International Journal of Pure and Applied Mathematics
Volume 110 No. 2 2016, 347-360
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v110i2.9
ijpam.eu

DIFFERENCE SEQUENCE OF FIBONACCI t NUMBERS

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Abstract: We study interrelationships of Fibonacci s and Fibonacci t numbers for any $s \neq t$. Let $d_n^{(s,t)}$ be the difference sequence of Fibonacci s and t numbers. We investigate recurrence formulas of $d_n^{(s,t)}$ and find patterns that both Fibonacci s and Fibonacci t numbers satisfy at the same time. Moreover we research increasing ratio of the sequence $d_n^{(s,t)}$.

AMS Subject Classification: 11B30, 11B37
Key Words: Fibonacci t number, difference sequence of Fibonacci numbers

1. Introduction

For any integer $t > 0$, a Fibonacci t number $f_n^{(t)}$ satisfies the recurrence $f_{n+1}^{(t)} = f_n^{(t)} + f_{n-t}^{(t)}$ with $t+1$ initials $1, \cdots, 1, 2$. If $t = 1$ then it is the ordinary Fibonacci number and if $t = 2$ then it is sometimes called a Narayana number, see [4]. Fibonacci t numbers have many analogous properties of Fibonacci numbers, like Binet formula, sums of numbers and associated matrices, etc (see [2], [3], or [5]). In particular the limit of ratio of consecutive Fibonacci t numbers satisfies the equation $P_t(x) = x^{t+1} - x^t - 1$ (see [6]), and its application to to data-hiding was discussed in [1].

Though Fibonacci t numbers were studied in various aspect, it seems no one has asked interrelationships between Fibonacci t and Fibonacci s numbers for $t \neq s$. In this work we define a difference sequence $d_n^{(s,t)}$ of Fibonacci s

Received: August 19, 2016
Revised: September 19, 2016
Published: November 4, 2016

and Fibonacci t numbers. We investigate recurrence formulas of $d_n^{(s,t)}$, and find a recurrence pattern that both Fibonacci s and Fibonacci t numbers satisfy, and moreover examine increasing ratios of difference sequences $d_n^{(s,t)}$. Since $f_n^{(1)}$ is well known, Fibonacci t numbers $f_n^{(t)}$ would be obtained from $d_n^{(1,t)}$. In particular we discuss about $d_n^{(1,4)}$ that has rather unique property.

2. Difference Sequence of $f_n^{(1)}$ and $f_n^{(t)}$

The Fibonacci t number $f_n^{(t)}$ starts from $1, \cdots, 1, 2$ and satisfies the recurrence $f_{n+1}^{(t)} = f_n^{(t)} + f_{n-t}^{(t)}$ for $n - t > 0$. The $f_n^{(t)}$ can be extended to negative parameter n so that we have the following table of $f_n^{(t)}$ for $1 \le t \le 4$:

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$f_n^{(1)}$	5	-3	2	-1	1	0	1	1	2	3	5	8	13	21
$f_n^{(2)}$	-1	1	0	0	1	0	1	1	1	2	3	4	6	9
$f_n^{(3)}$	0	0	1	0	0	0	1	1	1	1	2	3	4	5
$f_n^{(4)}$	0	1	0	0	0	0	1	1	1	1	1	2	3	4

The next lemma explore some identities on $f_n^{(t)}$ $(1 \le t \le 3)$.

Lemma 1. *For any $t \ge 1$, we have $f_{n+1}^{(t)} = \sum_{i=t-1}^{2t-1} f_{n-i}^{(t)}$. Moreover we have the following equalities:*

(1) $f_n^{(1)} = 4f_{n-3}^{(1)} + f_{n-6}^{(1)}$, $3f_n^{(1)} = 4f_{n-1}^{(1)} + f_{n-2}^{(1)} + f_{n-4}^{(1)}$.

(2) $f_n^{(2)} + 2f_{n-2}^{(2)} + f_{n-4}^{(2)} = f_{n+2}^{(2)}$, $f_n^{(3)} = 4f_{n-5}^{(3)} + 3f_{n-8}^{(3)} - f_{n-9}^{(3)} + f_{n-11}^{(3)}$.

(3) $f_n^{(3)} = 2f_{n-2}^{(3)} - f_{n-11}^{(3)} - f_{n-12}^{(3)}$, $f_n^{(3)} = 2f_{n-1}^{(3)} - f_{n-4}^{(3)} - f_{n-5}^{(3)} + f_{n-13}^{(3)} + f_{n-14}^{(3)}$.

Proof. Clearly: $f_{n+1}^{(1)} = \sum_{i=0}^{1} f_{n-i}^{(1)}$, $f_{n+1}^{(2)} = (f_{n-1}^{(2)} + f_{n-3}^{(2)}) + f_{n-2}^{(2)} = \sum_{i=1}^{3} f_{n-i}^{(2)}$ and $f_{n+1}^{(3)} = (f_{n-1}^{(3)} + f_{n-4}^{(3)}) + f_{n-3}^{(2)} = (f_{n-2}^{(3)} + f_{n-5}^{(3)}) + f_{n-4}^{(3)} + f_{n-3}^{(2)} = \sum_{i=2}^{5} f_{n-i}^{(3)}$.

Now for $t \ge 1$, consider $\sum_{i=t-1}^{2t-1} f_{n-i}^{(t)} = f_{n-(t-1)}^{(t)} + \cdots + f_{n-(2t-1)}^{(t)}$. Then by adding the last term to the first one, we have

$$\sum_{i=t-1}^{2t-1} f_{n-i}^{(t)} = [f_{n-(t-1)}^{(t)} + f_{n-(2t-1)}^{(t)}] + f_{n-t}^{(t)} + f_{n-(t+1)}^{(t)} + \cdots + f_{n-(2t-2)}^{(t)}$$

$$=f^{(t)}_{n-(t-2)} + f^{(t)}_{n-t} + f^{(t)}_{n-(t+1)} + \cdots + f^{(t)}_{n-(2t-3)} + f^{(t)}_{n-(2t-2)}.$$

Continuing to add the last to the first term, it follows that

$$\begin{aligned}\sum_{i=t-1}^{2t-1} f^{(t)}_{n-i} =& f^{(t)}_{n-(t-3)} + f^{(t)}_{n-t} + f^{(t)}_{n-(t+1)}\\ =&\cdots\\ =& f^{(t)}_{n-1} + f^{(t)}_{n-t} + f^{(t)}_{n-t-1}\\ =& f^{(t)}_{n} + f^{(t)}_{n-t}\\ =& f^{(t)}_{n+1}.\end{aligned}$$

Now the identities on $f^{(1)}_n$ are clear, and we have

$$f^{(2)}_n + 2f^{(2)}_{n-2} + f^{(2)}_{n-4} = f^{(2)}_{n+1} + f^{(2)}_{n-1} = f^{(2)}_{n+2}.$$

