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A NEW SUBCLASS OF P-VALENT FUNCTIONS DEFINED BY AOUF ET AL DERIVATIVE OPERATOR

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Abstract: In this present paper, we study a new subclass of $\delta_p(A, B, b, \omega, \lambda, l, n)$ of analytic $p-valent$ functions in the unit disk $U = \{z : |z| < 1\}$ defined by Aouf et al derivative operator. Coefficient estimates and relevant connection of this class to the famous Fekete-Szego were obtained.

AMS Subject Classification: 30C45

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1. Introduction

Let $S(\omega)$ be the class of functions $f(z)$ of the form

$$f(z) =(z-\omega)+\sum_{k=2}^{\infty} a_k(z-\omega)^k,$$
$$U =\{z : z \in C \text{ and } \mid z \mid< 1\},$$

which are analytic and univalent in the open unit disk U, normalized with

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$f(\omega) = f'(\omega) - 1 = 0$ and ω is an arbitrary fixed point in U.

Also, let $\delta_p(\omega)$ denote the class of analytic p-valent functions of the form:

$$h(z) = (z-\omega)^p + \sum_{k=1}^{\infty} a_{p+k}(z-\omega)^{p+k}, \tag{1}$$

defined in the unit disk where ω is an arbitrary fixed point in U and satisfying the conditions $h(\omega) = 0,\ |h(z)| < 1,\ z \in U$.

Wald in [11] established that, if $P(\omega) \subset P$ (i.e class of caratheodary functions) and $p(z)$ is of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_k(z-\omega)^k, \tag{2}$$

then

$$\left|p_k\right| \leq \frac{2}{(1+d)(1-d)^k}, \quad k \geq 1 \ and \ |\omega| = d. \tag{3}$$

This result is scattered in many literatures [1,2,5,6,7,8] and we also note that if A and B are arbitrarily fixed integers and $-1 \leq B < A \leq 1$, then we have

$$|p_k| \leq \frac{A-B}{(1+d)(1-d)^k}, \quad k \geq 1,\ -1 \leq B < A \leq 1, \text{ and } |\omega| = d.$$

For $f(z) \in \delta_p(\omega)$, Eker and Seker in [9] introduced the following differential operator

$$\begin{aligned} D^0 f(z) =& f(z), \\ D^1 f(z) =& D\Big(f(z)\Big) = \frac{(z-\omega)}{p} f'(z) = (z-\omega)^p + \sum_{k=1}^{\infty} \Big(\frac{p+k}{p}\Big) a_{p+k}(z-\omega)^{p+k}, \\ D^2 f(z) =& \frac{(z-\omega)}{p} f'(z) = (z-\omega)^p + \sum_{k=1}^{\infty} \Big(\frac{p+k}{p}\Big)^2 a_{p+k}(z-\omega)^{p+k}, \end{aligned}$$

and

$$D^{n+p} f(z) = D\Big(D^{n+p-1} f(z)\Big) = (z-\omega)^p + \sum_{k=1}^{\infty} \Big(\frac{p+k}{p}\Big)^n a_{p+k}(z-\omega)^{p+k},$$

$$m \in N_0.$$

Using Aouf et al derivative operator (see [3]), we can write for functions $f(z) \in \delta_p(\omega)$:

$$\begin{aligned} I^0_{\omega,p}(\lambda,l)f(z) =& f(z), \\ I^1_{\omega,p}(\lambda,l)f(z) =& \Big(I^0_{\omega,p}(\lambda,l)f(z)\Big)\left(\frac{1-\lambda+l}{1+l}\right) + \Big(I^0_{\omega,p}(\lambda,l)f(z)\Big)' \frac{\lambda z}{1+l} \\ =& \left(\frac{1+\lambda(p-1)+l}{1+l}\right)(z-\omega)^p \\ &+ \sum_{k=1}^{\infty}\left(\frac{1+\lambda(p+k-1)+l}{1+l}\right)a_{p+k}(z-\omega)^{p+k}, \\ I^2_{\omega,p}(\lambda,l)f(z) =& \left(\frac{1+\lambda(p-1)+l}{1+l}\right)(z-\omega)^p \\ &+ \sum_{k=1}^{\infty}\left(\frac{1+\lambda(p+k-1)+l}{1+l}\right)^2 a_{p+k}(z-\omega)^{p+k}, \end{aligned}$$

