}$ to obtain

$$Y - \alpha = (M+\zeta_N)^{-1}(\zeta_N Y - \eta_N). \tag{17}$$

Also, since M is strictly diagonally dominant [21], we know that

$$\| M^{-1} \|_\infty \leq [\min_{0\leq i\leq N}(|a_{i,i}| - \sum_{j\neq i} |a_{i,j}|)]^{-1} = \omega(say) < \infty.$$

from equation (17), we obtain

$$\|Y-\alpha\|_\infty \leq 2\omega(\|\zeta_N\|_\infty \|Y\|_\infty + \|\eta_N\|_\infty), \tag{18}$$

This shows the stability of the system.

6. Results and Discussions

Four examples are presented to demonstrate the accuracy of the proposed method. The results are compared with that of shooting method [1], cubic B-spline, and extended cubic B-spline methods [18], and analytical approximation. The results are also presented with different values of n. Numerical errors are calculated using infinite and two norms, as follows:

$$L_\infty = \max_i \mid y(x_i) - S(x_i) \mid$$

$$L_2 = \sqrt{\sum_{i=1}^{N} (y(x_i) - S(x_i))^2}$$

Example 1. Consider the following nonlinear boundary value problem [1]

$$\begin{cases} y''(x) = 2y^3 - 6y - 2x^3, \quad 1 < x < 2, \\ y(1) = 2, \quad y(2) = \dfrac{5}{2} \end{cases} \tag{19}$$

The analytic solution is $y(x) = x + \frac{1}{x}$. The results for $\gamma = 2.5$ are shown in Table 2. It is clear that the HCBSM is acceptable and more accurate than shooting method [1]. The numerical results obtained by HCBSM are shown in Figs.2.

Example 2. Consider the following nonlinear boundary value problem [1]

$$\begin{cases} y''(x) = \dfrac{3}{2}y^2, \quad 0 < x < 1, \\ y(0) = 4, \quad y(1) = 1 \end{cases} \tag{20}$$

The analytic solution is $y(x) = \frac{4}{(x+1)^2}$. The results for $\gamma = 5.5$ are shown in Table 3. It is clear that the HCBSM is acceptable and more accurate than cubic B-spline method (CBSM) [18] and extended cubic B-spline method (ECBSM) [18]. The numerical results obtained by HCBSM are shown in Figs.3.

Example 3. Consider the following nonlinear boundary value problem [1]

$$\begin{cases} y''(x) = y^3 - yy', \quad 1 < x < 2, \\ y(0) = \dfrac{1}{2}, \quad y(2) = \dfrac{1}{3} \end{cases} \tag{21}$$

The analytic solution is $y(x) = \frac{1}{x+1}$. The results for $\gamma = 1.577295$ are shown in Table 4. As before HCBSM is more accurate than shooting method [1], cubic B-spline method (CBSM) [18], and extended cubic B-spline method (ECBSM) [18]. The numerical results obtained by hybrid cubic B-spline method are shown in Figs.4.

Example 4. Consider the following nonlinear boundary value problem [22]

$$\begin{cases} y''(x) = \dfrac{\exp(2y) + (y')^2}{2}, \quad 0 < x < 1, \\ y(0) = 0, \quad y(1) = \log(\dfrac{1}{2}) \end{cases} \tag{22}$$

The analytic solution is $y(x) = \log(\frac{1}{x+1})$. The results for $\gamma = 2.525$ are shown in Table 5. Again, it is clear that the HCBSM is acceptable and more accurate than cubic B-spline method (CBSM) [18] and extended cubic B-spline method (ECBSM) [18]. The numerical results obtained by hybrid cubic B-spline method are shown in Figs.5.

7. Conclusions

In this study, the HCBSM has been developed, analysed and applied to solve second-order nonlinear two-point boundary value problems. It was shown that the linear system (11) is stable. Numerical tests on four nonlinear two point boundary value problems indicate that the method is a feasible and accurate method.

