SYMMETRIC IDENTITIES FOR CARLITZ-TYPE TWISTED 
(h, q)-BERNOULLI NUMBERS AND POLYNOMIALS 
ASSOCIATED WITH p-ADIC q-INTEGRAL ON \( \mathbb{Z}_p \)

C.S. Ryoo
Department of Mathematics
Hannam University
Daejeon, 306-791, KOREA

Abstract: In this paper, we obtain some interesting symmetric identities for Carlitz’s twisted (h, q)-Bernoulli polynomials in p-adic field. Some interesting results and relationships are obtained.

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1. Introduction

Many mathematicians have studied in the area of the q-Bernoulli numbers and polynomials. The Bernoulli numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics (see [1-6]). Recently, Y. Hu studied several identities of symmetry for Carlitz’s q-Bernoulli numbers and polynomials in complex field (see [1]). D. Kim et al.[3] derived some identities of symmetry for Carlitz’s q-Bernoulli numbers and polynomials by using the p-adic q-integrals on \( \mathbb{Z}_p \) in p-adic field. In this paper, we obtain some interesting symmetric identities for Carlitz’s twisted (h, q)-Bernoulli polynomials in p-adic field. In this paper, if we take \( \zeta = 1 \), then [3] is the special case of this paper.
Also in this paper, if we take $h = 1$, then [5] is the special case of this paper.

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of $p$-adic rational integers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1 - q^x}{1 - q} \quad (\text{cf. [1-6]}).$$

Hence, $\lim_{q \to 1}[x] = x$ for any $x$ with $|x|_p \leq 1$ in the present $p$-adic case. For $g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} g(x)q^x \quad (\text{cf. [2, 3]}).$$

Let

$$T_p = \bigcup_{m \geq 1} C_{p^m} = \lim_{m \to \infty} C_{p^m},$$

where $C_{p^m} = \{\zeta | \zeta^{p^m} = 1\}$ is the cyclic group of order $p^m$. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto \zeta^x$ (see [6]).

First we introduce the twisted $(h, q)$-Bernoulli numbers $\beta_{n, q, \zeta}^{(h)}$ and polynomials $\beta_{n, q, \zeta}^{(h)}(x)$. We also find generating functions of twisted $(h, q)$-Bernoulli numbers $\beta_{n, q, \zeta}^{(h)}$ and polynomials $\beta_{n, q, \zeta}^{(h)}(x)$. For $h \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|q - 1|_p < p^{-\frac{1}{p-1}}$, twisted $(h, q)$-Bernoulli numbers $\beta_{n, q, \zeta}^{(h)}$ are defined by

$$\beta_{n, q, \zeta}^{(h)} = \int_{\mathbb{Z}_p} \phi_\zeta(x)q^{(h-1)x}[x]_q^n d\mu_q(x). \quad (1.1)$$
By using $p$-adic $q$-integral on $\mathbb{Z}_p$, we obtain,

$$
\int_{\mathbb{Z}_p} \phi_\zeta(x)q^{(h-1)x}x^n d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} \zeta^x q^{(h-1)x}x^n q^x
$$

$$
= \left(\frac{1}{1-q}\right)^{n-1} \sum_{l=0}^{n} \left(\frac{n}{l}\right) (-1)^l \frac{h+l}{1-\zeta q^{h+l}}
$$

(1.2)

By (1.2), we have

$$
\beta^{(h)}_{n,q,\zeta} = \left(\frac{1}{1-q}\right)^{n-1} \sum_{l=0}^{n} \left(\frac{n}{l}\right) (-1)^l \frac{h+l}{1-\zeta q^{h+l}}.
$$

We set

$$
F_{q,\zeta}^{(h)}(t) = \sum_{n=0}^{\infty} B_{n,q,\zeta}^{(h)} \frac{t^n}{n!}.
$$

By using above equation and (1.2), we have

$$
F_{q,\zeta}^{(h)}(t) = \sum_{n=0}^{\infty} \beta_{n,q,\zeta}^{(h)} t^n = \int_{\mathbb{Z}_p} \phi_\zeta(y)q^{(h-1)y}x^n e^{x[t]} d\mu_q(x)
$$

