IDEAL SEMIRINGS AND THE IDEAL EXTENSION PROPERTY FOR SEMIRINGS

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Abstract: Suppose that $R$ is a semiring. In this paper, we define the ideal extension property, ideal semiring, the congruence extension property for semiring $R$ and prove that $R$ has the ideal extension property if and only if every subsemiring of $R$ has the ideal extension property. We prove that if $R$ has the ideal extension property then the homomorphic image of $R$ has the ideal extension property. Also, we show that if $R$ is an ideal semiring then the homomorphic image of $R$ is an ideal semiring and it is proved that each ideal semiring with the congruence extension property has the ideal extension property.

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1. Introduction

Definition 1.1. Let $R \neq \emptyset$ be a set with $+$ and $.$ as binary operations on $R$, named addition and multiplication, respectively. Then $(R, +, .)$ is called a semiring if the following conditions are satisfied:
1 \((R, +)\) is a commutative semigroup;
2 \((R, \cdot)\) is a semigroup;
3 Both operations are connected by the distributive laws \(a(b + c) = a.b + a.c\) and \((a + b).c = a.c + b.c\) for all \(a, b, c \in R\).

**Definition 1.2.** A subset \(H\) of a semiring \(R\) is called a subsemiring provided that \(H\) is a semiring under both binary operations on \(R\).

**Definition 1.3.** Let \(R\) be a semiring. An equivalence relation \(\rho\) on the semiring \(R\) is called congruence if
\[
(\forall s, t, a \in R)(s, t) \in \rho \implies (a + s, a + t) \in \rho
\]
and
\[
(\forall s, t, a \in R)(s, t) \in \rho \implies (a.s, a.t) \in \rho
\]
and \((s.a, t.a) \in \rho\).

**Definition 1.4.** For a semiring \((R, +, \cdot)\) and a non-empty subset \(A\) of \(R\), we define the following subset of \(R\):
\[
\langle A \rangle = \left\{ \sum_{v=1}^{n} a_v : n \in \mathbb{N}, a_v \in A \right\}
\]
The subset \(\langle A \rangle\) is called the subset of \(R\) generated by \(A\).

**Definition 1.5.** A non-empty subset \(I\) of a semiring \(R\) will be called an ideal if \(a, b \in I\) and \(r \in R\) imply that \(a + b \in I\) and \(ra \in I\) and \(ar \in I\).

2. Ideal Semirings and Ideal Extension Property for Semirings

**Definition 2.1.** Let \(R\) be a semiring. \(R\) has the ideal extension property provided that for each subsemiring \(H\) of \(R\) and each ideal \(J\) of \(H\) there exists an ideal \(K\) of \(R\) such that \(K \cap H = J\).

**Theorem 2.2.** Let \(R\) be a semiring. Then \(R\) has the ideal extension property if and only if every subsemiring of \(R\) has the ideal extension property.
Proof. Let every subsemiring of \( R \) has the ideal extension property. Since \( R \) is a subsemiring of itself, \( R \) has the ideal extension property. Now, Suppose that \( R \) has the ideal extension property. Let \( H \) be a subsemiring of \( R \) and \( L \) be a subsemiring of \( H \) and \( J \) be an ideal of \( L \). Since \( L \) is also a subsemiring of \( R \) and \( R \) has the ideal extension property, there exists an ideal \( K \) of \( R \) such that \( K \cap L = J, K \cap H \) is an ideal of \( H \) and we have \( K \cap HL = K \cap L = J \), it follows that \( H \) has the ideal extension property.

Example 2.3. Let \( R = \{a, b, c\} \). Then \( R \) with addition and multiplication defined by the following Cayley tables is a semiring.

\[
\begin{array}{ccc}
+ & a & b & c \\
\hline
a & a & b & c \\
b & b & b & c \\
c & c & c & c \\
\end{array}
\]

