SOME FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS OF TYPES IN MULTIPLICATIVE METRIC SPACES

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Abstract: In this paper, we proved the common fixed point theorems for compatible mappings of types in multiplicative metric spaces.

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1. Introduction and Preliminaries

It is well know that the set of positive real numbers $\mathbb{R}_+$ is not complete according to the usual metric. To overcome this problem, in 2008, Bashirov et al. [1] introduced the concept of multiplicative metric spaces as follows:

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Definition 1.1. Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \rightarrow \mathbb{R}_+$ satisfying the following conditions:

- (i) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) $d(x, y) \leq d(x, z) \cdot d(z, y)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

Then the mapping d together with X , that is, (X, d) is a multiplicative metric space.

Example 1.2. ([6]) Let $\mathbb{R}_+^n$ be the collection of all n -tuples of positive real numbers. Let $d : \mathbb{R}_+^n \times \mathbb{R}_+^n \rightarrow \mathbb{R}$ be defined as follows:

$$d(x, y) = \left| \frac{x_1}{y_1} \right| \cdot \left| \frac{x_2}{y_2} \right| \cdots \left| \frac{x_n}{y_n} \right|,$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}_+^n$ and $|\cdot| : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 1, \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$$

Then it is obvious that all conditions of a multiplicative metric are satisfied. Therefore $(\mathbb{R}_+^n, d)$ is a multiplicative metric space.

One can refer to [6] for detailed a multiplicative metric topology.

Definition 1.3. Let (X, d) be a multiplicative metric space. Then a sequence $\{x_n\}$ in X is said to be

(1) a *multiplicative convergent* to x if for every multiplicative open ball $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$, $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n \geq N$, that is, $d(x_n, x) \rightarrow 1$ as $n \rightarrow \infty$.

(2) a *multiplicative Cauchy sequence* if for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N$, that is, $d(x_n, x_m) \rightarrow 1$ as $n, m \rightarrow \infty$.

(3) We call a multiplicative metric space *complete* if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.

In 2012, Özavsar and Çevikel [6] gave the concept of multiplicative contractive mappings and proved some fixed point theorem of such mappings in a multiplicative metric space.

Definition 1.4. Let f be a mapping of a multiplicative metric space (X, d) into itself. Then f is said to be a *multiplicative contraction* if there exists a real number $\lambda \in [0, 1)$ such that

$$d(fx, fy) \leq d^\lambda(x, y) \quad \text{for all } x, y \in X.$$

In 2015, Kang et al. [2, 3] introduced the notion of compatible mappings and its variants in a multiplicative metric space as follows:

Definition 1.5. Let f and g be two mappings of a multiplicative metric space (X, d) into itself. Then f and g are called

(1) *compatible* if

$$\lim_n d(fgx_n, gfx_n) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$.

(2) *compatible of type (P)* if

$$\lim_n d(ffx_n, ggx_n) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$.

(3) *compatible of type (R)* if

$$\lim_n d(fgx_n, gfx_n) = 1 \quad \text{and} \quad \lim_n d(ffx_n, ggx_n) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$,

(4) *compatible of type (K)* if

$$\lim_n d(ffx_n, gt) = 1 \quad \text{and} \quad \lim_n d(ggx_n, ft) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$,

(5) *compatible of type (E)* if

$$\lim_n ffx_n = \lim_n fgx_n = gt \quad \text{and} \quad \lim_n ggx_n = \lim_n gfx_n = ft,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$.

Remark 1.6. Obviously, compatible mappings of type (R) are also compatible mappings and compatible mappings of type (P).

Definition 1.7. ([3]) Let f and g be mappings of a multiplicative metric space (X, d) . Then f and g are called *reciprocally continuous* if

$$\lim_n ffx_n = ft \quad \text{and} \quad \lim_n gfx_n = ft,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$,

Now we give some properties related to compatible mappings of types in a multiplicative metric space, see [3].

Proposition 1.8. *Let f and g compatible mappings of type (E) of a multiplicative metric space (X, d) itself. Let one of f and g be continuous. Suppose that $\lim_n f x_n = \lim_n g x_n = t$ for some $t \in X$. Then*

- (a) $ft = gt$ and $\lim_n f f x_n = \lim_n g g x_n = \lim_n f g x_n = \lim_n g f x_n$.
 (b) If there exists $u \in X$ such that $fu = gu = t$, then $fgu = gfu$.