and in general, we have

$$I^n_{\omega,p}(\lambda,l)f(z) = \left(\frac{1+\lambda(p-1)+l}{1+l}\right)^n (z-\omega)^p$$
$$+ \sum_{k=1}^{\infty}\left(\frac{1+\lambda(p+k-1)+l}{1+l}\right)^n a_{p+k}(z-\omega)^{p+k}, \quad n \in N_0,\ \lambda \geq 0,\ l \geq 0. \tag{4}$$

With $\omega = 0$, $l = 0$ and $\lambda = 1$. It is trivial to show that

$$\frac{1}{p^n} I^n_{0,p}(1,0)f(z) = D^{n+p}f(z).$$

Now, the authors wish to introduce the class $\delta_p(A,B,b,\omega,\lambda,l,n)$ consisting of functions $f(z)$ of the form (1) and satisfying

$$1 + \frac{1}{b}\left\{\frac{(z-\omega)\Big(\frac{1}{p^n}I^n_{\omega,p}(\lambda,l)f(z)\Big)'}{\Big(\frac{1}{p^n}I^n_{\omega,p}(\lambda,l)f(z)\Big)} - p\right\} \prec \frac{1+A(z-\omega)}{1+B(z-\omega)}$$
$$-1 \leq B < A \leq 1,\ n \in N_0,\ \lambda \geq 0,\ l \geq 0,\ z \in U,$$

where b is any non-zero complex number, $\prec$ denotes subordination and ω is an arbitrary fixed point in U. With various choices of parameters involved, this class of functions would give birth to several existing ones by several authors, see [1], [3], [4].

Hence, by the definition of subordination it follows that $f \in \delta_p(A, B, b, \omega, \lambda, l, n)$ if and only if

$$1 + \frac{1}{b}\left\{\frac{(z-\omega)\left(\frac{1}{p^n} I^n_{\omega,p}(\lambda, l) f(z)\right)'}{\left(\frac{1}{p^n} I^n_{\omega,p}(\lambda, l) f(z)\right)} - p\right\} = \frac{1 + Ah(z-\omega)}{1 + Bh(z-\omega)} = p(z), \quad z, h \in U, \quad (5)$$

where $p(z)$ is as earlier defined in (2).

2. Main Results

In this section, we give the coefficient estimates for the classes $\delta_p(A, B, b, \omega, \lambda, l, n)$.

Theorem 1. *Let $f(z) \in \delta_p(A, B, b, \omega, \lambda, l, n)$, then for $k \geq 1$, $\lambda \geq 0$, $l \geq 0$, $-1 \leq B < A \leq 1$, $p \in N$ and $z \in U$:*

$$|a_2| \leq \frac{|b|\big(A-B\big)}{\big(1-d^2\big)\left(\frac{1+\lambda+l}{1+l}\right)^n},$$

$$|a_3| \leq \frac{|b|^2\big(A-B\big)^2 + |b|\big(A-B\big)\big(1+d\big)}{2\big(1-d^2\big)^2\left(\frac{1+2\lambda+l}{1+l}\right)^n},$$

$$|a_4| \leq \frac{|b|^3\big(A-B\big)^3 + 3|b|^2\big(A-B\big)^2\big(1+d\big) + 2|b|\big(A-B\big)\big(1+d\big)}{2.3\big(1-d^2\big)^3\left(\frac{1+3\lambda+l}{1+l}\right)^n},$$

$$|a_5| \leq \Big(|b|^4\big(A-B\big)^4 + 4|b|^3\big(A-B\big)^3\big(1+d\big) + 4|b|^2\big(A-B\big)^2\big(1+d\big)^2 + |b|\big(A-B\big)\big(1+d\big)^3\Big)\left(2.3.4\big(1-d^2\big)^4\left(\frac{1+4\lambda+l}{1+l}\right)^n\right)^{-1}.$$

