International Journal of Pure and Applied Mathematics
Volume 110 No. 2 2016, 389-395
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v110i2.13

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SYMMETRIC IDENTITIES FOR CARLITZ-TYPE TWISTED (h, q)-BERNOULLI NUMBERS AND POLYNOMIALS ASSOCIATED WITH p-ADIC q-INTEGRAL ON $\mathbb{Z}_p$

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Abstract: In this paper, we obtain some interesting symmetric identities for Carlitz's twisted (h, q)-Bernoulli polynomials in p-adic field. Some interesting results and relationships are obtained.

AMS Subject Classification: 11B68, 11S40, 11S80
Key Words: Bernoulli numbers and polynomials, q-Bernoulli numbers and polynomials, twisted (h, q)-Bernoulli numbers and polynomials, symmetric identity

1. Introduction

Many mathematicians have studied in the area of the q-Bernoulli numbers and polynomials. The Bernoulli numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics (see [1-6]). Recently, Y. Hu studied several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field (see [1]). D. Kim *et al.*[3] derived some identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials by using the p-adic q-integrals on $\mathbb{Z}_p$ in p-adic field. In this paper, we obtain some interesting symmetric identities for Carlitz's twisted (h, q)-Bernoulli polynomials in p-adic field. In this paper, if we take $\zeta = 1$, then [3] is the special case of this paper.

Received: June 15, 2015
Revised: May 9, 2016
Published: November 4, 2016

Also in this paper, if we take $h=1$, then [5] is the special case of this paper.

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q} \text{ (cf. [1-6])} .$$

Hence, $\lim_{q \to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} g(x) q^x \text{ (cf. [2, 3])} .$$

Let

$$T_p = \cup_{m \geq 1} C_{p^m} = \lim_{m \to \infty} C_{p^m},$$

where $C_{p^m} = \{\zeta | \zeta^{p^m} = 1\}$ is the cyclic group of order p^m. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \zeta^x$ (see [6]).

First we introduce the twisted (h,q)-Bernoulli numbers $\beta_{n,q,\zeta}^{(h)}$ and polynomials $\beta_{n,q,\zeta}^{(h)}(x)$. We also find generating functions of twisted (h,q)-Bernoulli numbers $\beta_{n,q,\zeta}^{(h)}$ and polynomials $\beta_{n,q,\zeta}^{(h)}(x)$. For $h \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$, twisted (h,q)-Bernoulli numbers $\beta_{n,q,\zeta}^{(h)}$ are defined by

$$\beta_{n,q,\zeta}^{(h)} = \int_{\mathbb{Z}_p} \phi_\zeta(x) q^{(h-1)x} [x]_q^n d\mu_q(x). \tag{1.1}$$

By using p-adic q-integral on $\mathbb{Z}_p$, we obtain,

$$\begin{aligned}\int_{\mathbb{Z}_p} \phi_\zeta(x) q^{(h-1)x}[x]_q^n d\mu_q(x) &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} \zeta^x q^{(h-1)x}[x]_q^n q^x \\ &= \left(\frac{1}{1-q}\right)^{n-1} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{h+l}{1-\zeta q^{h+l}}\end{aligned} \tag{1.2}$$

By (1.2), we have

$$\beta_{n,q,\zeta}^{(h)} = \left(\frac{1}{1-q}\right)^{n-1} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{h+l}{1-\zeta q^{h+l}}.$$

We set

$$F_{q,\zeta}^{(h)}(t) = \sum_{n=0}^{\infty} B_{n,q,\zeta}^{(h)} \frac{t^n}{n!}.$$

By using above equation and (1.2), we have

$$F_{q,\zeta}^{(h)}(t) = \sum_{n=0}^{\infty} \beta_{n,q,\zeta}^{(h)} \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \zeta^x q^{(h-1)x} e^{[x]_q t} d\mu_q(x) \tag{1.3}$$

Next, we introduce twisted (h,q)-Bernoulli polynomials $\beta_{n,q,\zeta}^{(h)}(x)$. The twisted (h,q)-Bernoulli polynomials $\beta_{n,q,\zeta}^{(h)}(x)$ are defined by

$$\beta_{n,q,\zeta}^{(h)}(x) = \int_{\mathbb{Z}_p} \phi_\zeta(y) q^{(h-1)y}[x+y]_q^n d\mu_q(y).$$

By using p-adic q-integral, we obtain

$$\beta_{n,q,\zeta}^{(h)}(x) = \left(\frac{1}{1-q}\right)^{n-1} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} \frac{h+l}{1-\zeta q^{h+l}}. \tag{1.4}$$

We set

$$F_{q,\zeta}^{(h)}(t,x) = \sum_{n=0}^{\infty} \beta_{n,q,\zeta}^{(h)}(x) \frac{t^n}{n!}.$$

Since $[x+y]_q = [x]_q + q^x[y]_q$, we easily obtain that

$$\begin{aligned}\beta_{n,q,\zeta}^{(h)}(x) &= \int_{\mathbb{Z}_p} \phi_\zeta(y) q^{(h-1)y}[x+y]_q^n d\mu_q(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{xl} \beta_{l,q,\zeta}^{(h)} \\ &= \left([x]_q + q^x \beta_{q,\zeta}^{(h)}\right)^n.\end{aligned} \tag{1.5}$$

2. Symmetric Properties for Carlitz's Twisted (h,q)-Bernoulli Numbers and Polynomials

Our primary goal of this section is to obtain symmetric identities for Carlitz's twisted (h,q)-Bernoulli numbers $\beta_{n,q,\zeta}^{(h)}$ and polynomials $\beta_{n,q,\zeta}^{(h)}(x)$. By using the similar method of [3], expect for obvious modifications, we are going to obtain the main results of Carlitz's twisted (h,q)-Bernoulli polynomials.