2. Main Results

Now we give the following theorem for compatible mappings of type (R) .

Theorem 2.1. *Let A, B, S and T be mappings of a complete multiplicative metric space (X, d) satisfying the following conditions*

$$(C_1) \quad SX \subset BX, \quad TX \subset AX;$$

$$(C_2) \quad d(Sx, Ty) \leq M^\lambda(x, y)$$

for each $x, y \in X$ and $\lambda \in (0, 1/2)$, where

$$\begin{aligned} M(x, y) &= \max \left\{ d(Ax, Sx), d(By, Ty), d(By, Ax), \right. \\ &\quad (d(Ax, Ty) \cdot d(By, Sx))^{1/2}, \min \left\{ \frac{d(Ax, Sx) \cdot d(By, Ty)}{d(Ax, By)}, \right. \\ &\quad \left. \left. \frac{d(Ax, Ty) \cdot d(By, Sx)}{d(Ax, By)}, \frac{d(Ax, Ty) \cdot d(By, Sx)}{d(Sx, Ty)} \right\} \right\}; \end{aligned}$$

(C₃) one of A, B, S and T is continuous.

Assume that the pairs A, S and B, T are compatible of type (R) . Then A, B, S and T have a unique common fixed point.

Proof. It follows from Remark 1.6 and [4, Theorem 2.1] or [5, Theorem 2.4]. $\square$

Next we give the following theorem for compatible mappings of type (K) .

Theorem 2.2. *Let A, B, S and T be mappings of a complete multiplicative metric space (X, d) into itself satisfying the conditions (C_1) and (C_2) .*

Assume that the pairs A, S and B, T are compatible of type (K) , the pairs A, S and B, T are reciprocally continuous. Then A, B, S and T have a unique common fixed point in X .

Proof. Let $x_0 \in X$ be an arbitrary point. Since $SX \subset BX$ and $TX \subset AX$, there exists $x_1 \in X$ such that $Sx_0 = Bx_1 = y_0$ and for this point x_1 , there exists $x_2 \in X$ such that $Tx_1 = Ax_2 = y_1$. Continuing in this way, we can construct a sequence $\{y_n\}$ such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n+2}; \quad y_{2n} = Sx_{2n} = Bx_{2n+1}.$$

From the proof of [4, Theorem 2.2], $\{y_n\}$ is a multiplicative Cauchy sequence in X and hence it converges to some point $z \in X$. Consequently, the subsequence $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Tx_{2n+1}\}$ and $\{Ax_{2n+2}\}$ of sequence $\{y_n\}$ also converges to z .

Since the pairs A, S and B, T are compatible of type (K) on X , we have

$$AAx_{2n} \rightarrow Sz, \quad SSx_{2n} \rightarrow Az, \quad BBx_{2n+1} \rightarrow Tz, \quad TTx_{2n+1} \rightarrow Bz$$

as $n \rightarrow \infty$.

Now putting $x = Sx_{2n}$ and $y = Tx_{2n+1}$ in (C_2) , we have

$$d(SSx_{2n}, TTx_{2n+1}) \leq M^\lambda(Sx_{2n}, Tx_{2n+1}),$$

where

$$\begin{aligned} & M(Sx_{2n}, Tx_{2n+1}) \\ &= \max \left\{ d(ASx_{2n}, SSx_{2n}), d(BTx_{2n+1}, TTx_{2n+1}), d(BTx_{2n+1}, ASx_{2n}), \right. \\ & \quad (d(ASx_{2n}, TTx_{2n+1}) \cdot d(BTx_{2n+1}, SSx_{2n}))^{1/2}, \\ & \quad \min \left\{ \frac{d(ASx_{2n}, SSx_{2n}) \cdot d(BTx_{2n+1}, TTx_{2n+1})}{d(ASx_{2n}, BTx_{2n+1})}, \right. \\ & \quad \frac{d(ASx_{2n}, TTx_{2n+1}) \cdot d(BTx_{2n+1}, SSx_{2n})}{d(ASx_{2n}, BTx_{2n+1})}, \\ & \quad \left. \left. \frac{d(ASx_{2n}, TTx_{2n+1}) \cdot d(BTx_{2n+1}, SSx_{2n})}{d(SSx_{2n}, TTx_{2n+1})} \right\} \right\}. \end{aligned}$$

