ON $[a, b]$ -COMPACTNESS IN BITOPOLOGICAL SPACES

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Abstract: The purpose of this paper is to introduce the concept of $[a, b]$ -compactness in bitopological spaces. Several related results are obtained, concerning characterization, images and inverse image under pairwise continuous functions. Finally we give some product theorems in the sequel.

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1. Introduction

In 1963, Kelly [7] introduced the concept of bitopological space, also he defined p -regular and p -normal spaces. In 1969, Fletcher P. et. al.[4] gave the definitions of $\tau_1\tau_2$ -open and p -open covers in a bitopological space. A cover $\mathcal{U}$ of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$ -open if $\mathcal{U} \subset \tau_1 \cup \tau_2$. If, in addition, $\mathcal{U}$ contains at least one non-empty member of τ_1 and at least one non-empty member of τ_2 , it is called p -open. Also they defined the concept of p -compact space as follows:

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A bitopological space is called p -compact if every p -open cover of the space has a finite subcover, while in (1972) Datta [2], defined s -compact space if every $\tau_1\tau_2$ -open cover of the space has a finite subcover. and in 1969 Birsan [1] gave the following definitions.

A bitopological space (X, τ_1, τ_2) is called τ_1 compact with respect to τ_2 if for each τ_1 -open cover of X there is a finite τ_2 -open subcover. A bitopological space is called B -compact if it is τ_i -compact with respect to τ_j for $i \neq j; i, j = 1, 2$.

In (1983), Fora and Hdeib [5] introduced the definitions of p -Lindelöf, s -Lindelöf and B -Lindelöf spaces in analogue manner. They also gave the definitions of certain types of functions as follows: A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is p -continuous (p -open, p -closed, p -homeomorphism, respectively) if both $f : (X, \tau_1) \rightarrow (Y, \sigma_1)$ and $f : (X, \tau_2) \rightarrow (Y, \sigma_2)$ are continuous (open, closed, homeomorphism, respectively).

In this paper we introduce the notion of $[a, b]$ -compactness in bitopological spaces. When (X, τ_1, τ_2) has a topological property Q this means that means both τ_1 and τ_2 have this property. For instance (X, τ_1, τ_2) is $[a, b]$ -compact space if both of (X, τ_1) and (X, τ_2) are $[a, b]$ -compact spaces, the letters a, b denote infinite regular cardinals and ω_0, ω_1 stands for the cardinality of $\mathbb{Z}, \mathbb{R}$, respectively.

2. Preliminaries

Definition 2.1. A bitopological space $X = (X, \tau_1, \tau_2)$ is called $p - [a, b]$ -compact if every p -open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq b$ has a subcover with cardinality $< a$.

If (X, τ_1, τ_2) is $p - [a, b]$ -compact for all $b \geq a$, then it is called $p - [a, \infty]$ -compact.

Definition 2.2. A bitopological space $X = (X, \tau_1, \tau_2)$ is called $s - [a, b]$ -compact if every $\tau_1\tau_2$ -open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq b$ has a subcover with cardinality $< a$.

If (X, τ_1, τ_2) is $s - [a, b]$ -compact for all $b \geq a$, then it is called $s - [a, \infty]$ -compact.

Definition 2.3. A bitopological space $X = (X, \tau_1, \tau_2)$ is called $\tau_1 - [a, b]$ -compact with respect to τ_2 if every τ_1 open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq b$ has a subcover in τ_2 with cardinality $< a$.

Definition 2.4. A bitopological space $X = (X, \tau_1, \tau_2)$ is called $B - [a, b]$ -compact if it is $\tau_i - [a, b]$ -compact with respect to τ_j for $i \neq j; i, j = 1, 2$.

Remark 2.5. Consider a bitopological space $X = (X, \tau_1, \tau_2)$, then:

- (i) X is $p - [\omega_0, \infty]$ -compact if and only if it is p -compact ,
- (ii) X is $p - [\omega_1, \infty]$ -compact if and only if it is p -Lindelöf,
- (iii) X is $s - [\omega_0, \infty]$ -compact if and only if it is s -compact ,
- (iv) X is $s - [\omega_1, \infty]$ -compact if and only if it is s -Lindelöf,
- (v) X is $B - [\omega_0, \infty]$ -compact if and only if it is B -compact.

Theorem 2.6. A bitopological space $X = (X, \tau_1, \tau_2)$ is $s - [a, b]$ -compact if and only if it is $p - [a, b]$ -compact and $[a, b]$ -compact.

Proof. Let $\mathcal{U}$ be p -open cover of X with $|\mathcal{U}| \leq b$. Then $\mathcal{U}$ is a $\tau_1\tau_2$ -open cover and hence X is $p - [a, b]$ -compact, also for τ_i -open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq b, i = 1, 2$, then $\mathcal{U}$ is $\tau_1\tau_2$ -open cover of X , and hence X is $[a, b]$ -compact.

Conversely, let $\mathcal{U}$ be a $\tau_1\tau_2$ -open cover of X with $|\mathcal{U}| \leq b$, if $\mathcal{U}$ is p -open cover of X , then the result is follows. Otherwise $\mathcal{U}$ is τ_1 -open or τ_2 -open, since X is $[a, b]$ -compact, then $\mathcal{U}$ has a subcover of cardinality $< a$. $\square$

Theorem 2.7. If $X = (X, \tau_1, \tau_2)$ is hereditary $[a, b]$ -compact, then X is $s - [a, b]$ -compact.

