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ON GENERALIZED w-CLOSED SETS IN w-SPACES

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Abstract: The purpose of this note is to introduce the notions of generalized w-closed set and generalized w-open set in w-spaces. In fact, every w-open set is generalized w-open in a given w-space. We study some basic properties of such the notions and the conditions of W-continuous functions which preserve generalized w-closed sets or generalized w-open sets.

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1. Introduction

Siwiec [17] introduced the notions of weak neighborhoods and weak base in a topological space. We introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in [13]. The weak neighborhood system induces a weak neighborhood space which is independent of neighborhood spaces [4] and general topological spaces [2]. The notions of weak structure, w-space, W-continuity and W^{*}-continuity were investigated in [14]. In fact, the set of all g-closed subsets [5] in a topological space is a kind of weak structure.

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The one purpose of our research is to introduce the notion of generalize w-open sets (generalize w-closed sets)in w-spaces. Levine [5] introduced the notion of g-closed subsets in topological spaces. In the same way, we introduce the notion of generalized w-closed set (simply, gw-closed set) in weak spaces, and investigate some basic properties of such notions.

2. Preliminaries

Definition 2.1 ([14]). Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* on X if it satisfies the following:

(1) $\emptyset \in w_X$ and $X \in w_X$.

(2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a *w-space* on X. Then $V \in w_X$ is called a *w-open* set and the complement of a w-open set is a *w-closed* set.

The collection of all w-open sets (resp., w-closed sets) in a w-space X will be denoted by $WO(X)$ (resp., $WC(X)$). We set $W(x) = \{U \in WO(X) : x \in U\}$.

Let S be a subset of a topological space X. The closure (resp., interior) of S will be denoted by clS (resp., $intS$). A subset S of X is called a *preopen* set [11] (resp., α-open set [16], *semi-open* [6]) if $S \subset int(cl(S))$ (resp., $S \subset int(cl(int(S))), S \subset cl(int(S))$). The complement of a preopen set (resp., α-open set, *semi-open*) is called a *preclosed* set (resp., α-closed set, *semi-closed*). The family of all preopen sets (resp., α-open sets, semi-open sets) in X will be denoted by $PO(X)$ (resp., $\alpha(X), SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on X.

A subset A of a topological space (X, τ) is said to be:

(a) g-closed [5] if $cl(A) \subset U$ whenever $A \subset U$ and U is open in X;

(b) gp-closed [7] if $pCl(A) \subset U$ whenever $A \subset U$ and U is open in X;

(c) gs-closed [1, 3] if $sCl(A) \subset U$ whenever $A \subset U$ and U is open in X;

(d) $g\alpha$-closed [9] if $\tau^\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is α-open in X where $\tau^\alpha = \alpha(X)$;

(e) $g\alpha^{*}$-closed [8] if $\tau^\alpha Cl(A) \subset int(U)$ whenever $A \subset U$ and U is α-open in X;

(f) $g\alpha^{**}$-closed [8] if $\tau^\alpha Cl(A) \subset int(cl(U))$ whenever $A \subset U$ and U is α-open in X;

(g) αg-closed [9] if $\tau^\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is open in X;

(h) $\alpha^{**}g$-closed [9] if $\tau^\alpha Cl(A) \subset int(cl(U))$ whenever $A \subset U$ and U is open in X.

Then the family τ, $GO(X)$, $g\alpha O(X)$, $g\alpha^{\square}O(X)$, $g\alpha^{\square\square}O(X)$, $\alpha gO(X)$ and $\alpha^{\square\square}gO(X)$ are all weak structures on X. But $PO(X)$, $GPO(X)$ and $SO(X)$ are not weak structures on X. A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* on X [10] if $\emptyset \in w_X$ and $X \in w_X$. Thus clearly every weak structure is a minimal structure.

Definition 2.2 ([14]). Let (X, w_X) be a w-space. For a subset A of X, the *w-closure* of A and the *w-interior* of A are defined as follows:

(1) $wC(A) = \cap\{F : A \subseteq F, X - F \in w_X\}$.

(2) $wI(A) = \cup\{U : U \subseteq A, U \in w_X\}$.

