ON w_τ -GENERALIZED CLOSED SETS IN ASSOCIATED w -SPACES WITH TOPOLOGIES

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Abstract: The purpose of this paper is to introduce the notion of w_τ -generalized closed (simply, $w_\tau g$ -closed) sets in an associated w_τ -space with a topology, and to study its properties. In particular, we find the following results: (1) If f is W -continuous and closed, then for every $w_\tau g$ -open subset B in Y , $f^{-1}(B)$ is $w_\tau g$ -open: (2) If f is continuous and quasi- W^* -closed, for every $w_\tau g$ -closed set A in X , $f(A)$ is $w_\tau g$ -closed.

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1. Introduction

Siwiec [13] introduced the notions of weak neighborhoods and weak base in a topological space. We introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in [8]. The weak neighborhood system induces a weak neighborhood space which is independent of neighborhood spaces [2] and general topological spaces [1]. The notions of weak structure, w -space, W -continuity and W -continuity were investigated in [9]. In fact, the set of all g -closed subsets [3] in a topological space is a kind of weak structure. In

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[10], we introduced the notion of an associated weak space (simply, associated w_τ -space) containing a given topology τ , and moreover, we studied the notion of generalized w_τ -closed sets (simply, gw_τ -closed sets) [11] in an associated w_τ -space with a topology τ . Now, the one purpose of this research is to study w_τ -generalized closed sets in an associated weak space w_τ in the similar way introduced by Levine [3] in topological spaces. So we introduce the notion of w_τ -generalized closed (simply, $w_\tau g$ -closed) sets in an associated w_τ -space with a topology, and to study its properties. In particular, we find the following results: (1) If f is W -continuous and closed, then for every $w_\tau g$ -open subset B in Y , $f^{-1}(B)$ is $w_\tau g$ -open: (2) if f is continuous and quasi- W -closed, for every $w_\tau g$ -closed set A in X , $f(A)$ is $w_\tau g$ -closed.

2. Preliminaries

Definition 2.1 ([9]). Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* on X if it satisfies the following:

- (1) $\emptyset \in w_X$ and $X \in w_X$.
- (2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a *w-space* on X . Then $V \in w_X$ is called a *w-open* set and the complement of a *w-open* set is a *w-closed* set.

The collection of all *w-open* sets (resp., *w-closed* sets) in a *w-space* X will be denoted by $WO(X)$ (resp., $WC(X)$). We set $W(x) = \{U \in WO(X) : x \in U\}$.

Let S be a subset of a topological space X . The closure (resp., interior) of S will be denoted by clS (resp., $intS$). A subset S of X is called a *preopen* set [6] (resp., α -open set [12], *semi-open* [4]) if $S \subset int(cl(S))$ (resp., $S \subset int(cl(int(S)))$, $S \subset cl(int(S))$). The complement of a preopen set (resp., α -open set, *semi-open*) is called a *preclosed* set (resp., α -closed set, *semi-closed*). The family of all preopen sets (resp., α -open sets, semi-open sets) in X will be denoted by $PO(X)$ (resp., $\alpha(X)$, $SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on X .

Moreover, a subset S of X is said to be *g-closed* [3] if $cl(A) \subset U$ whenever $A \subset U$ and U is open in X .

Then the family $GO(X) = \{U \subseteq X : U \text{ is } g\text{-open}\}$, $O(X) = \{U \subseteq X : U \text{ is open}\}$ and $CL(X) = \{F \subseteq X : F \text{ is closed}\}$ are all weak structures on X . But $PO(X)$, $GPO(X)$ and $SO(X)$ are not weak structures on X . A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal*

structure on X [5] if $\emptyset \in w_X$ and $X \in w_X$. Thus clearly every weak structure is a minimal structure.

Let (X, w_X) be a w -space. For a subset A of X , the w -closure of A and the w -interior of A are defined as follows:

- (1) $wC(A) = \cap\{F : A \subseteq F, X - F \in w_X\}$.
- (2) $wI(A) = \cup\{U : U \subseteq A, U \in w_X\}$.