(1.3)

Next, we introduce twisted $(h,q)$-Bernoulli polynomials $\beta_{n,q,\zeta}^{(h)}(x)$. The twisted $(h,q)$-Bernoulli polynomials $\beta_{n,q,\zeta}^{(h)}(x)$ are defined by

$$
\beta_{n,q,\zeta}^{(h)}(x) = \int_{\mathbb{Z}_p} \phi_\zeta(y)q^{(h-1)y}x^n d\mu_q(y).
$$

By using $p$-adic $q$-integral, we obtain

$$
\beta_{n,q,\zeta}^{(h)}(x) = \left(\frac{1}{1-q}\right)^{n-1} \sum_{l=0}^{n} \left(\frac{n}{l}\right) (-1)^l q^{xl} \frac{h+l}{1-\zeta q^{h+l}}.
$$

(1.4)

We set

$$
F_{q,\zeta}^{(h)}(t, x) = \sum_{n=0}^{\infty} \beta_{n,q,\zeta}^{(h)}(x) \frac{t^n}{n!}.
$$

Since $[x+y]_q = [x]_q + q^x[y]_q$, we easily obtain that

$$
\beta_{n,q,\zeta}^{(h)}(x) = \int_{\mathbb{Z}_p} \phi_\zeta(y)q^{(h-1)y}x^n d\mu_q(y)
$$

$$
= \sum_{l=0}^{n} \left(\frac{n}{l}\right) [x]_q^{n-l} q^{xl} \beta_{l,q,\zeta}^{(h)}
$$

(1.5)

$$
= \left([x]_q + q^x \beta_{q,\zeta}^{(h)}\right)^n.
$$
2. Symmetric Properties for Carlitz’s Twisted \((h, q)\)-Bernoulli Numbers and Polynomials

Our primary goal of this section is to obtain symmetric identities for Carlitz’s twisted \((h, q)\)-Bernoulli numbers \(\beta_{n,q,\zeta}^{(h)}\) and polynomials \(\beta_{n,q,\zeta}^{(h)}(x)\). By using the similar method of [3], expect for obvious modifications, we are going to obtain the main results of Carlitz’s twisted \((h, q)\)-Bernoulli polynomials.

Let \(w_1\) and \(w_2\) be natural numbers. Then we have

\[
\frac{1}{[w_1]_q} \int_{\mathbb{Z}_p} \zeta^{w_1 y} q^{w_1 (h-1)y} e^{[w_1 w_2 x+w_2 j+w_1 y]_q t} d\mu_{q^{w_1}}(y)
\]

\[
= \lim_{N \to \infty} \frac{1}{[w_1]_q} \frac{1}{[p^N]_q} \sum_{y=0}^{p^N-1} e^{[w_1 w_2 x+w_2 j+w_1 y]_q t} \zeta^{w_1 y} q^{w_1 (h-1)y} q^{w_1 y}
\]

\[
= \lim_{N \to \infty} \frac{1}{[w_1]_q [w_2]_q} \sum_{i=0}^{w_2-1} \sum_{y=0}^{p_2-1} e^{[w_1 w_2 x+w_2 j+w_1 i+w_1 y]_q t} \zeta^{w_1 (i+w_2 y)} q^{w_1 h(i+w_2 y)}
\]

From (2.1), we can derive the following equation (2.2):

\[
\frac{1}{[w_1]_q} \sum_{j=0}^{w_1-1} \zeta^{w_2 j} q^{h w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} q^{w_1 (h-1)y} e^{[w_1 w_2 x+w_2 j+w_1 y]_q t} d\mu_{q^{w_1}}(y)
\]

\[
= \lim_{N \to \infty} \frac{1}{[w_1]_q [w_2]_q} \sum_{j=0}^{w_1-1} \sum_{i=0}^{w_2-1} \sum_{y=0}^{p_2-1} e^{[w_1 w_2 x+w_2 j+w_1 i+w_1 y]_q t} \zeta^{w_2 j} \zeta^{w_1 i} \zeta^{w_1 w_2 y} q^{w_2 h j} q^{w_1 h i} q^{w_1 w_2 h y}
\]