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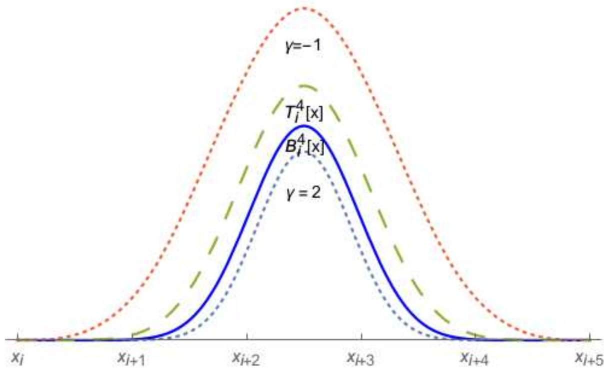

Figure 1: Hybrid cubic B-spline basis, $H_i^4(x)$, when γ=2, 1, 0, and -1

Table 1: Coefficient of H_i, $H_i^{'}$ and $H_i^{''}$

x	x_{i-1}	x_i	x_{i+1}
H_i	a_1	a_2	a_1
$H_j^{'}$	a_3	0	a_4
$H_i^{''}$	a_5	a_6	a_5

Table 2: Comparison of HCBSM results with shooting method [1] for example 1 with $h = 0.05$

x	Exact solution	Shooting method	HCBSM ($\gamma = 2.5$)
1.1	2.0090909	$6.723E-04$	$1.423E-05$
1.2	2.0333333	$1.434E-03$	$1.445E-05$
1.3	2.0692306	$2.168E-03$	$9.524E-06$
1.4	2.1142855	$2.807E-03$	$3.570E-06$
1.5	2.1666665	$3.301E-03$	$1.601E-06$
1.6	2.2249997	$3.598E-03$	$5.292E-06$
1.7	2.2882349	$3.621E-03$	$7.266E-06$
1.8	2.3555551	$3.234E-03$	$7.366E-06$
1.9	2.4263151	$2.185E-03$	$5.234E-06$

Table 3: Comparison of HCBSM results with CBSM [18] and ECBSM [18], for example 2 with $h = 0.1$

x	CBSM	ECBSM ($\lambda = 2.5771E-02$)	HCBSM ($\gamma = 5.5$)
0.1	$3.348E-03$	$2.933E-03$	$4.857E-04$
0.2	$4.617E-03$	$3.998E-03$	$3.307E-04$
0.3	$4.863E-03$	$4.165E-03$	$3.168E-06$
0.4	$4.594E-03$	$3.892E-03$	$3.091E-04$
0.5	$4.058E-03$	$3.404E-03$	$5.345E-04$
0.6	$3.381E-03$	$2.809E-03$	$6.497E-04$
0.7	$2.619E-03$	$2.157E-03$	$6.506E-04$
0.8	$1.799E-03$	$1.469E-03$	$5.399E-04$
0.9	$9.269E-04$	$7.517E-04$	$3.219E-04$

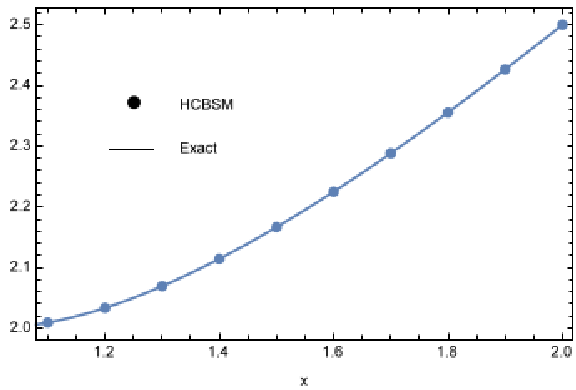

Figure 2: Numerical solution $S(x)$ and exact solution $y(x)$ for Example 1 with $h = 0.05$ and $\gamma = 2.5$

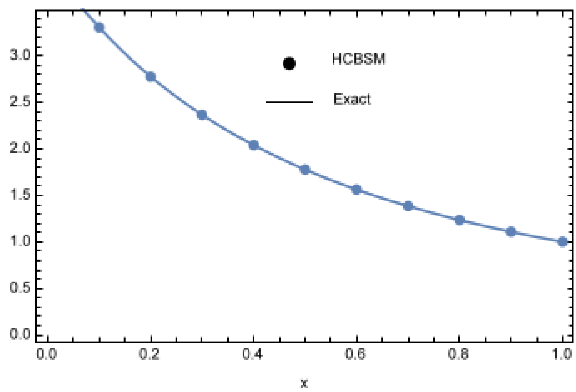

Figure 3: Numerical solution $S(x)$ and exact solution $y(x)$ for Example 2 with $h = 0.1$ and $\gamma = 5.5$