Proof. From (5)

$$1 + \frac{1}{b}\left\{\frac{(z-\omega)\left(\frac{1}{p^n} I^n_{\omega,p}(\lambda, l) f(z)\right)'}{\left(\frac{1}{p^n} I^n_{\omega,p}(\lambda, l) f(z)\right)} - p\right\} = 1 + \sum_{k=1}^{\infty} p_k (z-\omega)^k.$$

That is

$$\left\{ \frac{(z-\omega)\left(\frac{1}{p^n} I^n_{\omega,p}(\lambda,l) f(z)\right)'}{\left(\frac{1}{p^n} I^n_{\omega,p}(\lambda,l) f(z)\right)} - p \right\} = \sum_{k=1}^{\infty} bp_k (z-\omega)^k. \tag{6}$$

Using (4) on (6), we have

$$\frac{\sum_{k=1}^{\infty} \frac{k}{p^n}\left(\frac{1+\lambda(p+k-1)+l}{1+l}\right)^n a_{p+k}(z-\omega)^{p+k}}{\frac{1}{p^n}\left[\left(\frac{1+\lambda(p-1)+l}{1+l}\right)^n (z-\omega)^p + \sum_{k=1}^{\infty}\left(\frac{1+\lambda(p+k-1)+l}{1+l}\right)^n a_{p+k}(z-\omega)^{p+k}\right]} = \sum_{k=1}^{\infty} bp_k (z-\omega)^k.$$

Dividing the numerator denominator of the L.H.S. by the factor $(z-\omega)^p$, then we have

$$\frac{\sum_{k=1}^{\infty} \frac{k}{p^n}\left(\frac{1+\lambda(p+k-1)+l}{1+l}\right)^n a_{p+k}(z-\omega)^{k}}{\frac{1}{p^n}\left[\left(\frac{1+\lambda(p-1)+l}{1+l}\right)^n + \sum_{k=1}^{\infty}\left(\frac{1+\lambda(p+k-1)+l}{1+l}\right)^n a_{p+k}(z-\omega)^{k}\right]} = \sum_{k=1}^{\infty} bp_k (z-\omega)^k. \tag{7}$$

By expanding (7) and equating the coefficient of the like powers, we have

$$\begin{aligned}
\left(\frac{1+\lambda p+l}{1+l}\right)^n a_{p+1} &= b\left(\frac{1+\lambda(p-1)+l}{1+l}\right)^n p_1,\\
2\left(\frac{1+\lambda(p+1)+l}{1+l}\right)^n a_{p+2} &= b\left(\frac{1+\lambda p+l}{1+l}\right)^n a_{p+1}p_1 + b\left(\frac{1+\lambda(p-1)+l}{1+l}\right)^n p_2,\\
3\left(\frac{1+\lambda(p+2)+l}{1+l}\right)^n a_{p+3} &= b\left(\frac{1+\lambda(p+1)+l}{1+l}\right)^n a_{p+2}p_1\\
&\quad + b\left(\frac{1+\lambda p+l}{1+l}\right)^n a_{p+1}p_2\\
&\quad + b\left(\frac{1+\lambda(p-1)+l}{1+l}\right)^n p_3,\\
4\left(\frac{1+\lambda(p+3)+l}{1+l}\right)^n a_{p+4} &= b\left(\frac{1+\lambda(p+2)+l}{1+l}\right)^n a_{p+3}p_1
\end{aligned}$$

$$+ b\left(\frac{1+\lambda(p+1)+l}{1+l}\right)^n a_{p+2}p_2$$
$$+ b\left(\frac{1+\lambda p+l}{1+l}\right)^n a_{p+1}p_3$$
$$+ b\left(\frac{1+\lambda(p-1)+l}{1+l}\right)^n p_4.$$