For Fibonacci 3 numbers, it follows that

$$\begin{aligned}f^{(3)}_n =& f^{(3)}_{n-3} + f^{(3)}_{n-4} + f^{(3)}_{n-5} + f^{(3)}_{n-6}\\ =& 2f^{(3)}_{n-4} + f^{(3)}_{n-5} + f^{(3)}_{n-6} + f^{(3)}_{n-7}\\ =& 3f^{(3)}_{n-5} + f^{(3)}_{n-6} + f^{(3)}_{n-7} + 2f^{(3)}_{n-8},\end{aligned}$$

so that

$$\begin{aligned}f^{(3)}_n - 4f^{(3)}_{n-5} =& - f^{(3)}_{n-5} + f^{(3)}_{n-6} + f^{(3)}_{n-7} + 2f^{(3)}_{n-8}\\ =& - (f^{(3)}_{n-6} + f^{(3)}_{n-9}) + f^{(3)}_{n-6} + f^{(3)}_{n-7} + 2f^{(3)}_{n-8}\\ =& 3f^{(3)}_{n-8} - f^{(3)}_{n-9} + f^{(3)}_{n-11}.\end{aligned}$$

Moreover we also have

$$\begin{aligned}f^{(3)}_n - f^{(3)}_{n-2} + f^{(3)}_{n-11} + f^{(3)}_{n-12} =& (f^{(3)}_{n-1} + f^{(3)}_{n-4}) - f^{(3)}_{n-2} + (f^{(3)}_{n-7}\\ & - f^{(3)}_{n-8}) + (f^{(3)}_{n-8} - f^{(3)}_{n-9})\\ =& f^{(3)}_{n-1} - f^{(3)}_{n-2} + f^{(3)}_{n-4} + f^{(3)}_{n-7} - (f^{(3)}_{n-5} - f^{(3)}_{n-6})\\ =& [f^{(3)}_{n-1} - f^{(3)}_{n-2} - f^{(3)}_{n-5}] + f^{(3)}_{n-4} + f^{(3)}_{n-6} + f^{(3)}_{n-7}\\ =& f^{(3)}_{n-3} + f^{(3)}_{n-6}\\ =& f^{(3)}_{n-2},\end{aligned}$$

since $f^{(3)}_{n-1} - f^{(3)}_{n-2} - f^{(3)}_{n-5} = 0$. Finally it is not hard to see that

$$2f^{(3)}_{n-1} - f^{(3)}_{n-4} - f^{(3)}_{n-5} + f^{(3)}_{n-13} + f^{(3)}_{n-14} = f^{(3)}_{n-1} + (f^{(3)}_{n-2} - f^{(3)}_{n-4} + f^{(3)}_{n-13} + f^{(3)}_{n-14}).$$

But since $f^{(3)}_{n-2} = 2f^{(3)}_{n-4} - f^{(3)}_{n-13} - f^{(3)}_{n-14}$, it follows immediately that

$$2f^{(3)}_{n-1} - f^{(3)}_{n-4} - f^{(3)}_{n-5} + f^{(3)}_{n-13} + f^{(3)}_{n-14} = f^{(3)}_{n-1} + f^{(3)}_{n-4} = f^{(3)}_{n}. \qquad \square$$

For any $n, i \geq 0$, let $d^{(s,t)}_{n,(i)} = f^{(s)}_n - f^{(t)}_{n+i}$ and call $\{d^{(s,t)}_{n,(i)}\}$ the i step difference sequence of Fibonacci s and t numbers. In particular when $i = 0$, we denote the difference sequence by $\{d^{(s,t)}_{n,(i)}\} = \{d^{(s,t)}_n\}$. We begin to study recurrence formula of the difference sequence $d^{(1,2)}_{n,(i)} = f^{(1)}_n - f^{(2)}_{n+i}$.

Theorem 2. *$d^{(1,2)}_{n,(i)}$ satisfies*

$$d^{(1,2)}_{n+1,(i)} = 2d^{(1,2)}_{n,(i)} - d^{(1,2)}_{n-3,(i)} - d^{(1,2)}_{n-4,(i)},$$

for all $i \geq 0$.

Proof. For convenience, write $d^{(1,2)}_{n,(0)} = d_n$. Then $\{d_n\}_{n\geq 1} = \{0, 1, 1, 2, 4, 7, 12, \cdots\}$, so if $n = 5$ then $2d_5 - d_2 - d_1 = 7 = d_6$. Suppose the recurrence is true for all $k < n$. Then the induction hypothesis shows

$$\begin{aligned} 2d_n - d_{n-3} - d_{n-4} =& 2(2d_{n-1} - d_{n-4} - d_{n-5}) - (2d_{n-4} - d_{n-7} - d_{n-8}) \\ & - (2d_{n-5} - d_{n-8} - d_{n-9}) \\ =& 4d_{n-1} - 4d_{n-4} - 4d_{n-5} + d_{n-7} + 2d_{n-8} + d_{n-9}. \end{aligned}$$

So by substituting $d^{(1,2)}_n = f^{(1)}_n - f^{(2)}_n$, we have

$$2d_n - d_{n-3} - d_{n-4} = A^{(1)} - A^{(2)},$$

where

$$A^{(j)} = 4f^{(j)}_{n-1} - 4f^{(j)}_{n-4} - 4f^{(j)}_{n-5} + f^{(j)}_{n-7} + 2f^{(j)}_{n-8} + f^{(j)}_{n-9},$$

for $j = 1, 2$.

It is not hard to see from Lemma 1 that

$$\begin{aligned} A^{(1)} =& 4f^{(1)}_{n-1} - 4(f^{(1)}_{n-4} + f^{(1)}_{n-5}) + (f^{(1)}_{n-7} + f^{(1)}_{n-8}) + (f^{(1)}_{n-8} + f^{(1)}_{n-9}) \\ =& 4f^{(1)}_{n-2} + f^{(1)}_{n-6} + f^{(1)}_{n-7} \\ =& f^{(1)}_{n+1} - f^{(1)}_{n-5} + f^{(1)}_{n-6} + f^{(1)}_{n-7} = f^{(1)}_{n+1}. \end{aligned}$$

And Lemma 1 also shows

$$\begin{aligned} A^{(2)} =&4f_{n-1}^{(2)} - 4f_{n-4}^{(2)} - 4f_{n-5}^{(2)} + f_{n-7}^{(2)} + 2f_{n-8}^{(2)} + f_{n-9}^{(2)} \\ =&4(f_{n-1}^{(2)} - f_{n-4}^{(2)} - f_{n-5}^{(2)}) + (f_{n-7}^{(2)} + f_{n-8}^{(2)} + f_{n-9}^{(2)}) + f_{n-8}^{(2)} \\ =&4f_{n-3}^{(2)} + f_{n-5}^{(2)} + (f_{n-4}^{(2)} - f_{n-6}^{(2)} - f_{n-7}^{(2)}) \\ =&(f_{n-3}^{(2)} + f_{n-4}^{(2)} + f_{n-5}^{(2)}) + 3f_{n-3}^{(2)} - f_{n-6}^{(2)} - f_{n-7}^{(2)} \\ =&f_{n-1}^{(2)} + 2f_{n-3}^{(2)} + (f_{n-3}^{(2)} - f_{n-6}^{(2)} - f_{n-7}^{(2)}) \\ =&f_{n+1}^{(2)}. \end{aligned}$$

Therefore we have $2d_n - d_{n-3} - d_{n-4} = f_{n+1}^{(1)} - f_{n+1}^{(2)} = d_{n+1}$.