Theorem 2.3 ([14]). *Let (X, w_X) be a w-space and $A \subseteq X$.*

(1) $x \in wI(A)$ if and only if there exists an element $U \in W(x)$ such that $U \subseteq A$.

(2) $x \in wC(A)$ if and only if $A \cap V \neq \emptyset$ for all $V \in W(x)$.

(3) If $A \subset B$, then $wI(A) \subset wI(B)$; $wC(A) \subset wC(B)$.

(4) $wC(X - A) = X - wI(A)$; $wI(X - A) = X - wC(A)$.

(5) If A is w-closed (resp., w-open), then $wC(A) = A$ (resp., $wI(A) = A$).

3. Main results

Definition 3.1. Let (X, w_X) be a w-space and $A \subseteq X$. Then A is called *a generalized w-closed set* (simply, gw-closed set) if $wC(A) \subseteq U$, whenever $A \subseteq U$ and U is w-open.

Remark 3.2. (1) If the w_X-structure is a topology, the generalized w-closed set is exactly a generalized closed set in sense of Levine in [5].

(2) Obviously, every w-closed set is generalized w-closed, but in general, the converse is not true as the next example.

Example 3.3. Let $X = \{a, b, c, d\}$ and $w_X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, d\}, X\}$ be a w-structure in X. Now, consider $A = \{d\}$. Then $wC(A) = A$. So A is gw-closed but not w-closed.

Lemma 3.4. *Let (X, w_X) be a w-space and $A, B \subseteq X$. Then the following things hold:*

(1) $wI(A) \cap wI(B) = wI(A \cap B)$.

(2) $wC(A) \cup wC(B) = wC(A \cup B)$.

Theorem 3.5. *Let (X, w_X) be a w-space. Then the union of two gw-closed sets is a gw-closed set.*

Proof. Let A and B be any two gw-closed sets. Let G be any w-open set such that $A \cup B \subseteq G$. Then $A \subseteq G$ and $B \subseteq G$. Since A and B are gw-closed sets, $wC(A) \subseteq G$ and $wC(B) \subseteq G$. Now, by Lemma 3.4, $wC(A \cup B) = wC(A) \cup wC(B) \subseteq G$. Hence $A \cup B$ is gw-closed. □

In general, the intersection of two gw-closed sets is not gw-closed:

Example 3.6. For $X = \{a, b, c, d\}$, let $w = \{\emptyset, \{a, c\}, \{a\}, \{b\}, \{c\}, \{a, d\}, X\}$ be a w-structure in X. Now, consider $A = \{a, b, c\}$ and $B = \{a, c, d\}$. Then A and B are gw-closed. But $A \cap B = \{a, c\}$ is not gw-closed because $\{a, c\}$ is w-open and $wC(\{a, c\}) = \{a, c, d\}$.

Theorem 3.7. *Let (X, w_X) be a w-space. Then if A is a gw-closed set, then $wC(A) - A$ contains no non-empty w-closed set.*

Proof. Suppose that there is a w-closed set F such that $F \subseteq wC(A) - A$. Then $A \subseteq X - F$, and since $X - F$ is w-open and A is gw-closed, $wC(A) \subseteq X - F$. It implies that $F \subseteq X - wC(A)$, and so $F \subseteq wC(A) \cap (X - wC(A)) = \emptyset$. Hence, $F = \emptyset$. □

In general, the converse in Theorem 3.7 is not true as shown in the next example.

Example 3.8. Let $X = \{a, b, c, d\}$ and $w_X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, d\}, X\}$ be a w-structure in X. Consider $A = \{a\}$. Note $wC(A) = \{a, d\}$ and $wC(A) - A = \{a, d\} - \{a\} = \{d\}$. Since $\{d\}$ is not w-closed, $wC(A) - A$ contains no non-empty w-closed set, but A is not w-closed.

Theorem 3.9. *Let (X, w_X) be a w-space. Then if A is a gw-closed set and $A \subseteq B \subseteq wC(A)$, then B is gw-closed.*

Proof. Let U be any w-open set such that $B \subseteq U$. Since $A \subseteq B \subseteq wC(A)$ and A is a gw-closed set, $A \subseteq U$ and $wC(B) \subseteq wC(A) \subseteq U$. It implies that $wC(B) \subseteq U$, and so B is gw-closed set. □

Definition 3.10. Let (X, w_X) be a w-space and $A \subseteq X$. Then A is called *a generalized w-open set* (simply, gw-open set) if $X - A$ is gw-closed.