Theorem 2.2 ([9]). *Let (X, w_X) be a w -space and $A \subseteq X$. Then the following things hold:*

- (1) *If $A \subseteq B$, then $wI(A) \subseteq wI(B)$; $wC(A) \subseteq wC(B)$.*
- (2) *$wI(wI(A)) = wI(A)$; $wC(wC(A)) = wC(A)$.*
- (3) *$wC(X - A) = X - wI(A)$; $wI(X - A) = X - wC(A)$.*
- (4) *If A is w -closed (resp., w -open), then $wC(A) = A$ (resp., $wI(A) = A$).*

Let X be a nonempty set and let (X, τ) be a topological space. A subfamily w of the power set $P(X)$ is called an associated weak structure (simply, w_τ) [10] on X if $\tau \subseteq w$ and w is a weak structure. Then the pair (X, w_τ) is called an associated w -space with τ .

3. Main Results

Definition 3.1. Let (X, w_τ) be an associated w -space with a topology τ and $A \subseteq X$. Then A is called a w_τ -generalized closed set (simply, $w_\tau g$ -closed set) if $wC(A) \subseteq U$, whenever $A \subseteq U$ and U is open.

We recall that: A is called a generalized closed set (simply, g -closed set) [3] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is open.

Remark 3.2. (1) If $w_\tau = \tau$, then the w_τ -generalized closed set is exactly a g -closed set.

(2) If w_τ is the family of all g -open sets in sense of Levine, then the generalized w_τ -closed set is exactly a g -closed set [14].

Theorem 3.3. *Every g -closed set is w_τ -generalized closed.*

Proof. Since $wC(A) \subseteq cl(A)$, every gw_τ -closed set is $w_\tau g$ -closed. □

In general, the converse of the above theorem is not true as the next example.

Example 3.4. Let $X = \{a, b, c, d\}$, a topology $\tau = \{\emptyset, \{b\}, \{b, c\}, X\}$ and a w -structure $w_X = \{\emptyset, \{a\}, \{b\}, \{b, c\}, \{a, d\}, X\}$ in X . For $A = \{c\}$, since $wC(A) = A$ and $cl(A) = \{a, c, d\}$, obviously A is $w_\tau g$ -closed but not g -closed.

Let (X, w_τ) be an associated w -space with a topology τ and $A \subseteq X$. Then A is called a *generalized w_τ -closed set* (simply, gw_τ -closed set) [11] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is w -open. Then since $\tau \subseteq w$, every gw_τ -closed set is $w_\tau g$ -closed. But, the converse may not be true as the next example.

Example 3.5. In Example 3.3, let $A = \{a\}$. Then $wC(A) = \{a, d\}$, and clearly A is $w_\tau g$ -closed. Now, consider a w -open set $U = \{a\}$ such that $A \subseteq U$. Then A is not gw_τ -closed.

For a given associated w_τ -space with a topology τ , from the above theorems and examples, the following relations are obtained:

Lemma 3.6 ([9]). Let (X, w_τ) be an associated w -space with a topology τ and $A, B \subseteq X$. Then the following things hold:

- (1) $wI(A) \cap wI(B) = wI(A \cap B)$.
- (2) $wC(A) \cup wC(B) = wC(A \cup B)$.

Theorem 3.7. Let (X, w_τ) be an associated w -space with a topology τ . Then the union of two $w_\tau g$ -closed sets is a $w_\tau g$ -closed set.

Proof. Let A and B be any two $w_\tau g$ -closed sets. Let G be any open set such that $A \cup B \subseteq G$. Then $A \subseteq G$ and $B \subseteq G$. Since A and B are $w_\tau g$ -closed sets, $wC(A) \subseteq G$ and $wC(B) \subseteq G$. Now, by the above lemma, $wC(A \cup B) \subseteq G$, and so $A \cup B$ is $w_\tau g$ -closed. $\square$

In general, the intersection of two $w_\tau g$ -closed sets is not $w_\tau g$ -closed:

Example 3.8. Let $X = \{a, b, c, d\}$, a topology $\tau = \{\emptyset, \{a, c\}, X\}$ and a w -structure $w_X = \{\emptyset, \{a, c\}, \{a\}, \{b\}, \{c\}, \{a, d\}, X\}$ in X . Now, consider $A = \{a, b, c\}$ and $B = \{a, c, d\}$. Then A and B are $w_\tau g$ -closed. But for $A \cap B = \{a, c\}$, since $\{a, c\}$ is open and $wC(\{a, c\}) = \{a, c, d\}$, the intersection is not $w_\tau g$ -closed.