By the same method as (2.2), we obtain

\[
\frac{1}{[w_2]_q} \sum_{j=0}^{w_2-1} \zeta^{w_1 j} q^{h w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} q^{w_2 (h-1)y} e^{[w_1 w_2 x+w_1 j+w_2 y]_q t} d\mu_{q^{w_2}}(y)
\]

\[
= \lim_{N \to \infty} \frac{1}{[w_1]_q [w_2]_q} \sum_{j=0}^{w_2-1} \sum_{i=0}^{w_1-1} \sum_{y=0}^{p_1-1} e^{[w_1 w_2 x+w_1 j+w_2 i+w_1 y]_q t} \zeta^{w_1 j} \zeta^{w_2 i} \zeta^{w_1 w_2 y} q^{w_1 h j} q^{w_2 h i} q^{w_1 w_2 h y}
\]
Therefore, by (2.2) and (2.3), we have the following theorem.

**Theorem 1.** For \( w_1, w_2 \in \mathbb{N} \), we have

\[
\frac{1}{[w_1]_q} \sum_{j=0}^{w_1-1} \zeta^{w_2j} q^{h w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} q^{w_1 (h-1) y} e^{[w_1 w_2 x + w_2 j + w_1 y] q t} \, d\mu_q^{w_1} (y) = \frac{1}{[w_2]_q} \sum_{j=0}^{w_2-1} \zeta^{w_1 j} q^{h w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_2 y} q^{w_2 (h-1) y} e^{[w_1 w_2 x + w_1 j + w_2 y] q t} \, d\mu_q^{w_2} (y).
\]

(2.4)

By substituting Taylor series of \( e^{x t} \) into (2.4) and after elementary calculations, we obtain the following corollary.

**Corollary 2.** For \( w_1, w_2 \in \mathbb{N}, n \geq 0 \), we have

\[
[w_1]_q^{n-1} \sum_{j=0}^{w_1-1} \zeta^{w_2j} q^{h w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} q^{w_1 (h-1) y} \left[ w_2 x + \frac{w_2}{w_1} j + y \right]^{n} \, q^{w_1} d\mu_q^{w_1} (y) = [w_2]_q^{n-1} \sum_{j=0}^{w_2-1} \zeta^{w_1 j} q^{h w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_2 y} q^{w_2 (h-1) y} \left[ w_1 x + \frac{w_1}{w_2} j + y \right]^{n} \, q^{w_2} d\mu_q^{w_2} (y).
\]

(2.5)

By (1.5) and Corollary 2, we have the following theorem.

**Theorem 3.** For \( w_1, w_2 \in \mathbb{N}, n \geq 0 \), we have

\[
[w_1]_q^{n-1} \sum_{j=0}^{w_1-1} \zeta^{w_2j} q^{h w_2 j} \beta_n q^{w_1, \zeta^{w_1}} \left( w_2 x + \frac{w_2}{w_1} j \right) = [w_2]_q^{n-1} \sum_{j=0}^{w_2-1} \zeta^{w_1 j} q^{h w_1 j} \beta_n q^{w_2, \zeta^{w_2}} \left( w_1 x + \frac{w_1}{w_2} j \right).
\]

By (2.5), we can derive the following equation (2.6):

\[
\int_{\mathbb{Z}_p} \zeta^{w_1 y} q^{w_1 (h-1) y} \left[ w_2 x + \frac{w_2}{w_1} j + y \right]^{n} \, q^{w_1} d\mu_q^{w_1} (y) = \sum_{i=0}^{n} \binom{n}{i} \left( \frac{[w_2]_q}{[w_1]_q} \right)^i \int_{\mathbb{Z}_p} \zeta^{w_1 y} q^{w_1 (h-1) y} \left[ w_2 x + y \right]^{n-i} \, q^{w_1} d\mu_q^{w_1} (y)
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} \left( \frac{[w_2]_q}{[w_1]_q} \right)^i \int_{\mathbb{Z}_p} \zeta^{w_1 y} q^{w_1 (h-1) y} \beta_n^{w_2} \left( w_2 x \right).
\]