Proof. Let $\mathcal{U}$ be a $\tau_1\tau_2$ -open cover of X with $|\mathcal{U}| \leq b$, so $\mathcal{U}$ can be written as

$$\mathcal{U} = \{W_\alpha | \alpha \in \Delta\} \cup \{V_\gamma | \gamma \in \Gamma\}$$

where $W_\alpha \in \tau_1$ for all $\alpha \in \Delta$ and $V_\gamma \in \tau_2$ for all $\gamma \in \Gamma$.

Let

$$W = \bigcup \{W_\alpha | \alpha \in \Delta\},$$

then W is $[a, b]$ compact and hence there exist $\Delta_1 \subset \Delta$ with $|\Delta_1| < a$ such that

$$W = \bigcup \{W_\alpha | \alpha \in \Delta_1\}.$$

Similarly for

$$V = \bigcup \{V_\gamma | \gamma \in \Gamma\},$$

there exist $\Gamma_1 \subset \Gamma$ with $|\Gamma_1| < a$ such that

$$V = \{V_\gamma | \gamma \in \Gamma_1\},$$

hence

$$\{W_\alpha | \alpha \in \Delta_1\} \cup \{V_\gamma | \gamma \in \Gamma_1\}$$

is subcover of $\mathcal{U}$ with cardinality $< a$. $\square$

Theorem 2.8. *Let $X = (X, \tau_1, \tau_2)$ be a $p - [a, b]$ -compact and F be a τ_1 -closed or τ_2 -closed. Then F is $p - [a, b]$ -compact.*

Proof. Let $\mathcal{U}$ be a p -open cover of F with $|\mathcal{U}| \leq b$. Now

$$\mathcal{U} \cup \{X - F\}$$

is a p -open cover of X with cardinality $\leq b$, then there exists a subcover

$$\mathcal{W} = \mathcal{U}^* \cup \{X - F\}$$

of X with $|\mathcal{W}| < a$, hence $\mathcal{U}^*$ is a subcover of F with $|\mathcal{U}^*| < a$. $\square$

The pervious theorem is also true for $s - [a, b]$ -compact and $B - [a, b]$ -compact spaces.

Definition 2.9. [6] We say that U is a -open if for all $x \in U$ there exists open set V containing x such that $|V - U| < a$. U is a -closed if $X - U$ is a -open.

Theorem 2.10. *Let $X = (X, \tau_1, \tau_2)$ be a $p - [a, \infty]$ -compact and F be an a -closed proper subset of (X, τ_1) . Then F is $\tau_2 - [a, b]$ -compact.*

Proof. Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be τ_2 -open cover of F with $|\mathcal{U}| \leq b$. For $x \in X - F$, there exists τ_1 -open set V_x containing x such that $|F \cap V_x| < a$. Now consider

$$\mathcal{V} = \mathcal{U} \bigcup \{V_x | x \in X - F\},$$

then $\mathcal{V}$ is a p -open cover of X , so there exists a subcover

$$\mathcal{V}_a = \{U_\alpha | \alpha \in \Delta_1\} \bigcup \{V_x | x \in \{x_1, x_2, \dots\}\}$$

where $\Delta_1 \subset \Delta$ with $|\Delta_1| < a$ and $\{x_1, x_2, \dots\} \subset X - F$ with cardinality of $\{x_1, x_2, \dots\} < a$. Since $|F \cap V_x| < a$ and a is regular, we have $|F \cap (\bigcup V_{x_i})| < a$, so for each $y_i \in F \cap (\bigcup V_{x_i})$ there exists $\alpha_i \in \Delta$ such that $y_i \in U_{\alpha_i}$. Finally

$$\{U_\alpha | \alpha \in \Delta_1\} \bigcup \{U_{\alpha_i} | \alpha_i \in \Delta\}$$

is a subcover of F with cardinality $< a$. $\square$

Corollary 2.11. *Let $X = (X, \tau_1, \tau_2)$ be a $p - [a, b]$ -compact and F be an a -closed proper subset of (X, τ_1) with $|X - F| \leq b$. Then F is $\tau_2 - [a, b]$ -compact.*

Corollary 2.12. *Let $X = (X, \tau_1, \tau_2)$ be a $s - [a, b]$ -compact and F be an a -closed proper subset of (X, τ_1) with $|X - F| \leq b$. Then F is $\tau_2 - [a, b]$ -compact.*

Corollary 2.13. *Let $X = (X, \tau_1, \tau_2)$ be a $s - [a, \infty]$ -compact and F be an a -closed proper subset of (X, τ_1) . Then F is $\tau_2 - [a, b]$ -compact.*

Definition 2.14. A family $\mathcal{F} = \{F_\alpha | \alpha \in \Delta\}$ is said to have a -intersection property if the intersection of a members of $\mathcal{F}$ is nonempty set.

Definition 2.15. [9] A family $\mathcal{F}$ of nonempty τ_1 -or τ_2 -closed sets in X is p -closed if $\mathcal{F}$ contain members F_1 and F_2 such that F_1 is a τ_1 -closed proper subset of X and F_2 is a τ_2 -closed proper subset of X .

Definition 2.16. [5] A family $\mathcal{F}$ of nonempty subsets of X is $\tau_1\tau_2$ -closed if every member of $\mathcal{F}$ is τ_1 -closed or τ_2 -closed.

