DIFFERENCE SEQUENCE OF FIBONACCI $t$ NUMBERS

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Abstract: We study interrelationships of Fibonacci $s$ and Fibonacci $t$ numbers for any $s \neq t$. Let $d^{(s,t)}_n$ be the difference sequence of Fibonacci $s$ and $t$ numbers. We investigate recurrence formulas of $d^{(s,t)}_n$ and find patterns that both Fibonacci $s$ and Fibonacci $t$ numbers satisfy at the same time. Moreover we research increasing ratio of the sequence $d^{(s,t)}_n$.

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Key Words: Fibonacci $t$ number, difference sequence of Fibonacci numbers

1. Introduction

For any integer $t > 0$, a Fibonacci $t$ number $f^{(t)}_n$ satisfies the recurrence $f^{(t)}_{n+1} = f^{(t)}_n + f^{(t)}_{n-t}$ with $t+1$ initials $1, \cdots, 1, 2$. If $t = 1$ then it is the ordinary Fibonacci number and if $t = 2$ then it is sometimes called a Narayana number, see [4]. Fibonacci $t$ numbers have many analogous properties of Fibonacci numbers, like Binet formula, sums of numbers and associated matrices, etc (see [2], [3], or [5]). In particular the limit of ratio of consecutive Fibonacci $t$ numbers satisfies the equation $P_t(x) = x^{t+1} - x^t - 1$ (see [6]), and its application to to data-hiding was discussed in [1].

Though Fibonacci $t$ numbers were studied in various aspect, it seems no one has asked interrelationships between Fibonacci $t$ and Fibonacci $s$ numbers for $t \neq s$. In this work we define a difference sequence $d^{(s,t)}_n$ of Fibonacci $s$
and Fibonacci $t$ numbers. We investigate recurrence formulas of $d_{n}^{(s, t)}$, and find a recurrence pattern that both Fibonacci $s$ and Fibonacci $t$ numbers satisfy, and moreover examine increasing ratios of difference sequences $d_{n}^{(s, t)}$. Since $f_{n}^{(1)}$ is well known, Fibonacci $t$ numbers $f_{n}^{(t)}$ would be obtained from $d_{n}^{(1, t)}$. In particular we discuss about $d_{n}^{(1, 4)}$ that has rather unique property.

2. Difference Sequence of $f_{n}^{(1)}$ and $f_{n}^{(t)}$

The Fibonacci $t$ number $f_{n}^{(t)}$ starts from $1, \cdots, 1, 2$ and satisfies the recurrence $f_{n+1}^{(t)} = f_{n}^{(t)} + f_{n-t}^{(t)}$ for $n-t > 0$. The $f_{n}^{(t)}$ can be extended to negative parameter $n$ so that we have the following table of $f_{n}^{(t)}$ for $1 \leq t \leq 4$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f_{n}^{(1)}$</th>
<th>$f_{n}^{(2)}$</th>
<th>$f_{n}^{(3)}$</th>
<th>$f_{n}^{(4)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$-6$</td>
<td>-3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$-5$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$-4$</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$-3$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$-2$</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$-1$</td>
<td>1</td>
<td>-3</td>
<td>2</td>
<td>-3</td>
</tr>
<tr>
<td>$0$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$1$</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$2$</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$3$</td>
<td>-1</td>
<td>2</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$4$</td>
<td>1</td>
<td>-3</td>
<td>2</td>
<td>-3</td>
</tr>
<tr>
<td>$5$</td>
<td>-1</td>
<td>5</td>
<td>-3</td>
<td>5</td>
</tr>
<tr>
<td>$6$</td>
<td>1</td>
<td>-1</td>
<td>5</td>
<td>-1</td>
</tr>
<tr>
<td>$7$</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The next lemma explore some identities on $f_{n}^{(t)}$ $(1 \leq t \leq 3)$.

**Lemma 1.** For any $t \geq 1$, we have

$$f_{n+1}^{(t)} = \sum_{i=t-1}^{n} f_{n-i}^{(t)}.$$ 

Moreover we have the following equalities:

1. $f_{n}^{(1)} = 4f_{n-3}^{(1)} + f_{n-6}^{(1)}$, $3f_{n}^{(1)} = 4f_{n-1}^{(1)} + f_{n-2}^{(1)} + f_{n}^{(1)}$.
2. $f_{n}^{(2)} + 2f_{n-2}^{(2)} + f_{n}^{(2)} = f_{n}^{(2)}$, $f_{n}^{(3)} = 4f_{n-5}^{(3)} + 3f_{n-8}^{(3)} - f_{n-9}^{(3)} + f_{n-11}^{(3)}$.
3. $f_{n}^{(3)} = 2f_{n-2}^{(3)} - f_{n-11}^{(3)} - f_{n-12}^{(3)}$, $f_{n}^{(3)} = 2f_{n-1}^{(3)} - f_{n-4}^{(3)} - f_{n-5}^{(3)} + f_{n-13}^{(3)} + f_{n-14}^{(3)}$.

**Proof.** Clearly: $f_{n+1}^{(1)} = \sum_{i=0}^{1} f_{n-i}^{(1)}$, $f_{n+1}^{(2)} = (f_{n-1}^{(2)} + f_{n-3}^{(2)}) + f_{n-2}^{(2)} = \sum_{i=1}^{3} f_{n-i}^{(2)}$ and $f_{n+1}^{(3)} = (f_{n-1}^{(3)} + f_{n-4}^{(3)}) + f_{n-3}^{(2)} = (f_{n-2}^{(3)} + f_{n-5}^{(3)}) + f_{n-4}^{(3)} + f_{n-3}^{(2)} = \sum_{i=2}^{5} f_{n-i}^{(3)}$.

Now for $t \geq 1$, consider $\sum_{i=t-1}^{2t-1} f_{n-i}^{(t)} = f_{n-(t-1)}^{(t)} + \cdots + f_{n-(2t-1)}^{(t)}$. Then by adding the last term to the first one, we have

$$\sum_{i=t-1}^{2t-1} f_{n-i}^{(t)} = [f_{n-(t-1)}^{(t)} + f_{n-(2t-1)}^{(t)}] + f_{n-t}^{(t)} + f_{n-(t+1)}^{(t)} + \cdots + f_{n-(2t-2)}^{(t)}.$$
Continuing to add the last to the first term, it follows that

\[ \sum_{i=t-1}^{2t-1} f^{(t)}_{n-i} = f^{(t)}_{n-(t-3)} + f^{(t)}_{n-t} + f^{(t)}_{n-(t+1)} = \ldots = f^{(t)}_{n-1} + f^{(t)}_{n-t} + f^{(t)}_{n-t-1} = f^{(t)}_{n} + f^{(t)}_{n-t} = f^{(t)}_{n+1}. \]