Now write $d_{n,(i)}^{(1,2)} = d_{n,(i)}$ for convenience.

If $i = 1$, then

$$\{d_{n,(1)} = f_n^{(1)} - f_{n+1}^{(2)}\} = \{0, 0, 0, 1, 2, 4, 8, 15, 27, 48, \cdots\}$$

and hence $d_{n+1,(1)} = 2d_{n,(1)} - d_{n-3,(1)} - d_{n-4,(1)}$ is true for $n \leq 10$.

Assume the identity is true for all $k < n$. Then

$$\begin{aligned} 2d_{n,(1)} - d_{n-3,(1)} - d_{n-4,(1)} =&4d_{n-1,(1)} - 4d_{n-4,(1)} - 4d_{n-5,(1)} + d_{n-7,(1)} \\ &+ 2d_{n-8,(1)} + d_{n-9,(1)} \\ =&(4f_{n-1}^{(1)} - 4f_{n-4}^{(1)} - 4f_{n-5}^{(1)} + f_{n-7}^{(1)} + 2f_{n-8}^{(1)} + f_{n-9}^{(1)}) \\ &- (4f_n^{(2)} - 4f_{n-3}^{(2)} - 4f_{n-4}^{(2)} + f_{n-6}^{(2)} + 2f_{n-7}^{(2)} + f_{n-8}^{(2)}). \end{aligned}$$

As seen above, it is easy to see that

$$4f_{n-1}^{(1)} - 4f_{n-4}^{(1)} - 4f_{n-5}^{(1)} + f_{n-7}^{(1)} + 2f_{n-8}^{(1)} + f_{n-9}^{(1)} = f_{n+1}^{(1)}$$

and

$$4f_n^{(2)} - 4f_{n-3}^{(2)} - 4f_{n-4}^{(2)} + f_{n-6}^{(2)} + 2f_{n-7}^{(2)} + f_{n-8}^{(2)} = f_{n+2}^{(2)},$$

hence it follows that

$$2d_{n,(1)} - d_{n-3,(1)} - d_{n-4,(1)} = f_{n+1}^{(1)} - f_{n+2}^{(2)} = d_{n+1,(1)}.$$

Now for any $i \geq 0$, we also have

$$\begin{aligned} 2d_{n,(i)} - d_{n-3,(i)} - d_{n-4,(i)} =&4d_{n-1,(i)} - 4d_{n-4,(i)} - 4d_{n-5,(i)} \\ &+ d_{n-7,(i)} + 2d_{n-8,(i)} + d_{n-9,(i)} \\ =&f_{n+1}^{(1)} - f_{n+1+i}^{(2)} = d_{n+1,(i)}. \end{aligned}$$

This completes the proof. □

Theorem 2 gives a relation that both $f_n^{(1)}$ and $f_n^{(2)}$ satisfy.

Corollary 3. *$f_n^{(j)}$ $(j = 1, 2)$ satisfies $f_{n+1}^{(j)} = 2f_n^{(j)} - f_{n-3}^{(j)} - f_{n-4}^{(j)}$.*

Proof. The identity is clear for $f_n^{(1)}$. And since $f_n^{(2)} = f_{n-2}^{(2)} + f_{n-3}^{(2)} + f_{n-4}^{(2)}$ by Lemma 1, we have
$f_{n+1}^{(2)} = f_n^{(2)} + f_{n-2}^{(2)} = 2f_n^{(2)} - f_{n-3}^{(2)} - f_{n-4}^{(2)}$. □

Theorem 4. *The difference $d_{n,(i)}^{(1,3)} = f_n^{(1)} - f_{n+i}^{(3)}$ satisfies $d_{n+1,(i)}^{(1,3)} = 2d_{n,(i)}^{(1,3)} - d_{n-2,(i)}^{(1,3)} + d_{n-3,(i)}^{(1,3)} - d_{n-4,(i)}^{(1,3)} - d_{n-5,(i)}^{(1,3)}$.*

Proof. Let $i = 0$ and $d_{n,(0)}^{(1,3)} = d_n$. Then $\{d_n\}_{n\geq 1} = \{0, 1, 2, 3, 5, 9, 16, 27, \cdots\}$ shows $2d_6 - d_4 + d_3 - d_2 - d_1 = 16 = d_7$. Hence if we assume the identity is true for all $k < n$ then similar to the proof of Theorem 2, we have

$$2d_n - d_{n-2} + d_{n-3} - d_{n-4} - d_{n-5} = 4d_{n-1} - 4d_{n-3} + 4d_{n-4} - 3d_{n-5} - 6d_{n-6} + 3d_{n-7} + d_{n-10} - (d_{n-9} - d_{n-10} - d_{n-11}).$$

So due to the induction hypothesis $d_{n-5} = 2d_{n-6} - d_{n-8} + d_{n-9} - d_{n-10} - d_{n-11}$, we have

$$\begin{aligned} 2d_n - d_{n-2} + d_{n-3} - d_{n-4} - d_{n-5} &= 4d_{n-1} - 4d_{n-3} + 4d_{n-4} - 3d_{n-5} - 6d_{n-6} \\ &\quad + 3d_{n-7} + d_{n-10} - (d_{n-5} - 2d_{n-6} + d_{n-8}) \\ &= 4d_{n-1} - 4d_{n-3} + 4d_{n-4} - 4d_{n-5} \\ &\quad - 4d_{n-6} + 3d_{n-7} - d_{n-8} + d_{n-10}. \end{aligned}$$

Now substitute $d_n = f_n^{(1)} - f_n^{(3)}$. Then it follows that $2d_n - d_{n-2} + d_{n-3} - d_{n-4} - d_{n-5} = A^{(1)} - A^{(3)}$ where,

$$A^{(j)} = 4f_{n-1}^{(j)} - 4f_{n-3}^{(j)} + 4f_{n-4}^{(j)} - 4f_{n-5}^{(j)} - 4f_{n-6}^{(j)} + 3f_{n-7}^{(j)} - f_{n-8}^{(j)} + f_{n-10}^{(j)}$$

for $j = 1, 3$. We shall show $A^{(j)} = f_{n+1}^{(j)}$.