Let \( R_1 = \{a\}, R_2 = \{b\}, R_3 = \{c\}, R_4 = \{a, b\}, R_5 = \{a, c\}, R_6 = \{b, c\} \) and \( R_7 = \{a, b, c\} \). Then \( R_1, R_2, ..., R_7 \) are subsemirings of \( R \). Let \( I_1 = \{a\}, I_2 = \{b\}, I_3 = \{c\}, I_5 = \{c\}, I_6 = \{c\} \) and \( I_7 = \{a, b, c\} \). Then \( I_1 \) is the only ideal of \( R_1 \), \( I_2 \) is the only ideal of \( R_2 \), \( I_3 \) is the only ideal of \( R_3 \), \( R_4 \) has not any ideal, \( I_5 \) is the only ideal of \( R_5 \), \( I_6 \) is the only ideal of \( R_6 \) and \( I_7 \) is the only ideal of \( R_7 \). Let \( J_1 = \{c\}, J_2 = \{a, b, c\} \). Then \( J_1 \), \( J_2 \) and \( J_3 \) are ideals of \( R \) and we have \( J_2 \cap R_1 = I_1, J_2 \cap R_2 = I_2, J_2 \cap R_3 = I_3, J_1 \cap R_5 = I_5, J_1 \cap R_6 = I_6, J_2 \cap R_7 = I_7 \). It follows that \( R \) has the ideal extension property.

Example 2.4. Let \( R = (\mathbb{Z}, +, \cdot) \) be the semiring of all integer number under usual addition and multiplication. Let \( H = \{0, 1, 2, 3, \ldots\} \) and \( J = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\} = H - \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, \ldots\} \). Then \( H \) is a subsemiring of \( R \) and \( J \) is an ideal of \( H \). Suppose that there exists an ideal \( K \) of \( R \) such that \( K \cap H = J \). Then there exists \( m \geq 0 \) in \( \mathbb{Z} \) such that \( K = m\mathbb{Z} \). It follows that \( K \cap H = mH \), but \( J \neq mH \). This contradiction establishes that \( R \) does not have the ideal extension property.
Corollary 2.5. In view of 2.2 and 2.4, the rational numbers $\mathbb{Q}$ does not have the ideal extension property.

Theorem 2.6. Let $R$ be a semiring if $R$ has the ideal extension property then any homomorphic image of $R$ has the ideal extension property.

Proof. Let $R$ be a semiring with the ideal extension property. Let $\Psi : R \rightarrow R^*$ be a homomorphism of semiring $R$ onto a semiring $R^*$. Suppose that $H^*$ be a subsemiring of $R^*$ and $J^*$ be an ideal of $H^*$. Let $H = \Psi^{-1}(H^*)$ and let $J = \Psi^{-1}(J^*)$. Then $H$ is a subsemiring of $R$ and $J$ is an ideal of $H$. By hypothesis, there exists an ideal $K$ of $R$ such that $K \cap H = J$. Let $K^* = \psi(K)$. Then since $\Psi$ is onto, we have that $K^*$ is an ideal of $R^*$.

Now, we show that $K^* \cap H^* = J^*$. Let $x \in K^* \cap H^*$. Then $\Psi^{-1}(x) \subseteq K$ and $\Psi^{-1}(x) \subseteq H$. Therefore, $\Psi^{-1}(x) \subseteq K \cap H = J$. Then $x \in \Psi(J) = J^*$.

To establish the other inclusion let $a \in J^*$. Then $\Psi^{-1}(a) \subseteq J = K \cap H$. Therefore, $a \in \Psi(K \cap H) \subseteq \Psi(K \cap \Psi H) = K^* \cap H^*$. It follows that $K^* \cap H^* = J^*$. We conclude that $R^*$ has the ideal extension property.

Definition 2.7. Let $R$ be a semiring and let $\rho$ be a congruence on $R$. Then $\rho$ is called an ideal congruence provided that there exists an ideal $J$ of $R$ such that $\rho = (J \times J) \cup \Delta_R$ (where $\Delta_R$ denotes the diagonal relation on $R$).

Definition 2.8. A semiring $R$ is said to be an ideal semiring provided that each congruence on $R$ is an ideal congruence.

Theorem 2.9. Let $R$ be a semiring. If $R$ is an ideal semiring then the homomorphic image of $R$ is an ideal semiring.