Let w_1 and w_2 be natural numbers. Then we have

$$\begin{aligned}
&\frac{1}{[w_1]_q}\int_{\mathbb{Z}_p}\zeta^{w_1y}q^{w_1(h-1)y}e^{[w_1w_2x+w_2j+w_1y]_qt}d\mu_{q^{w_1}}(y)\\
&=\lim_{N\to\infty}\frac{1}{[w_1]_q}\frac{1}{[p^N]_{q^{w_1}}}\sum_{y=0}^{p^N-1}e^{[w_1w_2x+w_2j+w_1y]_qt}\zeta^{w_1y}q^{w_1(h-1)y}q^{w_1y}\\
&=\lim_{N\to\infty}\frac{1}{[w_1]_q}\frac{1}{[w_2p^N]_{q^{w_1}}}\sum_{y=0}^{p^N-1}e^{[w_1w_2x+w_2j+w_1y]_qt}\zeta^{w_1y}q^{w_1hy}\\
&=\lim_{N\to\infty}\frac{1}{[w_1w_2p^N]_q}\sum_{i=0}^{w_2-1}\sum_{y=0}^{p^N-1}e^{[w_1w_2x+w_2j+w_1(i+w_2y)]_qt}\zeta^{w_1(i+w_2y)}q^{w_1h(i+w_2y)}
\end{aligned}\tag{2.1}$$

From (2.1), we can derive the following equation (2.2):

$$\begin{aligned}
&\frac{1}{[w_1]_q}\sum_{j=0}^{w_1-1}\zeta^{w_2j}q^{hw_2j}\int_{\mathbb{Z}_p}\zeta^{w_1y}q^{w_1(h-1)y}e^{[w_1w_2x+w_2j+w_1y]_qt}d\mu_{q^{w_1}}(y)\\
&=\lim_{N\to\infty}\frac{1}{[w_1w_2p^N]_q}\sum_{j=0}^{w_1-1}\sum_{i=0}^{w_2-1}\sum_{y=0}^{p^N-1}e^{[w_1w_2x+w_2j+w_1i+w_1w_2y)]_qt}\\
&\qquad\times\zeta^{w_2j}\zeta^{w_1i}\zeta^{w_1w_2y}q^{w_2hj}q^{w_1hi}q^{w_1w_2hy}
\end{aligned}\tag{2.2}$$

By the same method as (2.2), we obtain

$$\begin{aligned}
&\frac{1}{[w_2]_q}\sum_{j=0}^{w_2-1}\zeta^{w_1j}q^{hw_1j}\int_{\mathbb{Z}_p}\zeta^{w_2y}q^{w_2(h-1)y}e^{[w_1w_2x+w_1j+w_2y]_qt}d\mu_{q^{w_2}}(y)\\
&=\lim_{N\to\infty}\frac{1}{[w_1w_2p^N]_q}\sum_{j=0}^{w_2-1}\sum_{i=0}^{w_1-1}\sum_{y=0}^{p^N-1}e^{[w_1w_2x+w_1j+w_2i+w_1w_2y)]_qt}\\
&\qquad\times\zeta^{w_1j}\zeta^{w_2i}\zeta^{w_1w_2y}q^{w_1hj}q^{w_2hi}q^{w_1w_2hy}
\end{aligned}\tag{2.3}$$

Therefore, by (2.2) and (2.3), we have the following theorem.

Theorem 1. *For $w_1, w_2 \in \mathbb{N}$, we have*

$$\frac{1}{[w_1]_q}\sum_{j=0}^{w_1-1}\zeta^{w_2 j}q^{hw_2 j}\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1(h-1)y}e^{[w_1w_2x+w_2j+w_1y]_q t}d\mu_{q^{w_1}}(y)$$
$$=\frac{1}{[w_2]_q}\sum_{j=0}^{w_2-1}\zeta^{w_1 j}q^{hw_1 j}\int_{\mathbb{Z}_p}\zeta^{w_2 y}q^{w_2(h-1)y}e^{[w_1w_2x+w_1j+w_2y]_q t}d\mu_{q^{w_2}}(y). \tag{2.4}$$

By substituting Taylor series of e^{xt} into (2.4) and after elementary calculations, we obtain the following corollary.