Now the identities on \( f^{(1)}_n \) are clear, and we have

\[ f^{(2)}_n + 2f^{(2)}_{n-2} + f^{(2)}_{n-4} = f^{(2)}_{n+1} + f^{(2)}_{n-1} = f^{(2)}_{n+2}. \]

For Fibonacci 3 numbers, it follows that

\[ f^{(3)}_n = f^{(3)}_{n-3} + f^{(3)}_{n-4} + f^{(3)}_{n-5} + f^{(3)}_{n-6} = 2f^{(3)}_{n-4} + f^{(3)}_{n-5} + f^{(3)}_{n-6} + f^{(3)}_{n-7} = 3f^{(3)}_{n-5} + f^{(3)}_{n-6} + f^{(3)}_{n-7} + 2f^{(3)}_{n-8}, \]

so that

\[ f^{(3)}_n - 4f^{(3)}_{n-5} = -f^{(3)}_{n-5} + f^{(3)}_{n-6} + f^{(3)}_{n-7} + 2f^{(3)}_{n-8} = -(f^{(3)}_{n-6} + f^{(3)}_{n-9}) + f^{(3)}_{n-6} + f^{(3)}_{n-7} + 2f^{(3)}_{n-8} = 3f^{(3)}_{n-8} - f^{(3)}_{n-9} + f^{(3)}_{n-11}. \]

Moreover we also have

\[ f^{(3)}_n - f^{(3)}_{n-2} + f^{(3)}_{n-11} + f^{(3)}_{n-12} = (f^{(3)}_{n-1} + f^{(3)}_{n-4}) - f^{(3)}_{n-2} + (f^{(3)}_{n-7} - f^{(3)}_{n-8}) + (f^{(3)}_{n-8} - f^{(3)}_{n-9}) = f^{(3)}_{n-1} - f^{(3)}_{n-2} + f^{(3)}_{n-4} + f^{(3)}_{n-7} - (f^{(3)}_{n-5} - f^{(3)}_{n-6}) = [f^{(3)}_{n-1} - f^{(3)}_{n-2} - f^{(3)}_{n-5}] + f^{(3)}_{n-4} + f^{(3)}_{n-6} + f^{(3)}_{n-7} = f^{(3)}_{n-3} + f^{(3)}_{n-6} = f^{(3)}_{n-2}, \]
since \( f_{n-1}^{(3)} - f_{n-2}^{(3)} - f_{n-5}^{(3)} = 0 \). Finally it is not hard to see that

\[
2f_{n-1}^{(3)} - f_{n-4}^{(3)} - f_{n-5}^{(3)} + f_{n-13}^{(3)} + f_{n-14}^{(3)} = f_{n-1}^{(3)} + (f_{n-2}^{(3)} - f_{n-4}^{(3)} + f_{n-13}^{(3)} + f_{n-14}^{(3)}).
\]

But since \( f_{n-2}^{(3)} = 2f_{n-4}^{(3)} - f_{n-13}^{(3)} - f_{n-14}^{(3)} \), it follows immediately that

\[
2f_{n-1}^{(3)} - f_{n-4}^{(3)} - f_{n-5}^{(3)} + f_{n-13}^{(3)} + f_{n-14}^{(3)} = f_{n-1}^{(3)} + f_{n-4}^{(3)} = f_{n}^{(3)}.
\]

For any \( n, i \geq 0 \), let \( d_{n,(i)}^{(s,t)} = f_{n}^{(s)} - f_{n+i}^{(t)} \) and call \( \{d_{n,(i)}^{(s,t)}\} \) the \( i \) step difference sequence of Fibonacci \( s \) and \( t \) numbers. In particular when \( i = 0 \), we denote the difference sequence by \( \{d_{n,(i)}^{(s,t)}\} = \{d_{n}^{(s,t)}\} \). We begin to study recurrence formula of the difference sequence \( d_{n,(i)}^{(1,2)} = f_{n}^{(1)} - f_{n+i}^{(2)} \).

**Theorem 2.** \( d_{n,(i)}^{(1,2)} \) satisfies

\[
d_{n+1,(i)}^{(1,2)} = 2d_{n,(i)}^{(1,2)} - d_{n-3,(i)}^{(1,2)} - d_{n-4,(i)}^{(1,2)},
\]

for all \( i \geq 0 \).

**Proof.** For convenience, write \( d_{n,(0)}^{(1,2)} = d_n \). Then \( \{d_n\}_{n \geq 1} = \{0, 1, 1, 2, 4, 7, 12, \cdots\} \), so if \( n = 5 \) then \( 2d_5 - d_2 - d_1 = 7 = d_6 \). Suppose the recurrence is true for all \( k < n \). Then the induction hypothesis shows

\[
2d_n - d_{n-3} - d_{n-4} = 2(2d_{n-1} - d_{n-4} - d_{n-5}) - (2d_{n-4} - d_{n-7} - d_{n-8}) = 4d_{n-1} - 4d_{n-4} - 4d_{n-5} + d_{n-7} + 2d_{n-8} + d_{n-9}.
\]

So by substituting \( d_{n,(i)}^{(1,2)} = f_{n}^{(1)} - f_{n+i}^{(2)} \), we have

\[
2d_n - d_{n-3} - d_{n-4} = A^{(1)} - A^{(2)},
\]

where

\[
A^{(j)} = 4f_{n-1}^{(j)} - 4f_{n-4}^{(j)} - 4f_{n-5}^{(j)} + f_{n-7}^{(j)} + 2f_{n-8}^{(j)} + f_{n-9}^{(j)},
\]

for \( j = 1, 2 \).

It is not hard to see from Lemma 1 that

\[
A^{(1)} = 4f_{n-1}^{(1)} - 4(f_{n-4}^{(1)} + f_{n-5}^{(1)}) + (f_{n-7}^{(1)} + f_{n-8}^{(1)}) + (f_{n-8}^{(1)} + f_{n-9}^{(1)}) = 4f_{n-2}^{(1)} + f_{n-6}^{(1)} + f_{n-7}^{(1)} = f_{n+1}^{(1)} - f_{n-5}^{(1)} + f_{n-6}^{(1)} + f_{n-7}^{(1)} = f_{n+1}^{(1)}.
\]
And Lemma 1 also shows
\[ A^{(2)} = 4f_{n-1}^{(2)} - 4f_{n-4}^{(2)} - 4f_{n-5}^{(2)} + f_{n-7}^{(2)} + 2f_{n-8}^{(2)} + f_{n-9}^{(2)} \]
\[ = 4(f_{n-1}^{(2)} - f_{n-4}^{(2)} - f_{n-5}^{(2)}) + (f_{n-7}^{(2)} + f_{n-8}^{(2)} + f_{n-9}^{(2)}) + f_{n-8}^{(2)} \]
\[ = 4f_{n-3}^{(2)} + f_{n-5}^{(2)} + (f_{n-4}^{(2)} - f_{n-6}^{(2)} - f_{n-7}^{(2)}) \]
\[ = (f_{n-3}^{(2)} + f_{n-4}^{(2)} + f_{n-5}^{(2)}) + 3f_{n-3}^{(2)} - f_{n-6}^{(2)} - f_{n-7}^{(2)} \]
\[ = f_{n-1}^{(2)} + 2f_{n-3}^{(2)} + (f_{n-3}^{(2)} - f_{n-6}^{(2)} - f_{n-7}^{(2)}) \]
\[ = f_{n+1}^{(2)}. \]

Therefore we have \( 2d_n - d_{n-3} - d_{n-4} = f_{n+1}^{(1)} - f_{n+1}^{(2)} = d_{n+1}. \)

Now write \( d_{n,(i)}^{(1,2)} = d_{n,(i)} \) for convenience.