Theorem 3.9. *Let (X, w_τ) be an associated w -space with a topology τ . Then if A is a $w_\tau g$ -closed set, then $wC(A) - A$ contains no non-empty closed set.*

Proof. Let F be a closed set such that $F \subseteq wC(A) - A$. Then $A \subseteq X - F$. Since A is $w_\tau g$ -closed, $wC(A) \subseteq X - F$. It implies that $F \subseteq wC(A) \cap (X - wC(A)) = \emptyset$. So, $F = \emptyset$. $\square$

In general, the converse in Theorem 3.9 is not true as shown in the next example.

Example 3.10. Let $X = \{a, b, c, d\}$, a topology $\tau = \{\emptyset, \{a\}, \{a, d\}, X\}$ and $w_X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, d\}, X\}$ be a w -structure in X . Consider $A = \{a\}$. Note $wC(A) = \{a, d\}$ and $wC(A) - A = \{a, d\} - \{a\} = \{d\}$. Since $\{d\}$ is not closed, $wC(A) - A$ contains no non-empty closed set, but A is not $w_\tau g$ -closed.

Theorem 3.11. *Let (X, w_τ) be an associated w -space with a topology τ . Then if A is a $w_\tau g$ -closed set and $A \subseteq B \subseteq wC(A)$, then B is $w_\tau g$ -closed.*

Proof. Let U be any open set such that $B \subseteq U$. Since A is a $w_\tau g$ -closed set and $wC(B) \subseteq wC(A)$, $wC(B) \subseteq U$, and so B is $w_\tau g$ -closed set. $\square$

Definition 3.12. Let (X, w_τ) be an associated w -space with a topology τ and $A \subseteq X$. Then A is called a w_τ -generalized open set (simply, $w_\tau g$ -open set) if $X - A$ is $w_\tau g$ -closed.

Theorem 3.13. *Let (X, w_τ) be an associated w -space with a topology τ and $A \subseteq X$. Then A is $w_\tau g$ -open if and only if $F \subseteq wI(A)$ whenever $F \subseteq A$ and F is closed.*

Proof. From Definition 3.1, it is obvious. $\square$

Theorem 3.14. *Let (X, w_τ) be an associated w -space with a topology τ . Then the intersection of two $w_\tau g$ -open sets is a $w_\tau g$ -open set.*

Proof. It is obvious from Theorem 3.7 and Theorem 2.2. $\square$

In general, the union of two $w_\tau g$ -open sets is not $w_\tau g$ -open (See Example 3.8).

Theorem 3.15. *Let (X, w_τ) be an associated w -space with a topology τ and $A \subseteq X$. Then if A is $w_\tau g$ -open, then $U = X$, whenever $wI(A) \cup (X - A) \subseteq U$ and U is open.*

Proof. For $w_\tau g$ -open set A , let U be any open set satisfying $wI(A) \cup (X - A) \subseteq U$. Then $X - U \subseteq (X - wI(A)) \cap A = wC(X - A) \cap A = wC(X - A) - (X - A)$. Since $X - A$ is $w_\tau g$ -closed, by Theorem 3.9, $X - U$ must be empty. Thus, $U = X$. $\square$

Theorem 3.16. *Let (X, w_τ) be an associated w -space with a topology τ . Then if A is a $w_\tau g$ -open set and $wI(A) \subseteq B \subseteq A$, then B is $w_\tau g$ -open.*

Proof. It is similar to the proof of Theorem 3.11. $\square$

Theorem 3.17. *Let (X, w_τ) be an associated w -space with a topology τ . Then if A is a $w_\tau g$ -closed set, then $wC(A) - A$ is $w_\tau g$ -open.*

Proof. If A is a $w_\tau g$ -closed set, then by Theorem 3.11, the empty set is the only one closed subset of $wC(A) - A$. So, for the only closed subset $\emptyset$ of $wC(A) - A$, $\emptyset \subseteq wC(A) - A$ and $\emptyset \subseteq wI(wC(A) - A)$. Hence, $wC(A) - A$ is $w_\tau g$ -open. $\square$

Theorem 3.18. *Let (X, w_τ) be an associated w -space with a topology τ . Then if A is a $w_\tau g$ -open set, then $wI(A) \cup (X - A)$ is $w_\tau g$ -closed.*