Theorem 3.11. *Let (X, w_X) be a w-space and $A \subseteq X$. Then A is generalized w-open if and only if $F \subseteq wI(A)$ whenever $F \subseteq A$ and F is w-closed.*

Proof. From Definition 3.1, it is obvious. □

Theorem 3.12. *Let (X, w_X) be a w-space. Then the intersection of two gw-open sets is a gw-open set.*

Proof. It is obvious from Theorem 3.5. □

In general, the union of two gw-open sets is not gw-open (See Example 3.6).

Theorem 3.13. *Let (X, w_X) be a w-space and $A \subseteq X$. Then if A is gw-open, then $U = X$, whenever $wI(A) \cup (X - A) \subseteq U$ and U is w-open.*

Proof. Let U be any w-open set and $wI(A) \cup (X - A) \subseteq U$. Then $X - U \subseteq (X - wI(A)) \cap A = wC(X - A) \cap A = wC(X - A) - (X - A)$. Since $X - A$ is gw-closed, by Theorem 3.7, the w-closed set $X - U$ must be empty. Hence, $U = X$. □

Theorem 3.14. *Let (X, w_X) be a w-space. Then if A is a gw-open set and $wI(A) \subseteq B \subseteq A$, then B is gw-open.*

Proof. It is similar to the proof of Theorem 3.9. □

Theorem 3.15. *Let (X, w_X) be a w-space. Then if A is a gw-closed set, then $wC(A) - A$ is gw-open.*

Proof. Suppose that A is a gw-closed set. Then by Theorem 3.9, the empty set is the only one w-closed subset of $wC(A) - A$. So, for the only w-closed subset $\emptyset$ of $wC(A) - A$, $\emptyset \subseteq wC(A) - A$ and $\emptyset \subseteq wI(wC(A) - A)$. Hence, $wC(A) - A$ is gw-open. □

Theorem 3.16. *Let (X, w_X) be a w-space. Then if A is a gw-open set, then $wI(A) \cup (X - A)$ is gw-closed.*

Proof. Suppose that A is a gw-open set. Then by Theorem 3.13, the whole set X is the only one w-open set containing $wI(A) \cup (X - A)$. So, $wI(A) \cup (X - A) \subseteq X$ and $wC(wI(A) \cup (X - A)) \subseteq X$. Hence, $wI(A) \cup (X - A)$ is gw-closed. □

Definition 3.17. Let (X, w_X) be a w-space. For a subset A of X, *gw-closure* of A and *gw-interior* of A are defined as the following:
(1) $gwC(A) = \cap\{F : A \subseteq F, F \text{ is } gw\text{-closed}\}$.
(2) $gwI(A) = \cup\{U : U \subseteq A, U \text{ is } gw\text{-open}\}$.

Theorem 3.18. *Let (X, w_X) be a w-space and $A \subseteq X$.*
(1) If A is gw-open, then $gwI(A) = A$.
(2) If A is gw-closed, then $gwC(A) = A$.

Proof. Obvious. □

But the converses in the above theorem are not always true as shown in the next example.

Example 3.19. In Example 3.6, let $F = \{a, c\}$. Since $\{a, b, c\}$ and $\{a, c, d\}$ are gw-closed sets, $gwC(F) = \{a, c\}$. But from the fact that $\{a, c\}$ is w-open and $wC(\{a, c\}) = \{a, c, d\}$, F is not gw-closed. Similarly, we can show that the converse of (2) in Theorem 3.18 is not true, in general.

Theorem 3.20. *Let (X, w_X) be a w-space and $A, B \subseteq X$.*
(1) If $A \subseteq B$, then $gwI(A) \subseteq gwI(B)$ and $gwC(A) \subseteq gwC(B)$.
(2) $gwC(X - A) = X - gwI(A)$; $gwI(X - A) = X - gwC(A)$.