If so, we finish to prove $2d_n - d_{n-2} + d_{n-3} - d_{n-4} - d_{n-5} = f_{n+1}^{(1)} - f_{n+1}^{(3)} = d_{n+1}$.

It is not hard to see

$$\begin{aligned} A^{(1)} &= 4(f_{n-1}^{(1)} - f_{n-3}^{(1)}) + 4(f_{n-4}^{(1)} - f_{n-6}^{(1)}) - 4(f_{n-5}^{(1)} - f_{n-7}^{(1)}) - f_{n-7}^{(1)} - (f_{n-8}^{(1)} - f_{n-10}^{(1)}) \\ &= 4f_{n-2}^{(1)} + (4f_{n-5}^{(1)} - 4f_{n-6}^{(1)} - f_{n-7}^{(1)} - f_{n-9}^{(1)}). \end{aligned}$$

But since $3f_n^{(1)} = 4f_{n-1}^{(1)} + f_{n-2}^{(1)} + f_{n-4}^{(1)}$ (Lemma 1), we have $4f_n^{(1)} - 4f_{n-1}^{(1)} - f_{n-2}^{(1)} - f_{n-4}^{(1)} = f_n^{(1)}$, hence

$$A^{(1)} = 4f_{n-2}^{(1)} + f_{n-5}^{(1)} = 3f_{n-2}^{(1)} + 2f_{n-3}^{(1)} = f_{n-2}^{(1)} + 2f_{n-1}^{(1)} = f_{n+1}^{(1)}.$$

Similarly due to Lemma 1, we also have

$$\begin{aligned}
A^{(3)} &= 4(f^{(3)}_{n-1} + f^{(3)}_{n-4}) - 4(f^{(3)}_{n-3} + f^{(3)}_{n-6}) - (f^{(3)}_{n-5} + f^{(3)}_{n-8}) \\
&\quad - 3f^{(3)}_{n-5} + (f^{(3)}_{n-7} + f^{(3)}_{n-10}) + 2f^{(3)}_{n-7} \\
&= 4f^{(3)}_{n} - 3(f^{(3)}_{n-2} + f^{(3)}_{n-5}) - f^{(3)}_{n-2} + 2(f^{(3)}_{n-4} + 2f^{(3)}_{n-7}) - 3f^{(3)}_{n-4} + f^{(3)}_{n-6} \\
&= 4f^{(3)}_{n} - 3(f^{(3)}_{n-1} + f^{(3)}_{n-4}) - f^{(3)}_{n-2} + (f^{(3)}_{n-3} + f^{(3)}_{n-6}) + f^{(3)}_{n-3} \\
&= 4f^{(3)}_{n} - 3f^{(3)}_{n} - f^{(3)}_{n-2} + f^{(3)}_{n-2} + f^{(3)}_{n-3} \\
&= f^{(3)}_{n} + f^{(3)}_{n-3} \\
&= f^{(3)}_{n+1}.
\end{aligned}$$

Now if $i > 0$ then the proof follows similar to Theorem 2. □

The recurrence of $d^{(1,3)}_{n+1}$ yields an identity that both $f^{(1)}_n$ and $f^{(3)}_n$ hold.

Corollary 5. *$f^{(j)}_{n+1} = 2f^{(j)}_n - f^{(j)}_{n-2} + f^{(j)}_{n-3} - f^{(j)}_{n-4} - f^{(j)}_{n-5}$ for $j = 1, 3$.*

Proof. The identity for $f^{(1)}_n$ is clear. On the other hand, Lemma 1 says $f^{(3)}_n = f^{(3)}_{n-3} + f^{(3)}_{n-4} + f^{(3)}_{n-5} + f^{(3)}_{n-6} = f^{(3)}_{n-2} + f^{(3)}_{n-4} + f^{(3)}_{n-5}$, hence we have

$$\begin{aligned}
&2f^{(3)}_{n} - f^{(3)}_{n-2} + f^{(3)}_{n-3} - f^{(3)}_{n-4} - f^{(3)}_{n-5} \\
&\quad = 2f^{(3)}_{n} - f^{(3)}_{n-2} + f^{(3)}_{n-3} - f^{(3)}_{n} + f^{(3)}_{n-2} \\
&\quad = f^{(3)}_{n} + f^{(3)}_{n-3} \\
&\quad = f^{(3)}_{n+1}. \quad \square
\end{aligned}$$

Besides Theorem 4, another recurrence of $d^{(1,3)}_{n,(i)}$ is as follows.

Theorem 6. *$d^{(1,3)}_{n+1,(i)} = 2d^{(1,3)}_{n,(i)} - d^{(1,3)}_{n-3,(i)} - d^{(1,3)}_{n-4,(i)} - f^{(3)}_{n-12} - f^{(3)}_{n-13}$ for all $i \geq 0$. Moreover $f^{(3)}_n + f^{(3)}_{n-1} = d^{(1,3)}_{n+10} - 2d^{(1,3)}_{n+9} + d^{(1,3)}_{n+7}$.*

Proof. Without loss of generality we may assume $i = 0$. Then

$$\begin{aligned}
&2d^{(1,3)}_{n} - d^{(1,3)}_{n-3} - d^{(1,3)}_{n-4} - f^{(3)}_{n-12} - f^{(3)}_{n-13} \\
&\quad = 2(f^{(1)}_{n} - f^{(3)}_{n}) - (f^{(1)}_{n-3} - f^{(3)}_{n-3}) - (f^{(1)}_{n-4} - f^{(3)}_{n-4}) - (f^{(3)}_{n-12} - f^{(3)}_{n-13}) \\
&\quad = (2f^{(1)}_{n} - f^{(1)}_{n-3} - f^{(1)}_{n-4}) - (2f^{(3)}_{n} - f^{(3)}_{n-3} - f^{(3)}_{n-4} + f^{(3)}_{n-12} + f^{(3)}_{n-13}) \\
&\quad = f^{(1)}_{n+1} - f^{(3)}_{n+1} \\
&\quad = d^{(1,3)}_{n+1},
\end{aligned}$$

by Lemma 1. Hence together with Theorem 4, it follows that

$$\begin{aligned}2d_n^{(1,3)} - d_{n-2}^{(1,3)} + d_{n-3}^{(1,3)} - d_{n-4}^{(1,3)} - d_{n-5}^{(1,3)} &= d_{n+1}^{(1,3)} \\ &= 2d_n^{(1,3)} - d_{n-3}^{(1,3)} - d_{n-4}^{(1,3)} - f_{n-12}^{(3)} - f_{n-13}^{(3)},\end{aligned}$$

so we have $f_{n-12}^{(3)} + f_{n-13}^{(3)} = d_{n-2}^{(1,3)} - 2d_{n-3}^{(1,3)} + d_{n-5}^{(1,3)}$. □

We further study $f_n^{(4)}$ and difference sequence $d_n^{(1,4)} = f_n^{(1)} - f_n^{(4)}$.