Proof. If A is a $w_\tau g$ -open set, then by Theorem 3.15, X is the only one open set containing $wI(A) \cup (X - A)$. So, $wI(A) \cup (X - A) \subseteq X$ and $wC(wI(A) \cup (X - A)) \subseteq X$. Hence, $wI(A) \cup (X - A)$ is $w_\tau g$ -closed. $\square$

Let (X, w_τ) be an associated w -space with a topology τ . For a subset A of X , $w_\tau g$ -closure of A and $w_\tau g$ -interior of A are defined as the following:

- (1) $w_\tau gC(A) = \cap \{F : A \subseteq F, F \text{ is } w_\tau g\text{-closed}\}.$
- (2) $w_\tau gI(A) = \cup \{U : U \subseteq A, U \text{ is } w_\tau g\text{-open}\}.$

Theorem 3.19. *Let (X, w_τ) be an associated w -space with a topology τ and $A \subseteq X$.*

- (1) *If A is $w_\tau g$ -open, then $w_\tau gI(A) = A$.*
- (2) *If A is $w_\tau g$ -closed, then $w_\tau gC(A) = A$.*

Proof. Obvious. $\square$

But the converses in the above theorem are not always true as shown in the next example.

Example 3.20. In Example 3.8, let $F = \{a, c\}$. Since $\{a, b, c\}$ and $\{a, c, d\}$ are $w_\tau g$ -closed sets, $w_\tau gC(F) = \{a, c\}$. But F is not $w_\tau g$ -closed. Similarly, we can show that the converse of (2) in Theorem 3.19 may not be true.

The following things are easily obtained by the notions of $w_\tau g$ -closure and $w_\tau g$ -interior:

Theorem 3.21. Let (X, w_τ) be an associated w -space with a topology τ and $A, B \subseteq X$.

- (1) If $A \subseteq B$, then $w_\tau gI(A) \subseteq w_\tau gI(B)$ and $w_\tau gC(A) \subseteq w_\tau gC(B)$.
- (2) $w_\tau gC(X - A) = X - w_\tau gI(A)$; $w_\tau gI(X - A) = X - w_\tau gC(A)$.
- (3) $x \in w_\tau gI(A)$ if and only if there exists a $w_\tau g$ -open set U containing x such that $U \subseteq A$.
- (4) $x \in w_\tau gC(A)$ if and only if $A \cap V \neq \emptyset$ for all $w_\tau g$ -open set V containing x .

Theorem 3.22. Let (X, w_τ) be an associated w -space with a topology τ and $A, B \subset X$.

- (1) $\emptyset = w_\tau gC(\emptyset)$.
- (2) $A \subseteq w_\tau gC(A)$.
- (3) $w_\tau gC(A \cup B) = w_\tau gC(A) \cup w_\tau gC(B)$.
- (4) $w_\tau gC(w_\tau gC(A)) = w_\tau gC(A)$.

Proof. (1) and (2) are obvious.

(3) Now, we only show that $w_\tau gC(A \cup B) \subseteq w_\tau gC(A) \cup w_\tau gC(B)$. For the proof, let $x \notin w_\tau gC(A) \cup w_\tau gC(B)$. Then there exist $w_\tau g$ -closed sets F_1 and F_2 such that $x \notin F_1$ and $A \subseteq F_1$; $x \notin F_2$ and $B \subseteq F_2$. So $x \notin F_1 \cup F_2$ and $A \cup B \subseteq F_1 \cup F_2$. Since $F_1 \cup F_2$ is $w_\tau g$ -closed, we have that $x \notin w_\tau gC(A \cup B)$. Hence, $w_\tau gC(A \cup B) \subseteq w_\tau gC(A) \cup w_\tau gC(B)$.