Proof. Suppose that $R$ is an ideal semiring. Let $\Psi : R \rightarrow S$ be a homomorphism of $R$ onto a semiring $S$ and let $\rho$ be a congruence on $S$. Define $\alpha = \{(a, b) \in (R \times R) : (\Psi(a), \Psi(b)) \in \rho\}$. Then $\alpha$ is a congruence on $R$. By assumption, $R$ is an ideal semiring. Consequently there exists an ideal $J$ of $R$ such that $\alpha = (J \times J) \cup \Delta_R$. Let $K = \Psi(J)$. Then $K$ is an ideal of $S$.

We claim that $\rho = (K \times K) \cup \Delta_S$. Assume that $(\Psi(a), \Psi(b)) \in \rho$. Therefore $(a, b) \in \alpha$. If $a \neq b$, then $(a, b) \in (J \times J)$ and $(\Psi(a), \Psi(b)) \in (K \times K)$. If $a = b$ then $(\Psi(a), \Psi(b)) \in \Delta_S$. Thus $\rho \subseteq (K \times K) \cup \Delta_S$.

Let $(s, t) \in (K \times K)$. Then there exists $(a, b) \in (J \times J)$ such that $s = \Psi(a)$ and $t = \Psi(b)$. Then $(a, b) \in \alpha$ and we have $(s, t) \in \rho$. It follows that $\rho = (K \times K) \cup \Delta_S$ and $S$ is an ideal semiring.
3. Congruence Extension Property for Semirings

Definition 3.1. Let $R$ be a semiring. $R$ has the congruence extension property provided that for each subsemiring $H$ of $R$ and each congruence $\rho$ on $H$ there exists a congruence $\alpha$ on $R$ such that $\alpha \cap (H \times H) = \rho$. The congruence $\alpha$ is called an extension of $\rho$.

Theorem 3.2. Let $H$ be a subsemiring of a semiring $R$ and $\rho$ be a congruence on $H$ and $\langle \rho \rangle$ be a congruence on $R$ generated by $\rho$. Then $\rho$ has an extension to $R$ if and only if $\langle \rho \rangle$ is an extension.

Proof. If $\langle \rho \rangle$ is an extension of congruence $\rho$ to $R$, it is clear that $\rho$ has an extension to $R$. Let $\alpha$ be an extension of congruence $\rho$ to $R$. We must establish that $\rho = \langle \rho \rangle \cap (H \times H)$. Since $\rho \subset \langle \rho \rangle$ and $\rho \subset (H \times H)$ it follows that $\rho \subset \langle \rho \rangle \cap (H \times H)$. Since $\alpha$ is an extension of congruence $\rho$ to $R$, it follows that $\alpha \cap (H \times H) = \rho \subset (\rho) \cap (H \times H)$.

We claim that $\langle \rho \rangle \subset \alpha$. Let there exists $x \in (R \times R)$ such that $x \in \langle \rho \rangle$ and $x \notin \alpha$. Then we have $x = \sum_{v=1}^{n} a_v$ for some $n \in \mathbb{N}$, where $a_v \in \rho$, $1 \leq v \leq n$. Since $\alpha \cap (H \times H) = \rho$, $a_v \in \alpha$, $1 \leq v \leq n$ and so $x = \sum_{v=1}^{n} a_v \in \alpha$. This contradiction implies that $\langle \rho \rangle \subset \alpha$ and it follows that $\langle \rho \rangle \cap (H \times H) \subset \alpha \cap (H \times H) = \rho$. We conclude that $\rho = \langle \rho \rangle \cap (H \times H)$. Therefore, to establish the existence of an extension it suffices to show that $\langle \rho \rangle \cap (H \times H) \subset \rho$.

Theorem 3.3. Let $R$ be an ideal semiring. If $R$ has the congruence extension property, then $R$ has the ideal extension property.

Proof. Let $R$ be an ideal semiring which has the congruence extension property, let $H$ be a subsemiring of $R$ and $J$ be an ideal of $H$. Then $\rho = (J \times J) \cup \Delta_H$ is a congruence on $H$. Since $R$ has the congruence extension property, there exists an extension $\langle \rho \rangle$ of $\rho$ to $R$. Since $R$ is an ideal semiring there exists an ideal $K$ of $R$ such that $\langle \rho \rangle = (K \times K) \cup \Delta_R$. Since $\rho = \langle \rho \rangle \cap (H \times H)$, we conclude that $J = K \cap H$. Therefore $R$ has the ideal extension property.

References