Theorem 7. $d_n^{(1,4)}$ *satisfies* $d_{n+1}^{(1,4)} = 2d_n^{(1,4)} - d_{n-2}^{(1,4)} + d_{n-4}^{(1,4)} - d_{n-5}^{(1,4)} - d_{n-6}^{(1,4)}$.

Proof. We first verify $2f_n^{(j)} - f_{n-2}^{(j)} + f_{n-4}^{(j)} - f_{n-5}^{(j)} - f_{n-6}^{(j)} = f_{n+1}^{(j)}$ for $j = 1, 4$. If $j = 1$ then the identity is clear. If $j = 4$ then

$$\begin{aligned}&2f_n^{(4)} - f_{n-2}^{(4)} + f_{n-4}^{(4)} - f_{n-5}^{(4)} - f_{n-6}^{(4)} \\ &= f_n^{(4)} + (f_n^{(4)} + f_{n-4}^{(4)}) - (f_{n-2}^{(4)} + f_{n-6}^{(4)}) - f_{n-5}^{(4)} \\ &= f_n^{(4)} + f_{n+1}^{(4)} - f_{n-1}^{(4)} - f_{n-5}^{(4)} \\ &= f_n^{(4)} + f_{n+1}^{(4)} - f_n^{(4)} \\ &= f_{n+1}^{(4)}.\end{aligned}$$

It thus follows immediately that

$$\begin{aligned}&2d_n^{(1,4)} - d_{n-2}^{(1,4)} + d_{n-4}^{(1,4)} - d_{n-5}^{(1,4)} - d_{n-6}^{(1,4)} \\ &= (2f_n^{(1)} - f_{n-2}^{(1)} + f_{n-4}^{(1)} - f_{n-5}^{(1)} - f_{n-6}^{(1)}) - (2f_n^{(4)} - f_{n-2}^{(4)} + f_{n-4}^{(4)} - f_{n-5}^{(4)} - f_{n-6}^{(4)}) \\ &= f_{n+1}^{(1)} - f_{n+1}^{(4)} \\ &= d_{n+1}^{(1,4)}. \quad \square\end{aligned}$$

3. Ratio of Difference Sequence

The ratio $\frac{f_{n+1}^{(t)}}{f_n^{(t)}}$ of consecutive Fibonacci t numbers converges to a positive root of $P_t(x) = x^{t+1} - x^t - 1$ for all $t \geq 1$ (see [6]). In this section we shall investigate the limit of $\frac{d_{n+1,(i)}^{(1,t)}}{d_{n,(i)}^{(1,t)}}$ of difference sequence, moreover find their interrelationships with $P_t(x)$. Note that since $d_{n,(i)}^{(1,t)}$ satisfies the same recurrence pattern for all $i > 0$, we may enough to consider $\frac{d_{n+1}^{(1,t)}}{d_n^{(1,t)}}$.

Theorem 8. *Let* $\Delta^{(1,2)}(x) = x^5 - 2x^4 + x + 1$. *Then* $\lim_{n\to\infty} \frac{d_{n+1}^{(1,2)}}{d_n^{(1,2)}}$ *is a real positive root of* $\Delta^{(1,2)} = P_1(x)P_2(x)$. *Moreover* $\lim_{n\to\infty} \frac{d_{n+1}^{(1,2)}}{d_n^{(1,2)}} = \lim_{n\to\infty} \frac{f_{n+1}^{(1)}}{f_n^{(1)}}$.

Proof. The recurrence of $d_n^{(1,2)}$ in Theorem 2 gives rise to

$$\frac{d_{n+1}^{(1,2)}}{d_n^{(1,2)}} = 2 - \frac{1}{d_n^{(1,2)}/d_{n-3}^{(1,2)}} - \frac{1}{d_n^{(1,2)}/d_{n-4}^{(1,2)}}.$$

So by letting $\lim_{n\to\infty} \frac{d_{n+1}^{(1,2)}}{d_n^{(1,2)}} = \alpha$, we have $\alpha = 2 - \frac{1}{\alpha^3} - \frac{1}{\alpha^4}$ and α is a positive real root of $x^5 - 2x^4 + x + 1 = \Delta^{(1,2)}(x)$. Clearly

$$\begin{aligned} \Delta^{(1,2)}(x) &= x^5 - 2x^4 + x + 1 \\ &= (x^2 - x - 1)(x^3 - x^2 - 1) \\ &= P_1(x)P_2(x). \end{aligned}$$

So α is one of the roots of $P_i(x)$ $(i = 1, 2)$ where each $P_i(x)$ has only one positive real root that equals $\lim_{n\to\infty} \frac{f_{n+1}^{(1)}}{f_n^{(1)}} = 1.6180$ or $\lim_{n\to\infty} \frac{f_{n+1}^{(2)}}{f_n^{(2)}} = 1.4655$. Note that

$$\frac{d_{n+1}^{(1,2)}}{d_n^{(1,2)}} = \frac{f_{n+1}^{(1)} - f_{n+1}^{(2)}}{f_n^{(1)} - f_n^{(2)}} = \frac{(f_{n+1}^{(1)} - f_{n+1}^{(2)})/f_n^{(1)}}{(f_n^{(1)} - f_n^{(2)})/f_n^{(1)}} = \frac{\frac{f_{n+1}^{(1)}}{f_n^{(1)}} - \frac{1}{f_n^{(1)}/f_{n+1}^{(2)}}}{1 - \frac{1}{f_n^{(1)}/f_n^{(2)}}}.$$

But as n gets larger, $f_n^{(1)}$ increases much rapidly than $f_n^{(2)}$ so that $\lim_{n\to\infty} \frac{1}{f_n^{(1)}/f_n^{(2)}} = 0$. It thus follows that $\alpha = \lim_{n\to\infty} \frac{d_{n+1}^{(1,2)}}{d_n^{(1,2)}} = \lim_{n\to\infty} \frac{f_{n+1}^{(1)}}{f_n^{(1)}}$. □

In fact, $\frac{d_{n+1}^{(1,2)}}{d_n^{(1,2)}}$ equals 1.689, 1.640, 1.625, 1.619, and 1.618 when $n = 10, 20,$ 30, 40, 50. As n goes to infinite, the ratio $\frac{d_{n+1}^{(1,2)}}{d_n^{(1,2)}}$ corresponds to $\frac{f_{n+1}^{(1)}}{f_n^{(1)}}$.