If \( i = 1, \) then
\[ \{d_{n,(1)} = f_{n}^{(1)} - f_{n+1}^{(2)}\} = \{0, 0, 0, 1, 2, 4, 8, 15, 27, 48, \ldots\} \]
and hence \( d_{n+1,(1)} = 2d_{n,(1)} - d_{n-3,(1)} - d_{n-4,(1)} \) is true for \( n \leq 10. \)

Assume the identity is true for all \( k < n. \) Then
\[ 2d_{n,(1)} - d_{n-3,(1)} - d_{n-4,(1)} = 4d_{n-1,(1)} - 4d_{n-4,(1)} - 4d_{n-5,(1)} + d_{n-7,(1)} \]
\[ + 2d_{n-8,(1)} + d_{n-9,(1)} \]
\[ = (4f_{n-1}^{(1)} - 4f_{n-4}^{(1)} - 4f_{n-5}^{(1)} + f_{n-7}^{(1)} + 2f_{n-8}^{(1)} + f_{n-9}^{(1)}) \]
\[ - (4f_{n}^{(2)} - 4f_{n-3}^{(2)} - 4f_{n-4}^{(2)} + f_{n-6}^{(2)} + 2f_{n-7}^{(2)} + f_{n-8}^{(2)}). \]

As seen above, it is easy to see that
\[ 4f_{n-1}^{(1)} - 4f_{n-4}^{(1)} - 4f_{n-5}^{(1)} + f_{n-7}^{(1)} + 2f_{n-8}^{(1)} + f_{n-9}^{(1)} = f_{n+1}^{(1)} \]
and
\[ 4f_{n}^{(2)} - 4f_{n-3}^{(2)} - 4f_{n-4}^{(2)} + f_{n-6}^{(2)} + 2f_{n-7}^{(2)} + f_{n-8}^{(2)} = f_{n+2}^{(2)} \]

hence it follows that
\[ 2d_{n,(1)} - d_{n-3,(1)} - d_{n-4,(1)} = f_{n+1}^{(1)} - f_{n+2}^{(2)} = d_{n+1,(1)}. \]

Now for any \( i \geq 0, \) we also have
\[ 2d_{n,(i)} - d_{n-3,(i)} - d_{n-4,(i)} = 4d_{n-1,(i)} - 4d_{n-4,(i)} - 4d_{n-5,(i)} \]
\[ + d_{n-7,(i)} + 2d_{n-8,(i)} + d_{n-9,(i)} \]
\[ = f_{n+1}^{(1)} - f_{n+1+i}^{(2)} = d_{n+1,(i)}. \]

This completes the proof. \( \Box \)
Theorem 2 gives a relation that both \( f_n^{(1)} \) and \( f_n^{(2)} \) satisfy.

**Corollary 3.** \( f_n^{(j)} (j = 1, 2) \) satisfies \( f_{n+1}^{(j)} = 2f_n^{(j)} - f_{n-3}^{(j)} - f_{n-4}^{(j)} \).

**Proof.** The identity is clear for \( f_n^{(1)} \). And since \( f_n^{(2)} = f_{n-2}^{(2)} + f_{n-3}^{(2)} + f_{n-4}^{(2)} \) by Lemma 1, we have
\[
f_{n+1}^{(2)} = f_n^{(2)} + f_{n-2}^{(2)} = 2f_n^{(2)} - f_{n-3}^{(2)} - f_{n-4}^{(2)}.
\]
\( \square \)

**Theorem 4.** The difference \( d_{n,(i)}^{(1,3)} = f_n^{(1)} - f_n^{(3)} \) satisfies \( d_{n+1,(i)}^{(1,3)} = 2d_{n,(i)}^{(1,3)} - d_{n-2,(i)}^{(1,3)} - d_{n-3,(i)}^{(1,3)} - d_{n-4,(i)}^{(1,3)} - d_{n-5,(i)}^{(1,3)} \).

**Proof.** Let \( i = 0 \) and \( d_{n,(0)}^{(1,3)} = d_n \). Then \( \{d_n\}_{n \geq 1} = \{0, 1, 2, 3, 5, 9, 16, 27, \cdots \} \) shows \( 2d_6 - d_4 + d_3 - d_2 - d_1 = 16 = d_7 \). Hence if we assume the identity is true for all \( k < n \) then similar to the proof of Theorem 2, we have
\[
2d_n - d_{n-2} + d_{n-3} - d_{n-4} - d_{n-5} = 4d_{n-1} - 4d_{n-3} + 4d_{n-4} - 3d_{n-5} - 6d_{n-6} + 3d_{n-7} + d_{n-10} - (d_{n-9} - d_{n-10} - d_{n-11}).
\]
So due to the induction hypothesis \( d_{n-5} = 2d_{n-6} - d_{n-8} + d_{n-9} - d_{n-10} - d_{n-11} \), we have
\[
2d_n - d_{n-2} + d_{n-3} - d_{n-4} - d_{n-5} = 4d_{n-1} - 4d_{n-3} + 4d_{n-4} - 3d_{n-5} - 6d_{n-6} + 3d_{n-7} + d_{n-10} - (d_{n-9} - 2d_{n-6} + d_{n-8})
= 4d_{n-1} - 4d_{n-3} + 4d_{n-4} - 4d_{n-5}
- 4d_{n-6} + 3d_{n-7} - d_{n-8} + d_{n-10}.
\]

Now substitute \( d_n = f_n^{(1)} - f_n^{(3)} \). Then it follows that \( 2d_n - d_{n-2} + d_{n-3} - d_{n-4} - d_{n-5} = A^{(1)} - A^{(3)} \) where,
\[
A^{(j)} = 4f_{n-1}^{(j)} - 4f_{n-3}^{(j)} + 4f_{n-4}^{(j)} - 4f_{n-5}^{(j)} + 3f_{n-7}^{(j)} - f_{n-8}^{(j)} + f_{n-10}^{(j)}
\]
for \( j = 1, 3 \). We shall show \( A^{(j)} = f_{n+1}^{(j)} \).