Corollary 2. *For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have*

$$[w_1]_q^{n-1}\sum_{j=0}^{w_1-1}\zeta^{w_2 j}q^{hw_2 j}\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1(h-1)y}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y)$$
$$=[w_2]_q^{n-1}\sum_{j=0}^{w_2-1}\zeta^{w_1 j}q^{hw_1 j}\int_{\mathbb{Z}_p}\zeta^{w_2 y}q^{w_2(h-1)y}\left[w_1x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}}^n d\mu_{q^{w_2}}(y). \tag{2.5}$$

By (1.5) and Corollary 2, we have the following theorem.

Theorem 3. *For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have*

$$[w_1]_q^{n-1}\sum_{j=0}^{w_1-1}\zeta^{w_2 j}q^{hw_2 j}\beta_{n,q^{w_1},\zeta^{w_1}}^{(h)}\left(w_2x+\frac{w_2}{w_1}j\right)$$
$$=[w_2]_q^{n-1}\sum_{j=0}^{w_2-1}\zeta^{w_1 j}q^{hw_1 j}\beta_{n,q^{w_2},\zeta^{w_2}}^{(h)}\left(w_1x+\frac{w_1}{w_2}j\right).$$

By (2.5), we can derive the following equation (2.6):

$$\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1(h-1)y}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y)$$
$$=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i[j]_{q^{w_2}}^i q^{w_2(n-i)j}\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1(h-1)y}[w_2x+y]_{q^{w_1}}^{n-i}d\mu_{q^{w_1}}(y)$$
$$=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i[j]_{q^{w_2}}^i q^{w_2(n-i)j}\beta_{n-i,q^{w_1},\zeta^{w_1}}^{(h)}(w_2x). \tag{2.6}$$

By (2.6), and Theorem 3, we have

$$\begin{aligned}
&[w_1]_q^{n-1}\sum_{j=0}^{w_1-1}\zeta^{w_2j}q^{hw_2j}\int_{\mathbb{Z}_p}\zeta^{w_1y}q^{w_1(h-1)y}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y)\\
&=\sum_{j=0}^{w_1-1}\zeta^{w_2j}q^{hw_2j}\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i-1}[j]_{q^{w_2}}^i q^{w_2(n-i)j}\beta_{n-i,q^{w_1},\zeta^{w_1}}^{(h)}(w_2x)\\
&=\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i-1}\beta_{n-i,q^{w_1},\zeta^{w_1}}^{(h)}(w_2x)\sum_{j=0}^{w_1-1}\zeta^{w_2j}q^{w_2(n-i+h)j}[j]_{q^{w_2}}^i\\
&=\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i-1}\beta_{n-i,q^{w_1},\zeta^{w_1}}^{(h)}(w_2x)\,T_{n,i}^{(h)}(w_1,\zeta^{w_2},q^{w_2}),
\end{aligned}\tag{2.7}$$

where

$$T_{n,i}^{(h)}(w_1,\zeta,q)=\sum_{j=0}^{w_1-1}\zeta^j q^{(n-i+h)j}[j]_q^i.$$

By the same method as (2.7), we get

$$\begin{aligned}
&[w_2]_q^{n-1}\sum_{j=0}^{w_2-1}\zeta^{w_1j}q^{hw_1j}\int_{\mathbb{Z}_p}\zeta^{w_2y}q^{w_2(h-1)y}\left[w_1x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}}^n d\mu_{q^{w_2}}(y)\\
&=\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i-1}\beta_{n-i,q^{w_2},\zeta^{w_2}}^{(h)}(w_1x)\,T_{n,i}^{(h)}(w_2,\zeta^{w_1},q^{w_1}).
\end{aligned}\tag{2.8}$$

By (2.7) and (2.8), we have the following theorem.

Theorem 4. *For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have*

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i-1}\beta_{n-i,q^{w_1},\zeta^{w_1}}^{(h)}(w_2x)\,T_{n,i}^{(h)}(w_1,\zeta^{w_2},q^{w_2})\\
&=\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i-1}\beta_{n-i,q^{w_2},\zeta^{w_2}}^{(h)}(w_1x)\,T_{n,i}^{(h)}(w_2,\zeta^{w_1},q^{w_1}).
\end{aligned}$$

By (1.5) and Theorem 4, we have the following corollary.

Corollary 5. *For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have*

$$\sum_{i=0}^{n}\sum_{l=0}^{n-i}\binom{n}{i}\binom{n-i-l}{l}[w_2]_q^i[w_2]_{q^{w_1}}^{n-i-l}[w_1]_q^{n-i-1}T_{n,i}^{(h)}(w_1,\zeta^{w_2},q^{w_2})$$
$$\times\, q^{w_1w_2xl}[x]_{q^{w_1w_2}}^{n-i-l}\beta_{l,q^{w_1},\zeta^{w_1}}^{(h)}$$
$$=\sum_{i=0}^{n}\sum_{l=0}^{n-i}\binom{n}{i}\binom{n-i-l}{l}[w_1]_q^i[w_1]_{q^{w_2}}^{n-i-l}[w_2]_q^{n-i-1}T_{n,i}^{(h)}(w_2,\zeta^{w_1},q^{w_1})$$
$$\times\, q^{w_1w_2xl}[x]_{q^{w_1w_2}}^{n-i-l}\beta_{l,q^{w_2},\zeta^{w_2}}^{(h)}.$$

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