Theorem 9. *Let* $\Delta^{(1,3)}(x) = x^6 - 2x^5 + x^3 - x^2 + x + 1$. *Then* $\alpha = \lim_{n\to\infty} \frac{d_{n+1}^{(1,3)}}{d_n^{(1,3)}}$ *is a real root of* $\Delta^{(1,3)}(x) = P_1(x)P_3(x)$. *And* $\lim_{n\to\infty} \frac{d_{n+1}^{(1,3)}}{d_n^{(1,3)}} = \lim_{n\to\infty} \frac{f_{n+1}^{(1)}}{f_n^{(1)}}$.

Proof. Write $d_n^{(1,3)} = d_n$. Since

$$\frac{d_{n+1}}{d_n} = 2 - \frac{1}{d_n/d_{n-2}} + \frac{1}{d_n/d_{n-3}} - \frac{1}{d_n/d_{n-4}} - \frac{1}{d_n/d_{n-5}}$$

by Theorem 4, $\alpha = \lim_{n\to\infty} \frac{d_{n+1}^{(1,3)}}{d_n^{(1,3)}}$ satisfies $\alpha^6 = 2\alpha^5 - \alpha^3 + \alpha^2 - \alpha + 1$ so α is a zero of $\Delta^{(1,3)}(x)$. But since $P_1(x)P_3(x) = (x^2 - x - 1)(x^4 - x^3 - 1) = \Delta^{(1,3)}(x)$, α is a root of either $P_1(x)$ or $P_3(x)$. Note that

$$\frac{d_{n+1}^{(1,3)}}{d_n^{(1,3)}} = \frac{(f_{n+1}^{(1)} - f_{n+1}^{(3)})/f_n^{(1)}}{(f_n^{(1)} - f_n^{(3)})/f_n^{(1)}} = \frac{\frac{f_{n+1}^{(1)}}{f_n^{(1)}} - \frac{1}{f_n^{(1)}/f_{n+1}^{(3)}}}{1 - \frac{1}{f_n^{(1)}/f_n^{(3)}}}.$$

Since $f_n^{(1)}$ increases very faster than $f_n^{(3)}$ as n gets larger, $\lim_{n\to\infty} \frac{1}{f_n^{(1)}/f_n^{(3)}} = 0$ so that $\alpha = \lim_{n\to\infty} \frac{d_{n+1}^{(1,3)}}{d_n^{(1,3)}} = \lim_{n\to\infty} \frac{f_{n+1}^{(1)}}{f_n^{(1)}}$. □

The previous results on $d_n^{(1,t)}$ $(t = 2, 3, 4)$ yield the next theorem.

Theorem 10. *For any integer $t > 0$, we have the followings:*

(1) $d_{n+1}^{(1,t)} = d_n^{(1,t)} - d_{n-2}^{(1,t)} + d_{n-t}^{(1,t)} - d_{n-(t+1)}^{(1,t)} - d_{n-(t+2)}^{(1,t)}$.

(2) $\lim_{n\to\infty} \frac{d_{n+1}^{(1,t)}}{d_n^{(1,t)}}$ *is a real root of* $\Delta^{(1,t)}(x) = x^{t+3} - 2x^{t+2} + x^t - x^2 + x + 1$.

Moreover $\Delta^{(1,t)}(x) = P_1(x)P_t(x)$ *and* $\lim_{n\to\infty} \frac{d_{n+1}^{(1,t)}}{d_n^{(1,t)}} = \lim_{n\to\infty} \frac{f_{n+1}^{(1)}}{f_n^{(1)}}$.

Proof. When $t = 2, 3$, it is due to Theorem 8 and 9. When $t = 4$, if we write $d_n^{(1,4)} = d_n$ then

$$\frac{d_{n+1}}{d_n} = 2 - \frac{1}{d_n/d_{n-2}} + \frac{1}{d_n/d_{n-4}} - \frac{1}{d_n/d_{n-5}} - \frac{1}{d_n/d_{n-6}}$$

by Theorem 7.

Hence by letting $\alpha = \lim_{n\to\infty} \frac{d_{n+1}}{d_n}$, α satisfies $\alpha^7 = 2\alpha^6 - \alpha^4 + \alpha^2 - \alpha - 1$.

Consider the polynomial $P_t(x) = x^{t+1} - x^t - 1$. Clearly

$$\begin{aligned} P_1(x)P_4(x) &= (x^2 - x - 1)(x^5 - x^4 - 1) \\ &= x^7 - 2x^6 + x^4 - x^2 + x + 1 \\ &= \Delta^{(1,4)}(x), \end{aligned}$$

so α is a real root of either $P_1(x)$ or $P_4(x)$. But since $f_n^{(1)}$ increases rapidly than $f_n^{(4)}$ as n gets larger, we have $\lim_{n\to\infty} \frac{1}{f_n^{(1)}/f_n^{(4)}} = 0$ so that

$$\lim_{n\to\infty} \frac{d_{n+1}^{(1,4)}}{d_n^{(1,4)}} = \lim_{n\to\infty} \frac{\frac{f_{n+1}^{(1)}}{f_n^{(1)}} - \frac{1}{f_n^{(1)}/f_{n+1}^{(4)}}}{1 - \frac{1}{f_n^{(1)}/f_n^{(4)}}} = \lim_{n\to\infty} \frac{f_{n+1}^{(1)}}{f_n^{(1)}}.$$

Now for any $t > 0$, let $\Delta^{(1,t)}(x) = P_1(x)P_t(x)$. Then $\Delta^{(1,t)}(x) = 0$ implies $x^{t+3} = 2x^{t+2} - x^t + x^2 - x - 1$, so we may claim that $d_n^{(1,t)}$ satisfies

$$d_{n+1}^{(1,t)} = 2d_n^{(1,t)} - d_{n-2}^{(1,t)} + d_{n-t}^{(1,t)} - d_{n-t-1}^{(1,t)} - d_{n-t-2}^{(1,t)}.$$

Indeed when $1 \le t \le 5$, it is clear from the next table:

t	$P_1(x)P_t(x)$	recurrence of $d_n^{(1,t)} = d_n$
2	$x^5 - 2x^4 + x + 1$	$d_{n+1} = 2d_n - d_{n-3} - d_{n-4}$
3	$x^6 - 2x^5 + x^3 - x^2 + x + 1$	$d_{n+1} = 2d_n - d_{n-2} + d_{n-3} - d_{n-4} - d_{n-5}$
4	$x^7 - 2x^6 + x^4 - x^2 + x + 1$	$d_{n+1} = 2d_n - d_{n-2} + d_{n-4} - d_{n-5} - d_{n-6}$
5	$x^8 - 2x^7 + x^5 - x^2 + x + 1$	$d_{n+1} = 2d_n - d_{n-2} + d_{n-5} - d_{n-6} - d_{n-7}$