If so, we finish to prove \( 2d_n - d_{n-2} + d_{n-3} - d_{n-4} - d_{n-5} = f_{n+1}^{(1)} - f_{n+1}^{(3)} = d_{n+1} \).

It is not hard to see
\[
A^{(1)} = 4(f_{n-1}^{(1)} - f_{n-3}^{(1)}) + 4(f_{n-4}^{(1)} - f_{n-6}^{(1)}) - 4(f_{n-5}^{(1)} - f_{n-7}^{(1)}) - f_{n-8}^{(1)} - f_{n-10}^{(1)}
= 4f_{n-2}^{(1)} + (4f_{n-5}^{(1)} - 4f_{n-6}^{(1)} - f_{n-7}^{(1)} - f_{n-9}^{(1)}).
\]
But since \( 3f_n^{(1)} = 4f_{n-1}^{(1)} + f_{n-2}^{(1)} + f_{n-4}^{(1)} \) (Lemma 1), we have \( 4f_n^{(1)} - 4f_{n-1}^{(1)} - f_{n-2}^{(1)} - f_{n-4}^{(1)} = f_n^{(1)} \), hence
\[
A^{(1)} = 4f_{n-2}^{(1)} + f_{n-5}^{(1)} = 3f_{n-2}^{(1)} + 2f_{n-3}^{(1)} = f_{n-2}^{(1)} + 2f_{n-1}^{(1)} = f_{n+1}^{(1)}.
\]
Similarly due to Lemma 1, we also have

\[
A^{(3)} = 4(f_{n-1}^{(3)} + f_{n-4}^{(3)}) - 4(f_{n-3}^{(3)} + f_{n-6}^{(3)}) - (f_{n-5}^{(3)} + f_{n-8}^{(3)})
- 3f_{n-5}^{(3)} + (f_{n-7}^{(3)} + f_{n-10}^{(3)}) + 2f_{n-7}^{(3)}
\]
\[
= 4f_{n}^{(3)} - 3(f_{n-2}^{(3)} + f_{n-5}^{(3)}) - f_{n-2}^{(3)} + 2(f_{n-4}^{(3)} + 2f_{n-7}^{(3)}) - 3f_{n-4}^{(3)} + f_{n-6}^{(3)}
= 4f_{n}^{(3)} - 3(f_{n-1}^{(3)} + f_{n-4}^{(3)}) - f_{n-2}^{(3)} + (f_{n-3}^{(3)} + f_{n-6}^{(3)}) + f_{n-3}^{(3)}
= 4f_{n}^{(3)} - 3f_{n}^{(3)} - f_{n-2}^{(3)} + f_{n-2}^{(3)} + f_{n-3}^{(3)}
= f_{n}^{(3)} + f_{n-3}^{(3)}
= f_{n+1}^{(3)}.
\]

Now if \(i > 0\) then the proof follows similar to Theorem 2. \(\square\)

The recurrence of \(d_{n+1}^{(1,3)}\) yields an identity that both \(f_{n}^{(1)}\) and \(f_{n}^{(3)}\) hold.

**Corollary 5.** \(f_{n+1}^{(j)} = 2f_{n}^{(j)} - f_{n-2}^{(j)} + f_{n-3}^{(j)} - f_{n-4}^{(j)} - f_{n-5}^{(j)}\) for \(j = 1, 3\).

**Proof.** The identity for \(f_{n}^{(1)}\) is clear. On the other hand, Lemma 1 says
\[
f_{n}^{(3)} = f_{n-3}^{(3)} + f_{n-4}^{(3)} + f_{n-5}^{(3)} + f_{n-6}^{(3)} = f_{n-2}^{(3)} + f_{n-4}^{(3)} + f_{n-5}^{(3)},
\]
hence we have
\[
2f_{n}^{(3)} - f_{n-2}^{(3)} + f_{n-3}^{(3)} - f_{n-4}^{(3)} - f_{n-5}^{(3)}
= 2f_{n}^{(3)} - f_{n-2}^{(3)} + f_{n-3}^{(3)} - f_{n}^{(3)} + f_{n-2}^{(3)}
= f_{n}^{(3)} + f_{n-3}^{(3)}
= f_{n+1}^{(3)}.
\]

Besides Theorem 4, another recurrence of \(d_{n+1}^{(1,3)}\) is as follows.

**Theorem 6.** \(d_{n+1}^{(1,3)} = 2d_{n}^{(1,3)} - d_{n-3}^{(1,3)} + d_{n-4}^{(1,3)} + f_{n}^{(3)} - f_{n-2}^{(3)} \) for all \(i \geq 0\). Moreover \(f_{n}^{(3)} + f_{n-3}^{(3)} = d_{n+10}^{(1,3)} - 2d_{n+9}^{(1,3)} + d_{n+1}^{(1,3)}.\)

**Proof.** Without loss of generality we may assume \(i = 0\). Then
\[
2d_{n}^{(1,3)} - d_{n-3}^{(1,3)} - d_{n-4}^{(1,3)} + f_{n}^{(3)} - f_{n-2}^{(3)}
= 2(f_{n}^{(1)} - f_{n}^{(3)}) - (f_{n-3}^{(1)} - f_{n-3}^{(3)}) - (f_{n-4}^{(1)} - f_{n-4}^{(3)}) - (f_{n-12}^{(1)} - f_{n-12}^{(3)})
= (2f_{n}^{(1)} - f_{n-3}^{(1)} - f_{n-4}^{(1)}) - (2f_{n}^{(3)} - f_{n-3}^{(3)} - f_{n-4}^{(3)}) + f_{n-12}^{(3)} + f_{n-12}^{(3)}
= f_{n+1}^{(1)} - f_{n+1}^{(3)}
= d_{n+1}^{(1,3)}.
\]
by Lemma 1. Hence together with Theorem 4, it follows that
\[ 2d_{n-2}^{(1,3)} - d_{n-3}^{(1,3)} - d_{n-4}^{(1,3)} - d_{n-5}^{(1,3)} = d_{n+1}^{(1,3)} \]
\[ = 2d_{n-2}^{(1,3)} - d_{n-3}^{(1,3)} - d_{n-4}^{(1,3)} - f_{n-12}^{(3)} - f_{n-13}^{(3)} \]
so we have \[ f_{n-12}^{(3)} + f_{n-13}^{(3)} = d_{n-2}^{(1,3)} - 2d_{n-3}^{(1,3)} + d_{n-5}^{(1,3)}. \]

We further study \( f_n^{(4)} \) and difference sequence \( d_n^{(1,4)} = f_n^{(1)} - f_n^{(4)}. \)