Now, if we let $p = 1$ and using (3) on the above then

$$|a_2| \leq \frac{|b|\big(A-B\big)}{\big(1-d^2\big)\left(\frac{1+\lambda+l}{1+l}\right)^n},$$

$$|a_3| \leq \frac{|b|^2\big(A-B\big)^2 + |b|\big(A-B\big)\big(1+d\big)}{2\big(1-d^2\big)\left(\frac{1+2\lambda+l}{1+l}\right)^n},$$

$$|a_4| \leq \frac{|b|^3\big(A-B\big)^3 + 3|b|^2\big(A-B\big)^2\big(1+d\big) + 2|b|\big(A-B\big)\big(1+d\big)^2}{2.3\big(1-d^2\big)^3\left(\frac{1+3\lambda+l}{1+l}\right)^n},$$

$$|a_5| \leq \Big(|b|^4\big(A-B\big)^4 + 4|b|^3\big(A-B\big)^3\big(1+d\big) + 4|b|^2\big(A-B\big)^2\big(1+d\big)^2 + |b|\big(A-B\big)\big(1+d\big)^3\Big)\left(2.3.4\big(1-d^2\big)^4\left(\frac{1+4\lambda+l}{1+l}\right)^n\right)^{-1}. \qquad \square$$

This complete the proof of Theorem 2.1.

If $n = 0$ in Theorem 2.1, above, then we have for $k \geq 1$ and $p = 1$.

Corollary 2. *Let $f \in \delta_1(A, B, b, \omega, \lambda, l, 0)$ then:*

$$|a_2| \leq \frac{|b|\big(A-B\big)}{\big(1-d^2\big)},$$

$$|a_3| \leq \frac{|b|^2\big(A-B\big)^2 + |b|\big(A-B\big)\big(1+d\big)}{2\big(1-d^2\big)},$$

$$|a_4| \leq \frac{|b|^3\big(A-B\big)^3 + 3|b|^2\big(A-B\big)^2\big(1+d\big) + 2|b|\big(A-B\big)\big(1+d\big)^2}{2.3\big(1-d^2\big)^3},$$

$$|a_5| \leq \Big(|b|^4\big(A-B\big)^4 + 4|b|^3\big(A-B\big)^3\big(1+d\big) + 4|b|^2\big(A-B\big)^2\big(1+d\big)^2 + |b|\big(A-B\big)\big(1+d\big)^3\Big)\Big(2.3.4\big(1-d^2\big)^4\Big)^{-1}.$$

Also, setting $n = d = 0$ and $p = 1$ in Theorem 1, we receive.

Corollary 3. *Let $f \in \delta_1(A, B, b, \omega, \lambda, l, 0)$, then*

$$|a_2| \leq |b|\big(A - B\big),$$
$$|a_3| \leq \frac{|b|\big(A - B\big)\big[1 + |b|\big(A - B\big)\big]}{2},$$
$$|a_4| \leq \frac{|b|\big(A - B\big)\Big[2 + 3|b|\big(A - B\big) + |b|^2\big(A - B\big)^2\Big]}{2.3},$$
$$|a_5| \leq \frac{|b|\big(A - B\big)\Big[1 + 4|b|\big(A - B\big) + 4|b|^2\big(A - B\big)^2 + |b|^3\big(A - B\big)^3\Big]}{2.3.4}.$$

With several other choices of the parameters involved both the existing and many new coefficient estimate could be obtained.

Our next result shows the relevance of our class $\delta_1(A, B, b, \omega, \lambda, l, 0)$ to the famous Fekete-Zsego Theorem.