Theorem 2.17. *A bitopological space $X = (X, \tau_1, \tau_2)$ is a $p - [a, b]$ -compact if and only if every p -closed family $\mathcal{F} = \{F_\alpha | \alpha \in \Delta, |\Delta| \leq b\}$ with a -intersection property has a nonempty intersection.*

Proof. ($\Rightarrow$) Let $X = (X, \tau_1, \tau_2)$ be a $p - [a, b]$ -compact. Assume there exists a p -closed family $\mathcal{F} = \{F_\alpha | \alpha \in \Delta\}$, $|\Delta| \leq b$ with a -intersection property and

$$\bigcap_{\alpha \in \Delta} F_\alpha = \phi.$$

Now

$$X = X \setminus \bigcap_{\alpha \in \Delta} F_\alpha = \bigcup_{\alpha \in \Delta} (X - F_\alpha),$$

so $\{X - F_\alpha | \alpha \in \Delta\}$ is a p -open cover of X with cardinality $\leq b$, hence there exists $\Delta_1 \subset \Delta$ with $|\Delta_1| < a$ and

$$X = \bigcup_{\alpha \in \Delta} \{X - F_\alpha\}$$

which contradict with the fact that $\mathcal{F}$ has a -intersection property, hence $\mathcal{F}$ has a nonempty intersection.

($\Leftarrow$) Assume there exists a p -open cover $\mathcal{U}$ with $|\mathcal{U}| \leq b$ that has no subcover with cardinality $< a$, so

$$\bigcap_{\alpha \in \Delta_1} \{X - U_\alpha\} \neq \phi$$

and hence

$$\bigcap_{\alpha \in \Delta} \{X - U_\alpha\} \neq \phi$$

which is a contradiction, hence X is $p - [a, b]$ -compact. $\square$

Theorem 2.18. A bitopological space $X = (X, \tau_1, \tau_2)$ is a $s - [a, b]$ -compact if and only if every $\tau_1\tau_2$ -closed family $\mathcal{F} = \{F_\alpha | \alpha \in \Delta, |\Delta| \leq b\}$ with a -intersection property has nonempty intersection.

Theorem 2.19. A bitopological space $X = (X, \tau_1, \tau_2)$ is a $p - [a, b]$ -compact if and only if every a -closed proper subset F of τ_1 with $|X - F| \leq b$ is $\tau_2 - [a, b]$ -compact.

Proof. ($\Rightarrow$) By Corollary 2.13.

($\Leftarrow$) Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be a p -open cover of X with $|\mathcal{U}| \leq b$. Take U_{α_0} a nonempty τ_1 -open subset of $\mathcal{U}$, so $X - U_{\alpha_0}$ is a τ_1 -closed and hence it is a -closed subset of (X, τ_1) , therefore $X - U_{\alpha_0}$ covered by less than a members of $\mathcal{U}$, say $\mathcal{V}$, consequently

$$\mathcal{W} = \mathcal{V} \cup U_{\alpha_0}$$

is subcover of X with $|\mathcal{W}| < a$. □

Corollary 2.20. A bitopological space $X = (X, \tau_1, \tau_2)$ is a $p - [a, \infty]$ -compact if and only if each a -closed proper subset F of (X, τ_1) is $\tau_2 - [a, \infty]$ -compact.

Theorem 2.21. Every p -regular, $p - [a, b]$ -compact bitopological space (X, τ_1, τ_2) with $|X| \leq b$, is p -normal.

Proof. Let E be a τ_1 -closed and F be a τ_2 -closed subsets of X with $E \cap F = \emptyset$. Since X is a p -regular space, for each $e \in E$ there exists a τ_2 -open subset of X G_e and τ_1 -closed subset of X H_e such that $e \in G_e \subseteq H_e \subseteq X - F$. Similarly, for $f \in F$ there exists τ_1 -open subset of X C_f and τ_2 -closed subset of X M_f with $f \in C_f \subseteq M_f \subseteq X - E$.

Now

$$\mathcal{U} = \{C_f | f \in F\} \cup \{X - F\}$$

and

$$\mathcal{V} = \{G_e | e \in E\} \cup \{X - E\}$$

are p -open covers of X with cardinality $\leq b$, so there exists subcovers $\mathcal{U}_a, \mathcal{V}_a$ of X with cardinality $< a$. It is clear that $E \subseteq \bigcup_{i \in \Delta_1} G_{e_i}$ and $F \subseteq \bigcup_{i \in \Delta_2} C_{f_i}$ with

$|\Delta_i| < a$ for $i = 1, 2$.

Let

$$V_1 = C_1$$

and

$$V_n = C_n - \bigcup_{i=1}^{n-1} H_i \quad \text{for each positive integer } n > 1$$

and

$$W_n = G_n - \bigcup_{i=1}^n M_i.$$

Finally

$$V = \bigcup_{i \in \Delta_2} V_{n_i}$$

and

$$W = \bigcup_{i \in \Delta_1} W_{n_i}$$

are disjoint τ_1 -open (τ_2 -open) sets respectively, with $E \subseteq W$ and $F \subseteq V$, hence the X is p -normal. $\square$

Corollary 2.22. *Every p -regular, p - $[a, \infty]$ -compact in a bitopological space (X, τ_1, τ_2) is p -normal.*

3. Examples

In this section we give some examples on different types of $[a, b]$ -compactness in bitopological spaces.