**Theorem 7.** \( d_n^{(1,4)} \) satisfies \( d_{n+1}^{(1,4)} = 2d_n^{(1,4)} - d_{n-2}^{(1,4)} - d_{n-4}^{(1,4)} - d_{n-5}^{(1,4)} - d_{n-6}^{(1,4)}. \)

**Proof.** We first verify \( 2f_n^{(j)} - f_{n-2}^{(j)} + f_{n-4}^{(j)} - f_{n-5}^{(j)} - f_{n-6}^{(j)} = f_{n+1}^{(j)} \) for \( j = 1, 4. \) If \( j = 1 \) then the identity is clear. If \( j = 4 \) then
\[ 2f_n^{(4)} - f_{n-2}^{(4)} + f_{n-4}^{(4)} - f_{n-5}^{(4)} - f_{n-6}^{(4)} = f_n^{(4)} + (f_n^{(4)} + f_{n-4}^{(4)}) - (f_{n-2}^{(4)} + f_{n-6}^{(4)}) - f_{n-5}^{(4)} \]
\[ = f_n^{(4)} + f_{n+1}^{(4)} - f_{n-1}^{(4)} - f_{n-5}^{(4)} \]
\[ = f_n^{(4)} + f_{n+1}^{(4)} - f_{n-5}^{(4)} \]
\[ = f_{n+1}^{(4)}. \]

It thus follows immediately that
\[ 2d_n^{(1,4)} - d_{n-2}^{(1,4)} + d_{n-4}^{(1,4)} - d_{n-5}^{(1,4)} - d_{n-6}^{(1,4)} \]
\[ = (2f_n^{(1)} - f_{n-2}^{(1)} + f_{n-4}^{(1)} - f_{n-5}^{(1)} - f_{n-6}^{(1)}) - (2f_n^{(4)} - f_{n-2}^{(4)} + f_{n-4}^{(4)} - f_{n-5}^{(4)} - f_{n-6}^{(4)}) \]
\[ = f_{n+1}^{(1)} - f_{n+1}^{(4)} \]
\[ = d_{n+1}^{(1,4)}. \]

### 3. Ratio of Difference Sequence

The ratio \( \frac{f_{n+1}^{(i)}}{f_n^{(i)}} \) of consecutive Fibonacci \( t \) numbers converges to a positive root of \( P_t(x) = x^t - x^{t-1} - 1 \) for all \( t \geq 1 \) (see [6]). In this section we shall investigate the limit of \( \frac{d_{n+1}^{(1,t)}}{d_n^{(1,t)}} \) of difference sequence, moreover find their interrelationships with \( P_t(x) \). Note that since \( d_{n,(i)}^{(1,t)} \) satisfies the same recurrence pattern for all \( i > 0 \), we may enough to consider \( \frac{d_{n+1}^{(1,t)}}{d_n^{(1,t)}} \).
Theorem 8. Let $\Delta^{(1,2)}(x) = x^5 - 2x^4 + x + 1$. Then $\lim_{n \to \infty} \frac{d^{(1,2)}_{n+1}}{d^{(1,2)}_n}$ is a real positive root of $\Delta^{(1,2)} = P_1(x)P_2(x)$. Moreover $\lim_{n \to \infty} \frac{d^{(1,2)}_{n+1}}{d^{(1,2)}_n} = \lim_{n \to \infty} \frac{f^{(1)}_{n+1}}{f^{(1)}_n}$.

Proof. The recurrence of $d^{(1,2)}_n$ in Theorem 2 gives rise to

$$\frac{d^{(1,2)}_{n+1}}{d^{(1,2)}_n} = 2 - \frac{1}{d^{(1,2)}_n/d^{(1,2)}_{n-3}} - \frac{1}{d^{(1,2)}_n/d^{(1,2)}_{n-4}}.$$  

So by letting $\lim_{n \to \infty} \frac{d^{(1,2)}_{n+1}}{d^{(1,2)}_n} = \alpha$, we have $\alpha = 2 - \frac{1}{\alpha^2} - \frac{1}{\alpha}$ and $\alpha$ is a positive real root of $x^5 - 2x^4 + x + 1 = \Delta^{(1,2)}(x)$. Clearly

$$\Delta^{(1,2)}(x) = x^5 - 2x^4 + x + 1 = (x^2 - x - 1)(x^3 - x^2 - 1) = P_1(x)P_2(x).$$

So $\alpha$ is one of the roots of $P_i(x)$ ($i = 1, 2$) where each $P_i(x)$ has only one positive real root that equals $\lim_{n \to \infty} \frac{f^{(1)}_{n+1}}{f^{(1)}_n} = 1.6180$ or $\lim_{n \to \infty} \frac{f^{(2)}_{n+1}}{f^{(2)}_n} = 1.6555$. Note that

$$\frac{d^{(1,2)}_{n+1}}{d^{(1,2)}_n} = \frac{f^{(1)}_{n+1} - f^{(2)}_{n+1}}{f^{(1)}_n - f^{(2)}_n} = \frac{(f^{(1)}_{n+1} - f^{(2)}_{n+1})/f^{(1)}_n}{(f^{(1)}_n - f^{(2)}_n)/f^{(1)}_n} = \frac{f^{(1)}_{n+1} - f^{(2)}_{n+1}}{f^{(1)}_n - f^{(2)}_n} = \frac{1}{f^{(1)}_n/f^{(2)}_n} - 1.$$  

But as $n$ gets larger, $f^{(1)}_n$ increases much rapidly than $f^{(2)}_n$ so that $\lim_{n \to \infty} \frac{1}{f^{(1)}_n/f^{(2)}_n} = 0$. It thus follows that $\alpha = \lim_{n \to \infty} \frac{d^{(1,2)}_{n+1}}{d^{(1,2)}_n} = \lim_{n \to \infty} \frac{f^{(1)}_{n+1}}{f^{(1)}_n}$.  

In fact, $\frac{d^{(1,2)}_{n+1}}{d^{(1,2)}_n}$ equals 1.689, 1.640, 1.625, 1.619, and 1.618 when $n = 10, 20, 30, 40, 50$. As $n$ goes to infinite, the ratio $\frac{d^{(1,2)}_{n+1}}{d^{(1,2)}_n}$ corresponds to $\frac{f^{(1)}_{n+1}}{f^{(1)}_n}$.

Theorem 9. Let $\Delta^{(1,3)}(x) = x^6 - 2x^5 + x^3 - x^2 + x + 1$. Then $\alpha = \lim_{n \to \infty} \frac{d^{(1,3)}_{n+1}}{d^{(1,3)}_n}$ is a real root of $\Delta^{(1,3)}(x) = P_1(x)P_3(x)$. And $\lim_{n \to \infty} \frac{d^{(1,3)}_{n+1}}{d^{(1,3)}_n} = \lim_{n \to \infty} \frac{f^{(1)}_{n+1}}{f^{(1)}_n}$.