Theorem 4. *Let $f \in \delta_1(A, B, b, \omega, \lambda, l, n)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq$$
$$\frac{|b|\big(A - B\big)\Big[|b|\big(A - B\big) + \big(1 + d\big)\left(\frac{1+\lambda+l}{1+l}\right)^n - 2\mu|b|\big(A - B\big)\left(\frac{1+2\lambda+l}{1+l}\right)^n\Big]}{2\big(1 - d^2\big)^2\left(\frac{1+2\lambda+l}{1+l}\right)^n\left(\frac{1+\lambda+l}{1+l}\right)^{2n}}.$$

Proof. The proof follows from Theorem 2.1. With various choices of parameters involved, several connections of our class to the well known Fekete-Zsego Theorem could be obtained. □

If $n = 0$ in Theorem 4, we have the following corollary.

Corollary 5. *Let $f \in \delta_1(A, B, b, \omega, \lambda, l, 0)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\big(A - B\big)\Big[|b|\big(A - B\big) + \big(1 + d\big) - 2\mu|b|\big(A - B\big)\Big]}{2\big(1 - d^2\big)^2}.$$

Setting $n = d = 0$, we have

Corollary 6. *Let $f \in \delta_1(A, B, b, \omega, \lambda, l, 0)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\big(A - B\big)\Big[1 + |b|\big(A - B\big) - 2\mu|b|\big(A - B\big)\Big]}{2}.$$

Also, if $n = \lambda = 1$, and $l = 0$, we have

Corollary 7. *Let* $f \in \delta_1(A, B, b, \omega, 1, 0, 1)$

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\big(A-B\big)\Big[|b|\big(A-B\big) + 2\big(1+d\big) - 6\mu|b|\big(A-B\big)\Big]}{2.2.3\big(1-d\big)^2}.$$

Setting $n = \lambda = 1$ and $l = d = 0$, we obtain

Corollary 8. *Let* $f \in \delta_1(A, B, b, \omega, 1, 0, 1)$

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\big(A-B\big)\Big[2 + |b|\big(A-B\big) - 6\mu|b|\big(A-B\big)\Big]}{2.2.3}.$$

References

[1] A. Akbarally, S. Cik Sohb and M. Ismail, On the properties of a class of p-valent functions defined by salagean differential operator, *Int. Journal of Math. Analysis*, **5**, No. 21 (2011), 1035-1045.

[2] M. Acu and S.Owa, On some subclasses of univalent functions, *Journal of Inequalities in Pure and Applied Mathematics*, **6**, No. 3, Article 70 (2005), 1-14.

[3] M.K Aouf, A.Shamandy, A.O Mostafa, and S.M Madian, A subclass of M-W starlike functions, *Acta Universitatis Apulensis*, No. 21 (2010), 135-142.

[4] R.M. Goel and B.C. Mehok, A subclass of starlike functions with respect to symmetric points, *Tamkang J. Math.*, **13**, No. 1 (1982), 11-24.

[5] S.Kanas, and F.Ronning, Uniformly starlike and convex functions and other related classes of univalent functions, *Ann. Univ. Mariae Curie-Sklodowska section A*, **53** (1999), 95-105.

[6] A.T. Oladipo, On subclasses of univalent functions, *Advances in Applied Mathematical Analysis*, **4**, No. 2 (2009), 87-93.

[7] A. T. Oladipo and D. Breaz, On the family of Bazilevic functions, *Acta Universitatis Apulensis*, No. 24 (2004).

[8] A. T. Oladipo and S.O. Olatunji, On new subclasses of analytic functions with respect to other points defined by aouf et al. operator, *ICASTOR Journal of Mathematical Sciences*, **5**, No. 2 (2011) 245-253.

[9] S.S. Seker and B. Eker, On a class of multivalent functions defined by Salagean Operator, *General Mathematics*, **15** (2007), 154-163.

[10] C. Selvaraj and N. Vasanthi, Subclasses of analytic functions with respect to symmetric and conjugate points, *Tamkang Journal of Mathematics*, **42**, No. 1 (2011), 87-94.

[11] J. K Wald, *On Starlike Functions*, Ph.D. Thesis, University of Delaware, New Ark, Delaware (1978).