Now for any $t > 0$, write $d_n^{(1,t)} = f_n^{(1)} - f_n^{(t)}$ by d_n for convenience. Then

$$\begin{aligned}
&2d_n - d_{n-2} + d_{n-t} - d_{n-t-1} - d_{n-t-2}\\
&\quad = (2f_n^{(1)} - f_{n-2}^{(1)} + f_{n-t}^{(1)} - f_{n-t-1}^{(1)} - f_{n-t-2}^{(1)})\\
&\qquad - (2f_n^{(t)} - f_{n-2}^{(t)} + f_{n-t}^{(t)} - f_{n-t-1}^{(t)} - f_{n-t-2}^{(t)})\\
&\quad = f_{n+1}^{(1)} - f_{n+1}^{(t)}\\
&\quad = d_{n+1}.
\end{aligned}$$

Moreover since $\lim_{n\to\infty} \frac{1}{f_n^{(1)}/f_n^{(t)}} = 0$, we finish to prove

$$\lim_{n\to\infty} \frac{d_{n+1}}{d_n} = \lim_{n\to\infty} \frac{\frac{f_{n+1}^{(1)}}{f_n^{(1)}} - \frac{1}{f_n^{(1)}/f_{n+1}^{(t)}}}{1 - \frac{1}{f_n^{(1)}/f_n^{(t)}}} = \lim_{n\to\infty} \frac{f_{n+1}^{(1)}}{f_n^{(1)}}.$$

□

4. Fibonacci 4 Numbers

We now give our special attention to $f_n^{(4)}$ and $d_n^{(1,4)}$, and will prove that for any n, $f_n^{(4)} + f_{n-1}^{(4)} - f_{n-2}^{(4)} - 2f_{n-3}^{(4)} - f_{n-4}^{(4)}$ is equal to one of ± 1, ± 2. Let us define a constant C_n that equals

$$\begin{cases} 1 & \text{if } n \equiv 0, 2 \\ 2 & \text{if } n \equiv 1 \end{cases} \quad \text{and} \quad \begin{cases} -1 & \text{if } n \equiv 3, 5 \\ -2 & \text{if } n \equiv 4 \end{cases}$$

for $n \pmod 6$.

Theorem 11. *For any integer $n > 0$, we have the followings:*

(1) $f_n^{(4)} + f_{n-1}^{(4)} - f_{n-2}^{(4)} - 2f_{n-3}^{(4)} - f_{n-4}^{(4)} = C_n$.

(2) $f_{n+1}^{(4)} - 4f_{n-1}^{(4)} + 4f_{n-3}^{(4)} - f_{n-5}^{(4)} + f_{n-8}^{(4)} + 2f_{n-9}^{(4)} - f_{n-11}^{(4)} = -2C_n$.

Proof. When $5 \le n \le 10$, the next table proves the theorem.

n	C_n	$f_n^{(4)} + f_{n-1}^{(4)} - f_{n-2}^{(4)} - 2f_{n-3}^{(4)} - f_{n-4}^{(4)}$	n	C_n	
5	-1	$2+1-1-2-1=-1$	8	1	$5+4-3-4-1=1$
6	1	$3+2-1-2-1=1$	9	-1	$6+5-4-6-2=-1$
7	2	$4+3-2-2-1=2$	10	-2	$8+6-5-8-3=-2$

First of all, we can observe $C_n = C_{n-1} + C_{n-5}$ (n mod 6). In fact, if $n \equiv 0$ (mod 6) then $C_{n-1} + C_{n-5} = -1 + 2 = 1 = C_n$, if $n \equiv 1$ then $1 + 1 = 2 = C_n$, and so on. We now assume the identity $f_k^{(4)} + f_{k-1}^{(4)} = f_{k-2}^{(4)} + 2f_{k-3}^{(4)} + f_{k-4}^{(4)} + C_k$ is true for all $k < n$. Then the induction hypothesis implies that

$$\begin{aligned}
f_n^{(4)} + f_{n-1}^{(4)} =& (f_{n-1}^{(4)} + f_{n-2}^{(4)}) + (f_{n-5}^{(4)} + f_{n-6}^{(4)}) \\
=& (f_{n-3}^{(4)} + 2f_{n-4}^{(4)} + f_{n-5}^{(4)} + C_{n-1}) \\
& + (f_{n-7}^{(4)} + 2f_{n-8}^{(4)} + f_{n-9}^{(4)} + C_{n-5}) \\
=& (f_{n-3}^{(4)} + f_{n-7}^{(4)}) + 2(f_{n-4}^{(4)} + f_{n-8}^{(4)}) + (f_{n-5}^{(4)} + f_{n-9}^{(4)}) + (C_{n-1} + C_{n-5}) \\
=& f_{n-2}^{(4)} + 2f_{n-3}^{(4)} + f_{n-4}^{(4)} + C_n,
\end{aligned}$$

so we have (1) that $f_n^{(4)} + f_{n-1}^{(4)} - f_{n-2}^{(4)} - 2f_{n-3}^{(4)} - f_{n-4}^{(4)} = C_n$ for all n (mod 6).

Moreover we also have

$$\begin{aligned}
&4f_{n-1}^{(4)} - 4f_{n-3}^{(4)} + f_{n-5}^{(4)} - f_{n-8}^{(4)} - 2f_{n-9}^{(4)} + f_{n-11}^{(4)} \\
&= (f_{n-1}^{(4)} + f_{n-5}^{(4)}) + 3f_{n-1}^{(4)} - 4f_{n-3}^{(4)} - f_{n-8}^{(4)} - 2f_{n-9}^{(4)} + f_{n-11}^{(4)} \\
&= f_n^{(4)} + 3f_{n-1}^{(4)} - 4f_{n-3}^{(4)} + f_{n-4}^{(4)} - (f_{n-4}^{(4)} + f_{n-8}^{(4)}) - 2f_{n-9}^{(4)} + f_{n-11}^{(4)} \\
&= (f_n^{(4)} + f_{n-4}^{(4)}) + 3f_{n-1}^{(4)} - 5f_{n-3}^{(4)} - 2f_{n-9}^{(4)} + f_{n-11}^{(4)} \\
&= f_{n+1}^{(4)} + 3f_{n-1}^{(4)} - 5f_{n-3}^{(4)} - 2f_{n-9}^{(4)} + f_{n-11}^{(4)}.
\end{aligned}$$