Proof. Write $d^{(1,3)}_n = d_n$. Since

$$\frac{d_{n+1}}{d_n} = 2 - \frac{1}{d_n/d_{n-2}} + \frac{1}{d_n/d_{n-3}} - \frac{1}{d_n/d_{n-4}} - \frac{1}{d_n/d_{n-5}}.$$  

Therefore
by Theorem 4, \( \alpha = \lim_{n \to \infty} \frac{d_{n+1}^{(1,3)}}{d_n^{(1,3)}} \) satisfies \( \alpha^6 = 2\alpha^5 - \alpha^3 + \alpha^2 - \alpha + 1 \) so \( \alpha \) is a zero of \( \Delta^{(1,3)}(x) \). But since \( P_1(x)P_3(x) = (x^2 - x - 1)(x^4 - x^3 - 1) = \Delta^{(1,3)}(x) \), \( \alpha \) is a root of either \( P_1(x) \) or \( P_3(x) \). Note that

\[
\frac{d_{n+1}^{(1,3)}}{d_n^{(1,3)}} = \frac{(f_{n+1}^{(1)} - f_{n+1}^{(3)})/f_n^{(1)}}{(f_{n+1}^{(1)} - f_{n+1}^{(3)})/f_n^{(1)}} = \frac{f_n^{(1)}}{f_n^{(1)}} - \frac{1}{f_n^{(1)}/f_n^{(3)}}.
\]

Since \( f_n^{(1)} \) increases very faster than \( f_n^{(3)} \) as \( n \) gets larger, \( \lim_{n \to \infty} \frac{1}{f_n^{(1)}/f_n^{(3)}} = 0 \) so that \( \alpha = \lim_{n \to \infty} \frac{d_{n+1}^{(1,3)}}{d_n^{(1,3)}} = \lim_{n \to \infty} \frac{f_n^{(1)}}{f_n^{(1)}}. \)

The previous results on \( d_{n+1}^{(1,t)} \) \( (t = 2, 3, 4) \) yield the next theorem.

**Theorem 10.** For any integer \( t > 0 \), we have the followings:

1. \( d_{n+1}^{(1,t)} = d_n^{(1,t)} - d_{n-2}^{(1,t)} + d_{n-t}^{(1,t)} - d_{n-(t+1)}^{(1,t)} - d_{n-(t+2)}^{(1,t)}. \)

2. \( \lim_{n \to \infty} \frac{d_{n+1}^{(1,t)}}{d_n^{(1,t)}} \) is a real root of \( \Delta^{(1,t)}(x) = x^{t+3} - 2x^{t+2} + x^t - x^2 + x + 1. \)

Moreover \( \Delta^{(1,t)}(x) = P_1(x)P_t(x) \) and \( \lim_{n \to \infty} \frac{d_{n+1}^{(1,t)}}{d_n^{(1,t)}} = \lim_{n \to \infty} \frac{f_n^{(1)}}{f_n^{(1)}}. \)

**Proof.** When \( t = 2, 3 \), it is due to Theorem 8 and 9. When \( t = 4 \), if we write \( d_{n+1}^{(1,4)} = d_n \) then

\[
\frac{d_{n+1}}{d_n} = 2 - \frac{1}{d_n/d_{n-2}} + \frac{1}{d_n/d_{n-4}} - \frac{1}{d_n/d_{n-5}} - \frac{1}{d_n/d_{n-6}}
\]

by Theorem 7.

Hence by letting \( \alpha = \lim_{n \to \infty} \frac{d_{n+1}}{d_n} \), \( \alpha \) satisfies \( \alpha^7 = 2\alpha^6 - \alpha^4 + \alpha^2 - \alpha - 1. \)

Consider the polynomial \( P_t(x) = x^{t+1} - x^t - 1. \) Clearly

\[
P_1(x)P_4(x) = (x^2 - x - 1)(x^5 - x^4 - 1) = x^7 - 2x^6 + x^4 - x^2 + x + 1 = \Delta^{(1,4)}(x),
\]

so \( \alpha \) is a real root of either \( P_1(x) \) or \( P_4(x) \). But since \( f_n^{(1)} \) increases rapidly than \( f_n^{(4)} \) as \( n \) gets larger, we have \( \lim_{n \to \infty} \frac{1}{f_n^{(1)}/f_n^{(4)}} = 0 \) so that

\[
\lim_{n \to \infty} \frac{d_{n+1}^{(1,4)}}{d_n^{(1,4)}} = \lim_{n \to \infty} \frac{f_{n+1}^{(1)}}{f_n^{(1)}} - \frac{1}{f_n^{(1)}/f_n^{(4)}} = \lim_{n \to \infty} \frac{f_{n+1}^{(1)}}{f_n^{(1)}}.
\]
Now for any \( t > 0 \), let \( \Delta^{(1,t)}(x) = P_1(x)P_t(x) \). Then \( \Delta^{(1,t)}(x) = 0 \) implies \( x^{t+3} = 2x^{t+2} - x^t + x^2 - x - 1 \), so we may claim that \( d_n^{(1,t)} \) satisfies

\[
d_{n+1}^{(1,t)} = 2d_n^{(1,t)} - d_{n-t}^{(1,t)} + d_{n-t-1}^{(1,t)} - d_{n-t-2}^{(1,t)}.
\]

Indeed when \( 1 \leq t \leq 5 \), it is clear from the next table:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( P_1(x)P_t(x) )</th>
<th>recurrence of ( d_n^{(1,t)} = d_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( x^2 - 2x^3 + x + 1 )</td>
<td>( d_{n+1} = 2d_n - d_{n-3} - d_{n-4} )</td>
</tr>
<tr>
<td>3</td>
<td>( x^6 - 2x^5 + x^3 - x^2 + x + 1 )</td>
<td>( d_{n+1} = 2d_n - d_{n-2} + d_{n-3} - d_{n-4} - d_{n-5} )</td>
</tr>
<tr>
<td>4</td>
<td>( x^7 - 2x^6 + x^4 - x^2 + x + 1 )</td>
<td>( d_{n+1} = 2d_n - d_{n-2} + d_{n-4} - d_{n-5} - d_{n-6} )</td>
</tr>
<tr>
<td>5</td>
<td>( x^8 - 2x^7 + x^5 - x^2 + x + 1 )</td>
<td>( d_{n+1} = 2d_n - d_{n-2} + d_{n-5} - d_{n-6} - d_{n-7} )</td>
</tr>
</tbody>
</table>