Example 3.1. Consider the two topologies τ_1 and τ_2 on $\mathbb{R}$ defined by the basis

$$\mathbf{B}_1 = \{(-\infty, a) \mid a > 0\} \cup \{\{x\} \mid x > 0\}$$

$$\mathbf{B}_2 = \{(a, \infty) \mid a < 0\} \cup \{\{x\} \mid x < 0\}$$

Then X is a $p - [\omega_1, \infty]$ -compact but not a $B - [\omega_1, \infty]$ -compact, since for the τ_2 -open cover $\{(-1, \infty)\} \cup \{\{x\} \mid x < 0\}$ has no τ_1 -open subcover with cardinality $< \omega_1$. Observe that X is not $\tau_i - [\omega_1, \infty]$ -compact for $i = 1, 2$, also X is not $s - [\omega_1, \infty]$ -compact.

Example 3.2. Consider the two topologies τ_1 and τ_2 on $\mathbb{R}$ defined by the basis

$$\mathbf{B}_1 = \{(-\infty, 0) - F \mid F \text{ Finite}\} \cup \{\{x\} \mid x \geq 0\}$$

$$\mathbf{B}_2 = \{[0, \infty) - F \mid F \text{ Finite}\} \cup \{\{x\} \mid x < 0\}$$

Then (X, τ_1, τ_2) is not $p - [\omega_1, \infty]$ -compact, since the p -open cover $\mathcal{U} = \{\{x\} | x \in \mathbb{R}\}$ has no subcover with cardinality $< \omega_1$, also X is not $B - [\omega_1, \infty]$ -compact, for instance, consider the τ_1 -open cover

$$\mathcal{U} = \{\{x\} | x \geq 0\} \cup \{(-\infty, 0) - \{-1\}\} \cup \{(-\infty, 0) - \{-2\}\},$$

then $\mathcal{U}$ has no τ_2 -open subcover.

Clearly, X is niether $s - [\omega_1, \infty]$ -compact nor $[\omega_1, \infty]$ -compact.

Example 3.3. Consider $X = (\mathbb{R}, \tau_u, \tau_s)$, where τ_u and τ_s are the usual and Sorgenfrey line topologies, respectively. Then X is $s - [\omega_1, \infty]$ -compact but not $B - [\omega_1, \infty]$ -compact, since $\mathcal{U} = \{[-n, n) | n \in \mathbb{N}\}$ is τ_s - open cover of X with no τ_u -open subcover with cardinality $< \omega_1$.

4. Mappings and $[a, b]$ -Compactness in Bitopological Spaces

In this section we obtain some results concerning mappings and $[a, b]$ -compactness in bitopological spaces.

Theorem 4.1. *If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is a p -continuous onto map, and (X, τ_1, τ_2) is $p - [a, b]$ -compact ($s - [a, b]$ -compact, $B - [a, b]$ -compact, respectively), then (Y, σ_1, σ_2) is $p - [a, b]$ -compact ($s - [a, b]$ -compact, $B - [a, b]$ -compact, respectively).*

Proof. Let $\mathcal{U} = \{V_\alpha | \alpha \in \Delta\} \cup \{W_\gamma | \gamma \in \Gamma\}$ be a p -open cover of Y such that $V_\alpha \in \sigma_1$ and $W_\gamma \in \sigma_2$ with $|\Delta \cup \Gamma| \leq b$, since f is p -continuous onto map, we have

$$\{f^{-1}(V_\alpha) | \alpha \in \Delta\} \cup \{f^{-1}(W_\gamma) | \gamma \in \Gamma\}$$

is a p -open cover of X with cardinality $\leq b$, but X is a $p - [a, b]$ -compact, so there exists $\Delta_1 \subseteq \Delta$ and $\Gamma_1 \subseteq \Gamma$, such that

$$\{f^{-1}(V_\alpha) | \alpha \in \Delta_1\} \cup \{f^{-1}(W_\gamma) | \gamma \in \Gamma_1\}$$

is a subcover of X with $|\Delta_1 \cup \Gamma_1| < a$, therefore $\{V_\alpha | \alpha \in \Delta_1\} \cup \{W_\gamma | \gamma \in \Gamma_1\}$ is a subcover of Y with cardinality $< a$, hence Y is $p - [a, b]$ -compact.

The remaining parts are similarly proved. $\square$

Theorem 4.2. [5] *A bitopological $X = (X, \tau_1, \tau_2)$ is a p -Hausdorff space if and only if for each $x \in X$, $\{x\} = \bigcap_{\alpha \in \Delta} \{\overline{U_\alpha}^{\tau_i}, U_\alpha \text{ is a } \tau_j \text{ open of } x \text{ for } i \neq j ; i, j = 1, 2\}$.*

Definition 4.3. A space (X, τ) is called P_a -space if the intersection of (a) members of τ is open.

Lemma 4.4. Let $X = (X, \tau_1, \tau_2)$ be a p -Hausdorff, P_a -space. Then every $\tau_i - [a, \infty]$ -compact subset of X is τ_j -closed for $i \neq j$; $i, j = 1, 2$.