But since

$$f_{n-11}^{(4)} = f_{n-6}^{(4)} - f_{n-7}^{(4)} = f_{n-1}^{(4)} - f_{n-2}^{(4)} - f_{n-7}^{(4)},$$

we have

$$\begin{aligned}&f^{(4)}_{n+1} - 4f^{(4)}_{n-1} + 4f^{(4)}_{n-3} - f^{(4)}_{n-5} + f^{(4)}_{n-8} + 2f^{(4)}_{n-9} - f^{(4)}_{n-11}\\ &\quad = f^{(4)}_{n+1} - f^{(4)}_{n+1} - 3f^{(4)}_{n-1} + 5f^{(4)}_{n-3} + 2f^{(4)}_{n-4} - 2f^{(4)}_{n-5} - f^{(4)}_{n-1} + f^{(4)}_{n-2} + f^{(4)}_{n-7}\\ &\quad = -2(f^{(4)}_{n-1} + f^{(4)}_{n-5}) - 2f^{(4)}_{n-1} + f^{(4)}_{n-2} + (f^{(4)}_{n-3} + f^{(4)}_{n-7}) + 4f^{(4)}_{n-3} + 2f^{(4)}_{n-4}\\ &\quad = -2(f^{(4)}_{n} + f^{(4)}_{n-1} - f^{(4)}_{n-2} - 2f^{(4)}_{n-3} - f^{(4)}_{n-4})\\ &\quad = -2C_n. \quad \square\end{aligned}$$

Theorem 11 gives rise to a more simple recurrence of $d^{(1,4)}_n$.

Theorem 12. *Let $\lambda_n = 1$ (if $n \equiv 0, 1$), 0 (if $n \equiv 2, 5$) and -1 (if $n \equiv 3, 4$) with n mod 6. Then $d^{(1,4)}_{n+1} = 2d^{(1,4)}_n - d^{(1,4)}_{n-2} + (f^{(4)}_{n-8} + \lambda_n)$.*

Proof. Write $d^{(1,4)}_n = d_n$. Then $\{d_n\}_{n\geq 1} = \{0, 1, 2, 4, 6, 10, 17, 29, 49, 81, \cdots\}$, so when $n = 9$, $\lambda_9 = -1$ and $2d_9 - d_7 + f^{(4)}_1 + \lambda_9 = 81 = d_{10}$. We assume the identity $d_{k+1} = 2d_k - d_{k-2} + f^{(4)}_{k-8} + \lambda_k$ for $k < n$. Then by means of the induction hypothesis, we have

$$\begin{aligned}2d_n - d_{n-2} + f^{(4)}_{n-8} + \lambda_n &= (4d_{n-1} - 4d_{n-3} + d_{n-5}) + (2f^{(4)}_{n-9} - f^{(4)}_{n-11} + f^{(4)}_{n-8})\\ &\quad + (2\lambda_{n-1} - \lambda_{n-3} + \lambda_n). \quad (a)\end{aligned}$$

On the other hand due to Lemma 1 we also have

$$4d_{n-1} - 4d_{n-3} + d_{n-5} = f^{(1)}_{n+1} - (4f^{(4)}_{n-1} - 4f^{(4)}_{n-3} + f^{(4)}_{n-5}) \quad \text{(b)}$$

Hence together with (a) and (b), we have

$$\begin{aligned}2d_n - d_{n-2} + f^{(4)}_{n-8} + \lambda_n &= f^{(1)}_{n+1} - (4f^{(4)}_{n-1} - 4f^{(4)}_{n-3} + f^{(4)}_{n-5})\\ &\quad + (2f^{(4)}_{n-9} - f^{(4)}_{n-11} + f^{(4)}_{n-8}) + (2\lambda_{n-1} - \lambda_{n-3} + \lambda_n)\\ &= f^{(1)}_{n+1} - (4f^{(4)}_{n-1} - 4f^{(4)}_{n-3} + f^{(4)}_{n-5} - f^{(4)}_{n-8} - 2f^{(4)}_{n-9} + f^{(4)}_{n-11})\\ &\quad + (2\lambda_{n-1} - \lambda_{n-3} + \lambda_n).\end{aligned}$$

But since

$$4f^{(4)}_{n-1} - 4f^{(4)}_{n-3} + f^{(4)}_{n-5} - f^{(4)}_{n-8} - 2f^{(4)}_{n-9} + f^{(4)}_{n-11} = f^{(4)}_{n+1} + 2C_n,$$

by Theorem 11, it therefore follows that

$$\begin{aligned}2d_n - d_{n-2} + f^{(4)}_{n-8} + \lambda_n &= f^{(1)}_{n+1} - f^{(4)}_{n+1} - 2C_n + (2\lambda_{n-1} - \lambda_{n-3} + \lambda_n)\\ &= d_{n+1} - 2C_n + (\lambda_n + 2\lambda_{n-1} - \lambda_{n-3}).\end{aligned}$$

If we can show $\lambda_n + 2\lambda_{n-1} - \lambda_{n-3} = 2C_n$ for all $n \pmod 6$ then it completes to prove $d_{n+1}^{(1,4)} = 2d_n^{(1,4)} - d_{n-2}^{(1,4)} + (f_{n-8}^{(4)} + \lambda_n)$. Indeed, if $n \equiv 0$ then $\lambda_n + 2\lambda_{n-1} - \lambda_{n-3} = 1 + 2 \cdot 0 + 1 = 2 = 2C_n$, and if $n \equiv 1$ then $1 + 2 \cdot 1 + 1 = 4 = 2C_n$, etc. □

Corollary 13. *For all n, $f_n^{(4)} = d_{n+4}^{(1,4)} - d_{n+3}^{(1,4)} - d_{n+2}^{(1,4)} - \lambda_{n+2}$.*

Proof. Theorem 8 and 12 give rise to

$$\begin{aligned} d_{n+1}^{(1,4)} &= 2d_n^{(1,4)} - d_{n-2}^{(1,4)} + d_{n-4}^{(1,4)} - d_{n-5}^{(1,4)} - d_{n-6}^{(1,4)} \\ &= 2d_n^{(1,4)} - d_{n-2}^{(1,4)} + f_{n-8}^{(4)} + \lambda_n, \end{aligned}$$

so $f_{n-8}^{(4)} + \lambda_n = d_{n-4}^{(1,4)} - d_{n-5}^{(1,4)} - d_{n-6}^{(1,4)}$ and $f_n^{(4)} = d_{n+4}^{(1,4)} - d_{n+3}^{(1,4)} - d_{n+2}^{(1,4)} - \lambda_{n+8}$. It finishes the proof since $\lambda_{n+8} = \lambda_{n+2}$. □

Indeed a Fibonacci 4-number $f_n^{(4)}$ can be obtained from the relation that, for instance $d_{24}^{(1,4)} - d_{23}^{(1,4)} - d_{22}^{(1,4)} - \lambda_{22} = 74594 - 46043 - 28412 + 1 = 140 = f_{20}^{(4)}$.

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