Proof. Obvious. □

Theorem 3.21. *Let (X, w_X) be a w-space and $A \subset X$.*
(1) $x \in gwI(A)$ if and only if there exists a gw-open set U containing x such that $U \subseteq A$.
(2) $x \in gwC(A)$ if and only if $A \cap V \neq \emptyset$ for all gw-open set V containing x.

Proof. (1) It is obvious.

(2) For each $x \in gwC(A)$, suppose that there exists a gw-open set V containing x such that $A \cap V = \emptyset$. Then $A \subseteq X - V$, so $gwC(A) \subseteq X - V$ and $x \notin gwC(A)$, which is a contradiction.

Conversely, suppose on the contrary that $x \notin wC(A)$. Then there exists a gw-closed set F containing A such that $x \notin F$. It implies that there exists a gw-open set $X - F$ containing x such that $(X - F) \cap A = \emptyset$. So, the proof is completed. □

Theorem 3.22. *Let (X, w_X) be a w-space and $A, B \subset X$.*
(1) $\emptyset = gwC(\emptyset)$.
(2) $A \subseteq gwC(A)$.
(3) $gwC(A \cup B) = gwC(A) \cup gwC(B)$.
(4) $gwC(gwC(A)) = gwC(A)$.

Proof. (1) and (2) are obvious.

(3) From Theorem 3.20, $gwC(A \cup B) \supseteq gwC(A) \cup gwC(B)$. Now, we show that $gwC(A \cup B) \subseteq gwC(A) \cup gwC(B)$. For the proof, let $x \notin gwC(A) \cup gwC(B)$. Then there exist gw-closed sets F_1 and F_2 such that $x \notin F_1$ and $A \subseteq F_1$; $x \notin F_2$ and $B \subseteq F_2$. So $x \notin F_1 \cup F_2$ and $A \cup B \subseteq F_1 \cup F_2$. Since $F_1 \cup F_2$

is gw-closed, we have that $x \notin gwC(A \cup B)$. Consequently, $gwC(A \cup B) = gwC(A) \cup gwC(B)$.

(4) It is sufficient to show that $gwC(gwC(A)) \subseteq gwC(A)$. For any gw-closed set F satisfying $A \subseteq F$, $gwC(A) \subseteq gwC(F) = F$. From the fact, $\{F : A \subseteq F, F \text{ is } gw\text{-closed}\} \subseteq \{K : gwC(A) \subseteq K, K \text{ is } gw\text{-closed}\}$. So $gwC(gwC(A)) = \cap\{K : gwC(A) \subseteq K, K \text{ is } gw\text{-closed}\} \subseteq \cap\{F : A \subseteq F, F \text{ is } gw\text{-closed}\} = gwC(A)$. □

Theorem 3.23. *Let (X, w_X) be a w-space and $A, B \subset X$.*
(1) $X = gwI(X)$.
(2) $gwI(A) \subseteq A$.
(3) $gwI(A \cap B) = gwI(A) \cap gwI(B)$.
(4) $gwI(gwI(A)) = gwI(A)$.

Proof. These are easily obtained by Theorem 3.20 and Theorem 3.22. □

Finally, we have a topology induced by generalized w-open sets as the following:

Theorem 3.24. *Let (X, w_X) be a w-space. Then the family $\tau^{\square} = \{U \subseteq X : U = gwI(U)\}$ is a topology containing w_X.*

Let X and Y be w-spaces. A function $f : (X, w_X) \to (Y, w_Y)$ is said to be $W^{\square}$-open [15] if for every w-open set G in X, $f(G)$ is a w-open set in Y.

In the analogous way, we define the following notions:

Definition 3.25. Let X and Y be w-spaces. A function $f : (X, w_X) \to (Y, w_Y)$ is said to be
(1) $W^{\square}$-*closed* if for every w-closed set F in X, $f(F)$ is a w-closed set in Y.
(2) *quasi-*$W^{\square}$-*closed* if for $A \subseteq X$, $wC(f(A)) \subseteq f(wC(A))$.

Remark 3.26. There is no any relation between the notions of $W^{\square}$-closed function is quasi-$W^{\square}$-closed function.

Example 3.27. Let $X = \{a, b, c\}$ and $w_1 = \{\emptyset, \{a\}, \{b\}, X\}$ be a w-structure in X.