(2.6)
By (2.6), and Theorem 3, we have

\[ [w_1]_{q}^{n-1} \sum_{j=0}^{w_1-1} \zeta^{w_2j} q^{hw_2j} \int_{Z_p} \zeta^{w_1y} q^{w_1(h-1)y} \left[ w_2 x + \frac{w_2}{w_1} j + y \right]^{n}_{q^{w_1}} d\mu_{q^{w_1}}(y) \]

\[ = \sum_{j=0}^{w_1-1} \zeta^{w_2j} q^{hw_2j} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [w_2]^i [w_1]_{q}^{n-i-1} [j]^i q^{w_2(n-i)j} \beta_{n-i,q^{w_1},\zeta^{w_1}}^{(h)} (w_2 x) \]

\[ = \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [w_2]^i [w_1]_{q}^{n-i-1} \beta_{n-i,q^{w_1},\zeta^{w_1}}^{(h)} \left( w_2 x \right) T_{n,i}^{(h)} (w_1, \zeta^{w_2}, q^{w_2}), \]

where

\[ T_{n,i}^{(h)} (w_1, \zeta, q) = \sum_{j=0}^{w_1-1} \zeta^j q^{(n-i+h)j} [j]^i. \]

By the same method as (2.7), we get

\[ [w_2]_{q}^{n-1} \sum_{j=0}^{w_2-1} \zeta^{w_1j} q^{hw_1j} \int_{Z_p} \zeta^{w_2y} q^{w_2(h-1)y} \left[ w_1 x + \frac{w_1}{w_2} j + y \right]^{n}_{q^{w_2}} d\mu_{q^{w_2}}(y) \]

\[ = \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [w_1]^i [w_2]_{q}^{n-i-1} \beta_{n-i,q^{w_2},\zeta^{w_2}}^{(h)} (w_1 x) T_{n,i}^{(h)} (w_2, \zeta^{w_1}, q^{w_1}). \]

By (2.7) and (2.8), we have the following theorem.

**Theorem 4.** For \( w_1, w_2 \in \mathbb{N}, n \geq 0 \), we have

\[ \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [w_2]^i [w_1]_{q}^{n-i-1} \beta_{n-i,q^{w_2},\zeta^{w_2}}^{(h)} (w_2 x) T_{n,i}^{(h)} (w_1, \zeta^{w_2}, q^{w_2}) \]

\[ = \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [w_1]^i [w_2]_{q}^{n-i-1} \beta_{n-i,q^{w_2},\zeta^{w_2}}^{(h)} (w_1 x) T_{n,i}^{(h)} (w_2, \zeta^{w_1}, q^{w_1}). \]

By (1.5) and Theorem 4, we have the following corollary.
Corollary 5. For \( w_1, w_2 \in \mathbb{N}, n \geq 0 \), we have

\[
\sum_{i=0}^{n} \sum_{l=0}^{n-i} \binom{n}{i} \binom{n - i - l}{l} [w_2]^i [w_2]^{n-i-l} [w_1]^{n-i-1} T_{n,i}^{(h)}(w_1, \zeta^{w_2}, q^{w_2}) \\
\times q^{w_1 w_2 x l} [x]^{n-i-l} [l, q^{w_1}, q^{w_2}]^{(h)} = 0
\]

\[
= \sum_{i=0}^{n} \sum_{l=0}^{n-i} \binom{n}{i} \binom{n - i - l}{l} [w_1]^i [w_1]^{n-i-l} [w_2]^{n-i-1} T_{n,i}^{(h)}(w_2, \zeta^{w_1}, q^{w_1}) \\
\times q^{w_1 w_2 x l} [x]^{n-i-l} [l, q^{w_1}, q^{w_2}]^{(h)}
\]

References

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