Proof. Let A be a $\tau_i - [a, \infty]$ compact subset of X and $x \in X - A$, then by Theorem 4.2, $\{x\} = \bigcap_{\alpha \in \Delta} \{\overline{U_\alpha}^{\tau_i}, U_\alpha \text{ is a } \tau_j \text{ open of } x \text{ for } i \neq j; i, j = 1, 2.\}$, then $A \subset X - \{x\}$, so

$$\{X - \overline{U_\alpha}^{\tau_i} | \alpha \in \Delta\}$$

is a τ_i - open cover of A , so there exists $\Delta_1 \subseteq \Delta$ with $|\Delta_1| < a$ and

$$A \subseteq \bigcup_{\alpha \in \Delta_1} \{X - \overline{U_\alpha}^{\tau_i} | \alpha \in \Delta_1\}.$$

Now let

$$U = \bigcap_{\alpha \in \Delta_1} U_\alpha,$$

since X is a P_a -space, we have U is a τ_j -open and $x \in U \subseteq X - A$, i.e. A is τ_j closed. $\square$

Theorem 4.5. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be p -continuous bijection map. If (Y, σ_1, σ_2) is p -Hausdorff, P_a -space and (X, τ_1, τ_2) is $p - [a, \infty]$ -compact, then f is p -homeomorphism .

Proof. Enough to show f is p -closed. Let E be a τ_i -closed proper subset of X , and hence E is an a -closed proper subset of τ_i , then by corollary 2.11 E is a proper $\tau_j - [a, \infty]$ compact for $i \neq j$; $i, j = 1, 2$, and hence $f(E)$ is a $\sigma_j - [a, \infty]$ -compact, but Y is a p -Hausdorff, P_a -space, so by lemma 4.4 $f(E)$ is σ_i -closed, i.e. f is p -homeomorphism function. $\square$

Definition 4.6. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called p - strong $[a, b]$ -function if for every p -open cover $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ of X with $|\Delta| \leq b$, there exists a p -open cover $\mathcal{V} = \{V_\gamma | \gamma \in \Gamma\}$ of Y with $|\Gamma| \leq b$ such that

$$f^{-1}(V) \subseteq \bigcup \{U_\alpha | \alpha \in \Delta_1 \subseteq \Delta, |\Delta_1| < a\}$$

for all $V \in \mathcal{V}$.

Definition 4.7. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called s -strong $[a, b]$ -function if for every $\tau_1\tau_2$ -open cover $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ of (X, τ_1, τ_2) with $|\Delta| \leq b$, there exists a $\sigma_1\sigma_2$ -open cover $\mathcal{V} = \{V_\gamma | \gamma \in \Gamma\}$ of Y with $|\Gamma| \leq b$ such that

$$f^{-1}(V) \subseteq \bigcup \{U_\alpha | \alpha \in \Delta_1 \subseteq \Delta, |\Delta_1| < a\}$$

for all $V \in \mathcal{V}$.

Definition 4.8. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called p -weak $[a, b]$ -function if for every p -open cover $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ of X with $|\Delta| \leq b$, there exists a p -open cover $\mathcal{V} = \{V_\gamma | \gamma \in \Gamma\}$ of (Y, σ_1, σ_2) such that

$$f^{-1}(V) \subseteq \bigcup \{U_\alpha | \alpha \in \Delta_1 \subseteq \Delta, |\Delta_1| < a\}$$

for all $V \in \mathcal{V}$.

Definition 4.9. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called s -weak $[a, b]$ -function if for every $\tau_1\tau_2$ -open cover $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ of X with $|\Delta| \leq b$, there exists a $\sigma_1\sigma_2$ -open cover $\mathcal{V} = \{V_\gamma | \gamma \in \Gamma\}$ of Y such that

$$f^{-1}(V) \subseteq \bigcup \{U_\alpha | \alpha \in \Delta_1 \subseteq \Delta, |\Delta_1| < a\}$$

for all $V \in \mathcal{V}$.

Definition 4.10. A p -continuous p -closed function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be s -perfect function if for all $y \in Y$, $f^{-1}(y)$ is $s - [\omega_0, \infty]$ -compact.

Definition 4.11. A p -continuous p -closed function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be p -perfect function if for all $y \in Y$, $f^{-1}(y)$ is $p - [\omega_0, \infty]$ -compact.

Theorem 4.12. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ and $b = \sum_c \{b^c : c \text{ is an infinite cardinal and } c < a\}$. Then f is s -strong $[a, b]$ -function if and only if f is s -weak $[a, b]$ -function.

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be a $\tau_1\tau_2$ -open cover of X with $|\Delta| \leq b$, so there exists a $\sigma_1\sigma_2$ -open cover $\mathcal{V} = \{V_\gamma | \gamma \in \Gamma\}$ of Y with

$$f^{-1}(V) \subseteq \bigcup \{U_\alpha | \alpha \in \Delta_1 \subseteq \Delta, |\Delta_1| < a\}$$

for all $V \in \mathcal{V}$.

Let

$$V_{\Delta_1} = \bigcup \left\{ V_0 \mid f^{-1}(V_0) \subseteq \bigcup \{U_\alpha \mid \alpha \in \Delta_1\} \right\},$$

since $b = \sum_c \{b^c : c \text{ is a cardinal and } c < a\}$, define $\mathcal{M} = \{V_{\Delta_1} \mid \Delta_1 \subset \Delta\}$, then $\mathcal{M}$ is a $\sigma_1\sigma_2$ -open cover of Y with $|\mathcal{M}| \leq b$ and $f^{-1}(V_{\Delta_1})$ contains in the union of members of $\mathcal{U}$ with cardinality $< a$, hence f is s -strong $[a, b]$ -function. $\square$

Theorem 4.13. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a p -strong $[a, b]$ -onto function. Then X is p - $[a, b]$ -compact if Y is so.*