(1) Let $w_2 = \{\emptyset, \{b\}, \{c\}, X\}$ be a w-structure in X. Consider a function $f : (X, w_1) \to (X, w_2)$ defined by $f(a) = f(b) = a; f(c) = b$. Then the function is obviously $W^{\square}$-closed. Now, note that for $A = \{c\}$, $wC(A) = A$ in w_1 and $f(wC(A)) = f(A) = \{b\}$; $wC(f(A)) = wC(\{b\}) = \{a, b\}$ in w_2. So, f is not quasi-$W^{\square}$-closed.

(2) Let $w_3 = \{\emptyset, \{a, b\}, X\}$ be a w-structure in X. Consider a function $f : (X, w_3) \to (X, w_1)$ defined by $f(a) = b; f(b) = a; f(c) = c$. Note that in the

w-space (X, w_3), for nonempty set A, if $A = \{c\}$, then $wC(A) = A$; otherwise, $wC(A) = X$. So, f is quasi-$W^{\square}$-closed. But for a w-closed set $A = \{c\}$ in (X, w_3), since $f(A) = \{c\}$ is not w-closed in (X, w_1), f is not $W^{\square}$-closed.

Recall the notion of W-continuity: Let (X, w_X) and (Y, w_Y) be two w-spaces. Then $f : X \to Y$ is said to be *W-continuous* [14] if for $x \in X$ and $V \in W(f(x))$, there is $U \in W(x)$ such that $f(U) \subseteq V$.

Theorem 3.28 ([14]). *Let $f : (X, w_X) \to (Y, w_Y)$ be a function in two w-spaces. Then the following statements are equivalent:*
(1) f is W-continuous.
(2) $f(wC(A)) \subseteq wC(f(A))$ for $A \subseteq X$.
(3) $wC(f^{-1}(B)) \subseteq f^{-1}(wC(B))$ for $B \subseteq Y$.
(4) $f^{-1}(wI(B)) \subseteq wI(f^{-1}(B))$ for $B \subseteq Y$.

Lemma 3.29. *Let $f : X \to Y$ be a function between w-spaces X and Y.*
(1) If f is $W^{\square}$-closed, then $f(wC(A)) \subseteq wC(f(A))$ for each $A \subseteq X$.
(2) f is W-continuous and quasi-$W^{\square}$-closed if and only if $wC(f(A)) = f(wC(A))$ for every subset A in X.

Proof. (1) Straightforward.
(2) From the notion of quasi $W^{\square}$-closed function and property of W-continuity in Theorem 3.28, directly the statement is obtained. □

Theorem 3.30. *Let $f : (X, w_X) \to (Y, w_Y)$ be a function on w-spaces X and Y. Then the following statements hold:*
(1) If f is W-continuous and $W^{\square}$-closed, then for every gw-open subset B in Y, $f^{-1}(B)$ is gw-open.
(2) If f is $W^{\square}$-continuous and quasi-$W^{\square}$-closed, for every gw-closed set A in X, $f(A)$ is gw-closed.

Proof. (1) Let B be any gw-open subset B in Y, and F be a w-closed set in X such that $F \subseteq f^{-1}(B)$. Now, we show that $F \subseteq wI(f^{-1}(B))$. Since f is $W^{\square}$-closed, $f(F)$ is w-closed. Moreover, since B is gw-open, $f(F) \subseteq wI(B)$. From Theorem 3.28, it follows that $F \subseteq f^{-1}(wI(B)) \subseteq wI(f^{-1}(B))$. Hence, $f^{-1}(B)$ is gw-open.

(2) Let A be any gw-closed subset A in X, and U be a w-open set in Y such that $f(A) \subseteq U$. Now, we show that $wC(f(A)) \subseteq U$. Since f is $W^{\square}$-continuous and A is gw-closed, $f^{-1}(U)$ is w-open and $wC(A) \subseteq f^{-1}(U)$. Since f is quasi-$W^{\square}$-closed, $wC(f(A)) \subseteq f(wC(A)) \subseteq ff^{-1}(U) \subseteq U$, and hence, $f(A)$ is gw-closed.

□

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