Table 4: Comparison of HCBSM results with shooting method [1], CBSM [18], and ECBSM [18], for example 3 with $h = 0.05$

x	shooting method	CBSM	ECBSM ($\lambda = 3.0966E - 03$)	HCBSM ($\gamma = 1.577295$)
1.1	$5.400E - 05$	$3.260E - 06$	$6.575E - 06$	$2.192E - 09$
1.2	$8.800E - 05$	$5.316E - 06$	$1.129E - 05$	$2.619E - 07$
1.3	$1.070E - 04$	$6.438E - 06$	$1.434E - 05$	$6.245E - 07$
1.4	$1.130E - 04$	$6.828E - 06$	$1.587E - 05$	$9.821E - 07$
1.5	$1.090E - 04$	$6.634E - 06$	$1.605E - 05$	$1.258E - 06$
1.6	$9.700E - 05$	$5.967E - 06$	$1.497E - 05$	$1.400E - 06$
1.7	$7.800E - 05$	$4.913E - 06$	$1.275E - 05$	$1.368E - 06$
1.8	$5.400E - 05$	$3.536E - 06$	$9.464E - 06$	$1.136E - 06$
1.9	$2.600E - 05$	$1.886E - 06$	$5.194E - 06$	$6.846E - 07$

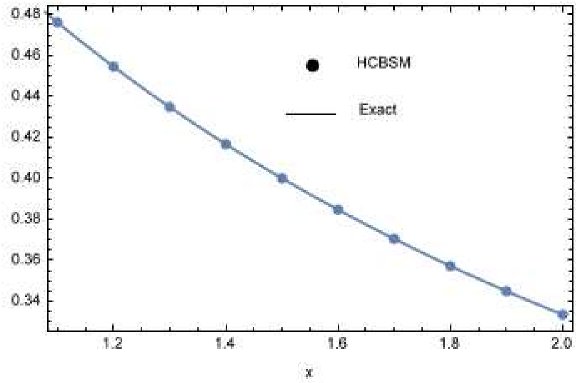

Figure 4: Numerical solution $S(x)$ and exact solution $y(x)$ for Example 3 with $h = 0.05$ and $\gamma = 1.577295$

Table 5: Comparison of HCBSM results with CBSM [18] and ECBSM [18] for example 4 with $h = 0.1$

x	CBSM	ECBSM ($\lambda = 6.6088E - 03$)	HCBSM ($\gamma = 2.525$)
0.1	$7.975E-05$	$4.327E-05$	$8.638E-06$
0.2	$1.252E-04$	$6.410E-05$	$8.487E-06$
0.3	$1.469E-04$	$7.098E-05$	$4.952E-06$
0.4	$1.516E-04$	$6.917E-05$	$8.633E-07$
0.5	$1.439E-04$	$6.198E-05$	$2.393E-06$
0.6	$1.269E-04$	$5.158E-05$	$4.262E-06$
0.7	$1.027E-04$	$3.938E-05$	$4.672E-06$
0.8	$7.276E-05$	$2.632E-05$	$3.833E-06$
0.9	$3.826E-05$	$1.305E-05$	$2.120E-06$

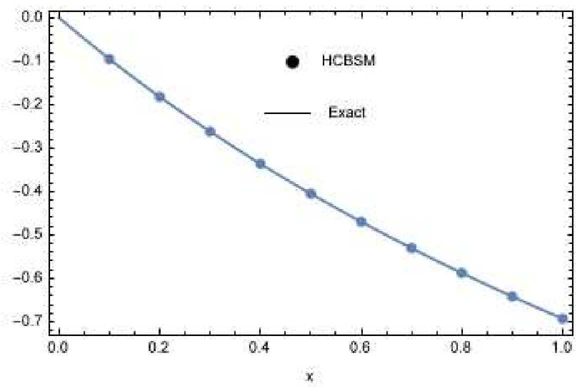

Figure 5: Numerical solution $S(x)$ and exact solution $y(x)$ for Example 4 with $h = 0.1$ and $\gamma = 2.525$