Proof. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be a p -open cover of X with $|\Delta| \leq b$. Since f is p -strong $[a, b]$ -function, there exists a p -open cover $\mathcal{V} = \{V_\gamma \mid \gamma \in \Gamma\}$ of Y with $|\Gamma| \leq b$ such that

$$f^{-1}(V) \subseteq \bigcup \{U_\alpha \mid \alpha \in \Delta_1 \subseteq \Delta, |\Delta_1| < a\} \quad \text{for all } V \in \mathcal{V}$$

but Y is p - $[a, b]$ -compact, so there exists $\Gamma_1 \subseteq \Gamma$ with $|\Gamma_1| < a$ such that

$$Y = \bigcup_{\gamma \in \Gamma_1} V_\gamma,$$

and hence

$$X = \bigcup_{\gamma \in \Gamma_1} f^{-1}(V_\gamma),$$

now each of $f^{-1}(V_\gamma)$ contains in $< a$ members of $\mathcal{U}$, but a is a regular cardinal, hence X is p - $[a, b]$ -compact. $\square$

Theorem 4.14. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a s -strong $[a, b]$ -onto function. Then X is s - $[a, b]$ -compact if Y is so.*

Theorem 4.15. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a s -weak $[a, b]$ -onto function. Then X is p - $[a, b]$ -compact if Y is p - $[a, \infty]$ -compact.*

Theorem 4.16. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a s -weak $[a, b]$ -onto function. Then X is s - $[a, b]$ -compact if Y is s - $[a, \infty]$ -compact.*

Theorem 4.17. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a p -closed onto function and $f^{-1}y$ is s - $[a, b]$ -compact for all $y \in Y$. Then f is s -weak $[a, b]$ -function.*

Proof. Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be a $\tau_1\tau_2$ -open cover of X , For $y \in Y$, $f^{-1}y$ is $s - [a, b]$ -compact, so there exists $\Delta_1 \subseteq \Delta$ with $|\Delta_1| < a$ such that $f^{-1}(y) \subseteq \bigcap_{\alpha \in \Delta_1} U_\alpha$. Let

$$O_y = Y - f(X - \bigcup_{\alpha \in \Delta_1} U_\alpha),$$

then O_y is $\sigma_1\sigma_2$ -open in Y . Define

$$\mathcal{O} = \{O_y | y \in Y\},$$

then $\mathcal{O}$ is $\sigma_1\sigma_2$ -open cover of Y , hence $f^{-1}(O_y)$ is contains in $< a$ of members of $\mathcal{U}$, hence f is s -weak $[a, b]$ -function. $\square$

Theorem 4.18. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a p -continuous s -strong $[a, b]$ -function, and let $K \in \sigma_1 \cup \sigma_2$ be $s - [a, b]$ -compact in Y . Then $f^{-1}K$ is $s - [a, b]$ -compact in X .*

Proof. Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be a $\tau_1\tau_2$ -open cover of $f^{-1}K$ with $|\Delta| \leq b$.

Let $\mathcal{W} = \mathcal{U} \cup \{X - f^{-1}K\}$, then $\mathcal{W}$ is a $\tau_1\tau_2$ -open cover of X with $|\Delta| \leq b$, since f is s -strong $[a, b]$ -function, there exists a $\sigma_1\sigma_2$ -open cover $\mathcal{V} = \{V_\gamma | \gamma \in \Gamma\}$ of Y with $|\Gamma| \leq b$ such that $f^{-1}K$ contains in union of $< a$ members of $\mathcal{U}$, but K is $s - [a, b]$ -compact, so K contains in union of $< a$ members of $\mathcal{V}$, hence $f^{-1}K$ is $s - [a, b]$ -compact. $\square$

Corollary 4.19. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a p -continuous s -weak $[a, b]$ -function, and let $K \in \sigma_1 \cup \sigma_2$ be $s - [a, \infty]$ -compact in Y . Then $f^{-1}K$ is $s - [a, b]$ -compact in X .*

Corollary 4.20. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a p -continuous s -weak $[a, b]$ -function, and Y is a T_1 -space. Then $f^{-1}y$ is $s - [a, b]$ -compact for all $y \in Y$.*

5. Product of $[a, b]$ Compactness in a Bitopological Spaces

In this section we introduced some results concerning the product of $[a, b]$ -compactness in bitopological spaces. The following example shows that the product of two $p - [a, b]$ -compact spaces need not be $p - [a, b]$ -compact.

Example 5.1. Let $(\mathbb{R}, \tau_{cof}, \tau_d)$ where τ_{cof}, τ_d are the cofinite and discrete topologies, respectively. Then $(\mathbb{R}, \tau_{cof}, \tau_d)$ is a $p - [\omega_0, \infty]$ -compact space but $(\mathbb{R} \times \mathbb{R}, \tau_{cof} \times \tau_{cof}, \tau_d \times \tau_d)$ is a not $p - [\omega_0, \infty]$ -compact space (in fact it is not $p - [\omega_1, \infty]$ -compact), since the p -open cover $\mathcal{U} = \{\mathbb{R} \times \mathbb{R} \setminus \{0\}\} \cup \{(x, 0) : x \in \mathbb{R}\}$ has no subcover with cardinality $< \omega_1$.

Also the product of two $s - [a, b]$ -compact spaces need not be $s - [a, b]$ -compact.

Example 5.2. Consider the basis $\mathbf{B} = \{[x, y) : x, y \in \mathbb{R}, a < b\}$ of τ_s , then $(\mathbb{R}, \tau_u, \tau_s)$ is $s - [\omega_1, \infty]$ compact space where τ_u is the usual topology, but $(\mathbb{R} \times \mathbb{R}, \tau_u \times \tau_u, \tau_s \times \tau_s)$ is not $s - [\omega_1, \infty]$ -compact space.