Now for any \( t > 0 \), write \( d_n^{(1,t)} = f_n^{(1)} - f_n^{(t)} \) by \( d_n \) for convenience. Then

\[
2d_n - d_{n-2} + d_{n-1} - d_{n-2} - \frac{f_n^{(1)} - f_n^{(t)}}{f_n^{(1)} / f_n^{(t)}} = (2f_n^{(1)} - f_{n-2}^{(1)} + f_{n-t}^{(1)} - f_{n-t-1}^{(1)} - f_{n-t-2}^{(1)}) - (2f_n^{(t)} - f_{n-2}^{(t)} + f_{n-t}^{(t)} - f_{n-t-1}^{(t)} - f_{n-t-2}^{(t)}) = f_{n+1}^{(1)} - f_{n+1}^{(t)} = d_{n+1}.
\]

Moreover since \( \lim_{n \to \infty} \frac{1}{f_n^{(1)} / f_n^{(t)}} = 0 \), we finish to prove

\[
\lim_{n \to \infty} \frac{d_{n+1}}{d_n} = \lim_{n \to \infty} \frac{f_n^{(1)} / f_n^{(t)}}{1 - \frac{1}{f_n^{(1)} / f_n^{(t)}}} = \lim_{n \to \infty} \frac{f_n^{(1)}}{f_n^{(t)}}. \quad \square
\]

4. Fibonacci 4 Numbers

We now give our special attention to \( f_n^{(4)} \) and \( d_n^{(1,4)} \), and will prove that for any \( n \), \( f_{n-1}^{(4)} + f_{n-2}^{(4)} - 2f_{n-3}^{(4)} - f_{n-4}^{(4)} \) is equal to one of \( \pm 1, \pm 2 \). Let us define a constant \( C_n \) that equals

\[
\begin{cases} 
1 & \text{if } n \equiv 0, 2 \\
2 & \text{if } n \equiv 1 \\
-1 & \text{if } n \equiv 3, 5 \\
-2 & \text{if } n \equiv 4 
\end{cases}
\]

for \( n \pmod{6} \).
Theorem 11. For any integer $n > 0$, we have the followings:

(1) $f_n^{(4)} + f_{n-1}^{(4)} - f_{n-2}^{(4)} - 2f_{n-3}^{(4)} - f_{n-4}^{(4)} = C_n$.

(2) $f_{n+1}^{(4)} - 4f_{n-1}^{(4)} + 4f_{n-3}^{(4)} - f_{n-5}^{(4)} + f_{n-8}^{(4)} + 2f_{n-9}^{(4)} - f_{n-11}^{(4)} = -2C_n$.

Proof. When $5 \leq n \leq 10$, the next table proves the theorem.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$C_n$</th>
<th>$f_n^{(4)} + f_{n-1}^{(4)} - f_{n-2}^{(4)} - 2f_{n-3}^{(4)} - f_{n-4}^{(4)}$</th>
<th>$n$</th>
<th>$C_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$-1$</td>
<td>$2 + 1 - 1 - 2 - 1 = -1$</td>
<td>8</td>
<td>$1$</td>
</tr>
<tr>
<td>6</td>
<td>$1$</td>
<td>$3 + 2 - 1 - 2 - 1 = 1$</td>
<td>9</td>
<td>$-1$</td>
</tr>
<tr>
<td>7</td>
<td>$2$</td>
<td>$4 + 3 - 2 - 2 - 1 = 2$</td>
<td>10</td>
<td>$-2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1 = 1$</td>
<td></td>
<td>$5 + 4 - 3 - 4 - 1 = 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$5 + 4 - 3 - 4 - 1 = 1$</td>
<td></td>
<td>$6 + 5 - 4 - 6 - 2 = -1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$6 + 5 - 4 - 6 - 2 = -1$</td>
<td></td>
<td>$8 + 6 - 5 - 8 - 3 = -2$</td>
</tr>
</tbody>
</table>

First of all, we can observe $C_n = C_{n-1} + C_{n-5} \pmod{6}$. In fact, if $n \equiv 0 \pmod{6}$ then $C_{n-1} + C_{n-5} = -1 + 2 = 1 = C_n$, if $n \equiv 1 \pmod{6}$ then $1 + 1 = 2 = C_n$, and so on. We now assume the identity $f_k^{(4)} + f_{k-1}^{(4)} = f_{k-2}^{(4)} + 2f_{k-3}^{(4)} + f_{k-4}^{(4)} + C_k$ is true for all $k < n$. Then the induction hypothesis implies that

$$f_n^{(4)} + f_{n-1}^{(4)} = (f_{n-1}^{(4)} + f_{n-2}^{(4)}) + (f_{n-5}^{(4)} + f_{n-6}^{(4)})$$

$$= (f_{n-3}^{(4)} + 2f_{n-4}^{(4)} + f_{n-5}^{(4)} + C_{n-1})$$

$$+ (f_{n-7}^{(4)} + 2f_{n-8}^{(4)} + f_{n-9}^{(4)} + C_{n-5})$$

$$= (f_{n-3}^{(4)} + f_{n-7}^{(4)}) + 2(f_{n-4}^{(4)} + f_{n-8}^{(4)}) + (f_{n-5}^{(4)} + f_{n-9}^{(4)}) + (C_{n-1} + C_{n-5})$$

$$= f_{n-2}^{(4)} + 2f_{n-3}^{(4)} + f_{n-4}^{(4)} + C_n,$$

so we have (1) that $f_n^{(4)} + f_{n-1}^{(4)} - f_{n-2}^{(4)} - 2f_{n-3}^{(4)} - f_{n-4}^{(4)} = C_n$ for all $n \pmod{6}$.

Moreover we also have

$$4f_{n-1}^{(4)} - 4f_{n-3}^{(4)} + f_{n-5}^{(4)} - f_{n-8}^{(4)} - 2f_{n-9}^{(4)} + f_{n-11}^{(4)}$$

$$= (f_{n-1}^{(4)} + f_{n-5}^{(4)}) + 3f_{n-1}^{(4)} - 4f_{n-3}^{(4)} - f_{n-8}^{(4)} - 2f_{n-9}^{(4)} + f_{n-11}^{(4)}$$

$$= f_n^{(4)} + 3f_{n-1}^{(4)} - 4f_{n-3}^{(4)} + f_{n-4}^{(4)} - (f_{n-4}^{(4)} + f_{n-8}^{(4)}) - 2f_{n-9}^{(4)} + f_{n-11}^{(4)}$$

$$= (f_n^{(4)} + f_{n-4}^{(4)}) + 3f_{n-1}^{(4)} - 5f_{n-3}^{(4)} - 2f_{n-9}^{(4)} + f_{n-11}^{(4)}$$

$$= f_{n+1}^{(4)} + 3f_{n-1}^{(4)} - 5f_{n-3}^{(4)} - 2f_{n-9}^{(4)} + f_{n-11}^{(4)}.$$
we have