(4) It is clear that $w_\tau gC(w_\tau gC(A)) \supseteq w_\tau gC(A)$. Now, we show that $w_\tau gC(w_\tau gC(A)) \subseteq w_\tau gC(A)$. First, for any $w_\tau g$ -closed set F satisfying $A \subseteq F$, we know that $w_\tau gC(A) \subseteq w_\tau gC(F) = F$. So, from this fact, $\{F : A \subseteq F, F \text{ is } w_\tau g\text{-closed}\} \subseteq \{K : w_\tau gC(A) \subseteq K, K \text{ is } w_\tau g\text{-closed}\}$. This implies that $w_\tau gC(w_\tau gC(A)) = \cap \{K : w_\tau gC(A) \subseteq K, K \text{ is } w_\tau g\text{-closed}\} \subseteq \cap \{F : A \subseteq F, F \text{ is } w_\tau g\text{-closed}\} = w_\tau gC(A)$. $\square$

From Theorem 3.21 and Theorem 3.22, the following are obtained:

Theorem 3.23. Let (X, w_τ) be an associated w -space with a topology τ and $A, B \subseteq X$.

- (1) $X = w_\tau gI(X)$.
- (2) $w_\tau gI(A) \subseteq A$.
- (3) $w_\tau gI(A \cap B) = w_\tau gI(A) \cap w_\tau gI(B)$.
- (4) $w_\tau gI(w_\tau gI(A)) = w_\tau gI(A)$.

Finally, from Theorem 3.3 and Theorem 3.23, we have a topology induced by $w_\tau g$ -open sets as the following:

Theorem 3.24. Let (X, w_τ) be an associated w -space with a topology τ . Then the family $\tau = \{U \subseteq X : U = w_\tau gI(U)\}$ is a topology such that $\tau \subseteq w_\tau \subseteq \tau$ and $GO(X) \subseteq \tau$.

Recall the notion of W -continuity: Let (X, w_X) and (Y, w_Y) be two w -spaces. Then $f : X \rightarrow Y$ is said to be:

- (1) W -continuous [9] if for $x \in X$ and $V \in W(f(x))$, there is $U \in W(x)$ such that $f(U) \subseteq V$;
- (2) W -closed [11] if for every w -closed set F in X , $f(F)$ is a w -closed set in Y .
- (3) quasi- W -closed [11] if for $A \subseteq X$, $wC(f(A)) \subseteq f(wC(A))$.

In fact, there is no any relation between the notions of W -closed function is quasi- W -closed function.

Theorem 3.25 ([9]). Let $f : (X, w_X) \rightarrow (Y, w_Y)$ be a function in two w -spaces. Then the following statements are equivalent:

- (1) f is W -continuous.
- (2) $f(wC(A)) \subseteq wC(f(A))$ for $A \subseteq X$.
- (3) $wC(f^{-1}(B)) \subseteq f^{-1}(wC(B))$ for $B \subseteq Y$.
- (4) $f^{-1}(wI(B)) \subseteq wI(f^{-1}(B))$ for $B \subseteq Y$.

Theorem 3.26. Let $f : (X, w_{\tau_X}) \rightarrow (Y, w_{\tau_Y})$ be a function on associated w -spaces X and Y . Then if f is W -continuous and closed, then for every $w_\tau g$ -open subset B in Y , $f^{-1}(B)$ is $w_\tau g$ -open.

Proof. Let B be any $w_\tau g$ -open subset in Y and F be a closed set in X such that $F \subseteq f^{-1}(B)$. Now, we show that $F \subseteq wI(f^{-1}(B))$. Since B is $w_\tau g$ -open and $f(F)$ is closed, $f(F) \subseteq wI(B)$. From (4) of Theorem 3.25, it follows that $F \subseteq f^{-1}(wI(B)) \subseteq wI(f^{-1}(B))$. Hence, $f^{-1}(B)$ is $w_\tau g$ -open. $\square$

Theorem 3.27. Let $f : (X, w_{\tau_X}) \rightarrow (Y, w_{\tau_Y})$ be a function on associated w -spaces X and Y . Then if f is continuous and quasi- W -closed, for every $w_\tau g$ -closed set A in X , $f(A)$ is $w_\tau g$ -closed.

Proof. For a $w_\tau g$ -closed subset A in X , let U be any open set in Y such that $f(A) \subseteq U$. Now, we show that $wC(f(A)) \subseteq U$. Since f is continuous and A is $w_\tau g$ -closed, $f^{-1}(U)$ is open and $wC(A) \subseteq f^{-1}(U)$. Since f is quasi- W -closed, $wC(f(A)) \subseteq f(wC(A)) \subseteq f f^{-1}(U) \subseteq U$. Hence, $f(A)$ is $w_\tau g$ -closed. $\square$

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