In [8], the authors state the definitions of $s - P$ - and $p - P$ -spaces. We give the generalized form of these notions.

Definition 5.3. A bitopological space $X = (X, \tau_1, \tau_2)$ is called $s - P_a$ space if for each $x \in X$ and $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ a $\tau_1 \tau_2$ -open cover of X with $|\Delta| \leq b$, we have $\bigcap_{\alpha \in \Delta_1} \{U_x^\alpha | x \in U_x^\alpha, U_x^\alpha \in \mathcal{U} \text{ for all } \Delta_1 \subseteq \Delta \text{ with } |\Delta_1| < a\} \in \tau_1 \cup \tau_2$.

Theorem 5.4. Let $X = (X, \tau_1, \tau_2)$ be $s - [a, b]$ -compact space, and $Y = (Y, \sigma_1, \sigma_2)$ is $s - [a, \infty]$ -compact and $s - P_a$ space. Then $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is $s - [a, b]$ -compact.

Proof. Let $\mathcal{U} = \{V_\alpha | \alpha \in \Delta\} \times \{W_\alpha | \alpha \in \Delta\}$ be a $(\tau_1 \times \sigma_1)(\tau_2 \times \sigma_2)$ -open cover of $X \times Y$ with cardinality $\leq b$, where $\{V_\alpha | \alpha \in \Delta\}$ is $\tau_1 \tau_2$ -open cover of X and $\{W_\alpha | \alpha \in \Delta\}$ is $\sigma_1 \sigma_2$ -open cover of Y . For $y \in Y$, $X_y = X \times \{y\}$ is a $s - [a, b]$ compact, so there exists a subcover of X_y say

$$\mathcal{U}_a = \{V_y^\alpha \times W_y^\alpha | \alpha \in \Delta_1 \subseteq \Delta, |\Delta_1| < a\}.$$

Now let $G_y = \bigcap_{\alpha \in \Delta_1} W_y^\alpha$, and $\mathcal{G} = \{G_y | y \in Y\}$, since Y is $s - P_a$ space, then

$$\left\{ \bigcap_{\alpha \in \Delta_1} W_y^\alpha, y \in Y \right\} \in \sigma_1 \cup \sigma_2,$$

and $\mathcal{G}$ is a $\sigma_1 \sigma_2$ -open cover of Y , but Y is $s - [a, \infty]$ -compact space, so there exists a subcover of Y say $\{G_{y_i} | i \in \Delta_2, |\Delta_2| < a\}$. Finally

$$\{V_y^{\alpha_j} \times W_y^{\alpha_j} | j \in \Delta_1\}$$

is subcover of $X \times G_y$, so

$$\{\{V_{y_i}^{\alpha_j} \times W_{y_i}^{\alpha_j} | j \in \Delta_1\}, i \in \Delta_2\}$$

is a subcover of $X \times Y$ with cardinality $< a$, hence the result. $\square$

Definition 5.5. A bitopological space $X = (X, \tau_1, \tau_2)$ is called $p - P_a$ space if for each $x \in X$ and $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ a p -open cover of X with $|\Delta| \leq b$, such that

$\bigcap_{\alpha \in \Delta_1} \{U_x^\alpha | x \in U_x^\alpha, U_x^\alpha \in \mathcal{U} \text{ for all } \Delta_1 \subseteq \Delta \text{ with } |\Delta_1| < a\} \in \tau_1 \cup \tau_2$. We have

$$U \cap \tau_1 \neq \phi \text{ and } U \cap \tau_2 \neq \phi \text{ for all } U \in \mathcal{U}.$$

Theorem 5.6. Let $X = (X, \tau_1, \tau_2)$ be $p - [a, b]$ -compact space, and $Y = (Y, \sigma_1, \sigma_2)$ is $p - [a, \infty]$ -compact and $p - P_a$ space. Then $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is $p - [a, b]$ -compact.

Corollary 5.7. Let $X = (X, \tau_1, \tau_2)$ and $Y = (Y, \sigma_1, \sigma_2)$ be $[a, \infty]$ -compact $p - [a, \infty]$ -compact spaces, and Y be a $s - P_a$ space. Then $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is $s - [a, \infty]$ -compact and hence $p - [a, \infty]$ -compact.

Proof. From Theorem 2.6. $\square$

Theorem 5.8. Let $X = (X, \tau_1, \tau_2)$ and $Y = (Y, \sigma_1, \sigma_2)$ be $p - [a, b]$ -compact spaces and $b = \sum_c \{b^c : c \text{ is an infinite cardinal and } c < a\}$. Then $\pi : (X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is p -strong $[a, b]$ -function if and only if for all $y \in Y$ there exists p -open set U_y containing y such that $X \times \overline{U_y}^{\sigma_i}$ is $p - [a, b]$ compact for $i = 1, 2$.

Proof. $(\Rightarrow)$ Let $X = (X, \tau_1, \tau_2)$ and $Y = (Y, \sigma_1, \sigma_2)$ be $p - [a, b]$ compact spaces and π is a p -strong $[a, b]$ -function, then by Theorem 4.13

$$(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$$

is $p - [a, b]$ -compact, hence by Theorem 2.8 $X \times \overline{U_y}^{\sigma_i}$ is $p - [a, b]$ -compact for $i = 1, 2$.