\[ \begin{align*}
    f_{n+1}^{(4)} &- 4f_{n}^{(4)} + 4f_{n-3}^{(4)} - f_{n-5}^{(4)} + f_{n-8}^{(4)} + 2f_{n-9}^{(4)} - f_{n-11}^{(4)} \\
    &= f_{n+1}^{(4)} - f_{n+1}^{(4)} - 3f_{n-1}^{(4)} + 5f_{n-3}^{(4)} + 2f_{n-4}^{(4)} - 2f_{n-5}^{(4)} - f_{n-1}^{(4)} + f_{n-2}^{(4)} + f_{n-7}^{(4)} \\
    &= -2(f_{n-1}^{(4)} + f_{n-5}^{(4)}) - 2f_{n-1}^{(4)} + f_{n-2}^{(4)} + f_{n-3}^{(4)} - f_{n-7}^{(4)} + 4f_{n-3}^{(4)} + 2f_{n-4}^{(4)} \\
    &= -2(f_{n-1}^{(4)} + f_{n-2}^{(4)} - 2f_{n-3}^{(4)} - f_{n-4}^{(4)}) \\
    &= -2C_n. \quad \Box
\end{align*} \]

Theorem 11 gives rise to a more simple recurrence of \( d_n^{(1,4)} \).

**Theorem 12.** Let \( \lambda_n = 1 \) (if \( n \equiv 0, 1 \)), 0 (if \( n \equiv 2, 5 \)) and \(-1\) (if \( n \equiv 3, 4 \)) with \( n \text{ mod } 6 \). Then \( d_{n+1}^{(1,4)} = 2d_n^{(1,4)} - d_{n-2}^{(1,4)} + (f_{n-8}^{(4)} + \lambda_n) \).

**Proof.** Write \( d_{n+1}^{(1,4)} = d_n \). Then \( \{d_n\}_{n \geq 1} = \{0, 1, 2, 4, 6, 10, 17, 29, 49, 81, \ldots\} \), so when \( n = 9 \), \( \lambda_9 = -1 \) and \( 2d_9 - d_7 + f_1^{(4)} + \lambda_9 = 81 = d_{10} \). We assume the identity \( d_{k+1} = 2d_k - d_{k-2} + f_{k-8}^{(4)} + \lambda_k \) for \( k < n \). Then by means of the induction hypothesis, we have

\[ \begin{align*}
    2d_n - d_{n-2} + f_{n-8}^{(4)} + \lambda_n &= (4d_{n-1} - 4d_{n-3} + d_{n-5}) + (2f_{n-9}^{(4)} - f_{n-11}^{(4)} + f_{n-8}^{(4)}) + (2\lambda_{n-1} - \lambda_{n-3} + \lambda_n). \quad (a)
\end{align*} \]

On the other hand due to Lemma 1 we also have

\[ \begin{align*}
    4d_{n-1} - 4d_{n-3} + d_{n-5} &= f_{n+1}^{(1)} - (4f_{n-1}^{(4)} - 4f_{n-3}^{(4)} + f_{n-5}^{(4)}) \quad (b)
\end{align*} \]

Hence together with (a) and (b), we have

\[ \begin{align*}
    2d_n - d_{n-2} + f_{n-8}^{(4)} + \lambda_n &= f_{n+1}^{(1)} - (4f_{n-1}^{(4)} - 4f_{n-3}^{(4)} + f_{n-5}^{(4)}) \\
    &\quad + (2f_{n-9}^{(4)} - f_{n-11}^{(4)} + f_{n-8}^{(4)}) + (2\lambda_{n-1} - \lambda_{n-3} + \lambda_n) \\
    &= f_{n+1}^{(1)} - (4f_{n-1}^{(4)} - 4f_{n-3}^{(4)} + f_{n-5}^{(4)} - 2f_{n-9}^{(4)} + f_{n-11}^{(4)}) + (2\lambda_{n-1} - \lambda_{n-3} + \lambda_n).
\end{align*} \]

But since

\[ \begin{align*}
    4f_{n-1}^{(4)} - 4f_{n-3}^{(4)} + f_{n-5}^{(4)} - f_{n-8}^{(4)} - 2f_{n-9}^{(4)} + f_{n-11}^{(4)} &= f_{n+1}^{(1)} + 2C_n,
\end{align*} \]

by Theorem 11, it therefore follows that

\[ \begin{align*}
    2d_n - d_{n-2} + f_{n-8}^{(4)} + \lambda_n &= f_{n+1}^{(1)} - f_{n+1}^{(4)} - 2C_n + (2\lambda_{n-1} - \lambda_{n-3} + \lambda_n) \\
    &= d_{n+1} - 2C_n + (\lambda_n + 2\lambda_{n-1} - \lambda_{n-3}).
\end{align*} \]
If we can show \( \lambda_n + 2\lambda_{n-1} - \lambda_{n-3} = 2C_n \) for all \( n \) (mod 6) then it completes to prove \( d_{n+1}^{(1,4)} = 2d_n^{(1,4)} - d_{n-2}^{(1,4)} + (f_n^{(4)} + \lambda_n) \). Indeed, if \( n \equiv 0 \) then \( \lambda_n + 2\lambda_{n-1} - \lambda_{n-3} = 1 + 2 \cdot 0 + 1 = 2 = 2C_n \), and if \( n \equiv 1 \) then \( 1 + 2 \cdot 1 + 1 = 4 = 2C_n \), etc.

Corollary 13. For all \( n \), \( f_n^{(4)} = d_{n+4}^{(1,4)} - d_{n+3}^{(1,4)} - d_{n+2}^{(1,4)} - \lambda_{n+2} \).

Proof. Theorem 8 and 12 give rise to

\[
 d_{n+1}^{(1,4)} = 2d_n^{(1,4)} - d_{n-2}^{(1,4)} + d_{n-4}^{(1,4)} - d_{n-6}^{(1,4)} - f_n^{(4)} + \lambda_n,
\]

so \( f_{n-8}^{(4)} + \lambda_n = d_{n-4}^{(1,4)} - d_{n-5}^{(1,4)} - d_{n-6}^{(1,4)} \) and \( f_n^{(4)} = d_{n+4}^{(1,4)} - d_{n+3}^{(1,4)} - d_{n+2}^{(1,4)} - \lambda_{n+8} \). It finishes the proof since \( \lambda_{n+8} = \lambda_{n+2} \).

Indeed a Fibonacci 4-number \( f_n^{(4)} \) can be obtained from the relation that, for instance \( d_{24}^{(1,4)} - d_{23}^{(1,4)} - d_{22}^{(1,4)} - \lambda_{22} = 74594 - 46043 - 28412 + 1 = 140 = f_{20}^{(4)} \).

References