$(\Leftarrow)$ Let $y \in Y$ and $\mathcal{U}$ be a p -open cover of $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ with $|\mathcal{U}| \leq b$, so $X \times \overline{U_y}^{\sigma_i}$ is contained in $< a$ members of $\mathcal{U}$, hence π is a p -weak $[a, b]$ -function, and by Theorem 4.12 π is a p -strong $[a, b]$ -function. $\square$

Theorem 5.9. Let (X, τ_1, τ_2) be a $[\omega_0, \infty]$ -compact space and (Y, σ_1, σ_2) be a T_1 -space, then the following are equivalent :

(i) The projection map $\pi : (X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is p -perfect,

(ii) The projection map $\pi : (X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is s -weak $[\omega_0, \infty]$,

(iii) X is $s - [\omega_0, \infty]$ -compact.

Proof. (i) $\Rightarrow$ (ii) Let π be a p -perfect function, since X is $[\omega_0, \infty]$ -compact and $\pi^{-1}y$ is p -compact, then by Theorem 2.6 $\pi^{-1}y$ is $s - [\omega_0, \infty]$ -compact. Finally the result comes from Theorem 4.17.

(ii) $\Rightarrow$ (iii) From corollary 4.19.

(iii) $\Rightarrow$ (i) Since X is $[\omega_0, \infty]$ -compact space, then π is a p -closed function. Also $\pi^{-1}y$ is a $s - [\omega_0, \infty]$ -compact and hence it is $p - [\omega_0, \infty]$ -compact space, therefore π is p -perfect function. $\square$

Lemma 5.10. Let (X, τ_1, τ_2) be a $s - [a, \infty]$ -compact space, and (Y, σ_1, σ_2) is P_a -space. Then the projection $\pi : (X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is p -closed, and hence it is s -weak $[a, \infty]$ -function.

Proof. It is clear that π is p -closed. and the result comes from Theorem 4.17. $\square$

Theorem 5.11. Let (X, τ_1, τ_2) be a $[a, \infty]$ -compact and (Y, σ_1, σ_2) is a T_1 -space. Then $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is $s - [a, \infty]$ -compact if and only if Y is P_a and $s - [a, \infty]$ -compact space.

Proof. ($\Rightarrow$) From Theorem 4.1.

($\Leftarrow$) Let $\pi : (X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2) \rightarrow (Y, \sigma_1, \sigma_2)$. Then by lemma 5.10 π is s -weak $[a, \infty]$ -function. hence the result comes from Theorem 4.16. $\square$

Corollary 5.12. Let (X, τ_1, τ_2) be a $s - [a, \infty]$ -compact and (Y, σ_1, σ_2) is a T_1 -space. Then $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is $s - [a, \infty]$ -compact if and only if Y is P_a and $s - [a, \infty]$ -compact space.

Theorem 5.13. Let (Y, σ_1, σ_2) be a T_1 -space such that each point in Y has σ_1 or σ_2 -open base with cardinality $\leq b$. Then (X, τ_1, τ_2) is $s - [\omega_0, b]$ -compact if and only if $\pi : (X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is s -weak $[\omega_0, \infty]$ -function.

Proof. ($\Rightarrow$) let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be a $(\tau_1 \times \sigma_1)(\tau_2 \times \sigma_2)$ -open cover of $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$. For $y \in Y$ there exists a $\sigma_1\sigma_2$ -base $\{V_i(y) | i \in \Delta\}$ of y , with $|\Delta| \leq b$.

Let $x \in X$, there exists a set $O_i(x, U) \subseteq \tau_1 \cup \tau_2$ containing x such that

$$(x, y) \in O_i(x, U) \times V_i(y) \subseteq U$$

for some $U \in \mathcal{U}$ and $i \in \Delta$.

Define

$$O_i(U) = \bigcup \{O_i(x, U) | O_i(x, U) \times V_i(y) \subseteq U\}$$

and

$$\mathcal{O} = \{O_i(U) | U \in \mathcal{U}, i \in \Delta\},$$

then $\mathcal{O}$ is a $\tau_1\tau_2$ -open cover of X , but X is $s - [\omega_0, b]$ -compact, therefore $\mathcal{O}$ has a subcover

$$\mathcal{O}^* = \{O_i^*(U) | U \in \mathcal{U}, i \in \Delta^*, |\mathcal{O}^*| < \omega_0\},$$

and

$$\pi^{-1} \left(\bigcap_{i \in \Delta^*} V_i(y) \right) \subseteq \bigcup_{i \in \Delta^*} (O_i^*(U) \times V_i(y)) \subseteq \bigcup_{i \in \Delta^*} U_i.$$

Let

$$V_i = \bigcap_{i \in \Delta^*} V_i(y)$$

and

$$\mathcal{V} = \{V_i | i \in \Delta^*\}$$

then $\mathcal{V}$ is a $\sigma_1\sigma_2$ -open cover of Y with cardinality $< \omega_0$, hence π is s -weak $[\omega_0, \infty]$ -function.

($\Leftarrow$) straightforward. □

Corollary 5.14. *Let (Y, σ_1, σ_2) is T_1 -space such that each point in Y has τ_1 or τ_2 -open base with cardinality $\leq b$ and (X, τ_1, τ_2) is $s - [\omega_0, b]$ -compact space. Then $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is $s - [\omega_0, b]$ -compact if and only if Y is $s - [\omega_0, \infty]$ -compact.*

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