Letting $n \rightarrow \infty$ and using reciprocal continuity of the pairs A, S and B, T , we

have

$$\begin{aligned}
 & \lim_n M(Sx_{2n}, x_{2n+1}) \\
 &= \max \left\{ d(Az, Az), d(Bz, Bz), d(Bz, Az), d(Bz, Az), \right. \\
 & \quad \left. \min \left\{ \frac{1}{d(Bz, Az)}, d(Az, Bz), d(Az, Bz) \right\} \right\} \\
 &= d(Bz, Az).
 \end{aligned}$$

Hence we have

$$d(Az, Bz) \leq d^\lambda(Bz, Az),$$

which implies that $Az = Bz$.

Next putting $x = z$ and $y = Tx_{2n+1}$ in (C_2) , we have

$$d(Sz, TTx_{2n+1}) \leq M^\lambda(z, Tx_{2n+1}),$$

where

$$\begin{aligned}
 & M(z, Tx_{2n+1}) \\
 &= \max \left\{ d(Bz, Sz), d(BTx_{2n+1}, TTx_{2n+1}), d(BTx_{2n+1}, Az), \right. \\
 & \quad (d(Az, TTx_{2n+1}) \cdot d(BTx_{2n+1}, Sz))^{1/2}, \\
 & \quad \min \left\{ \frac{d(Az, Sz) \cdot d(BTx_{2n+1}, TTx_{2n+1})}{d(Az, BTx_{2n+1})}, \right. \\
 & \quad \frac{d(Az, TTx_{2n+1}) \cdot d(BTx_{2n+1}, Sz)}{d(Az, BTx_{2n+1})}, \\
 & \quad \left. \left. \frac{d(Az, TTx_{2n+1}) \cdot d(BTx_{2n+1}, Sz)}{d(Sz, TTx_{2n+1})} \right\} \right\}.
 \end{aligned}$$

Letting $n \rightarrow \infty$ and using reciprocal continuity of the pairs A, S and B, T , we have

$$\begin{aligned}
 & \lim_n M(z, Tx_{2n+1}) \\
 &= \max \{ d(Bz, Sz), d(Bz, Bz), d(Bz, Bz), d(Bz, Sz), \\
 & \quad \min \{ d(Bz, Sz), d(Bz, Sz), d(Bz, Sz) \} \} \\
 &= d(Bz, Sz).
 \end{aligned}$$

This implies that $Sz = Bz$.

Next putting $x = z$ and $y = z$ in inequality (C_2) , we have

$$d(Sz, Tz) \leq M^\lambda(z, z),$$

where

$$\begin{aligned}
 M(z, z) &= \max \left\{ d(Az, Sz), d(Bz, Tz), d(Bz, Az), \right. \\
 &\quad (d(Az, Tz) \cdot d(Bz, Sz))^{1/2}, \min \left\{ \frac{d(Az, Sz) \cdot d(Bz, Tz)}{d(Az, Bz)}, \right. \\
 &\quad \left. \frac{d(Az, Tz) \cdot d(Bz, Sz)}{d(Az, Bz)}, \frac{d(Az, Tz) \cdot d(Bz, Sz)}{d(Sz, Tz)} \right\} \left. \right\} \\
 &= d(Tz, Sz),
 \end{aligned}$$

which implies that $Sz = Tz$. Hence we have $Az = Bz = Sz = Tz$.

Next, putting $x = x_{2n}$ and $y = z$ in inequality (C_2) , we have

$$d(Sx_{2n}, Tz) \leq M^\lambda(x_{2n}, z),$$

where

$$\begin{aligned}
 M(x_{2n}, z) &= \max \left\{ d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), d(Bz, Ax_{2n}), \right. \\
 &\quad (d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n}))^{1/2}, \min \left\{ \frac{d(Ax_{2n}, Sx_{2n}) \cdot d(Bz, Tz)}{d(Ax_{2n}, Bz)}, \right. \\
 &\quad \left. \frac{d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n})}{d(Ax_{2n}, Bz)}, \frac{d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n})}{d(Sx_{2n}, Tz)} \right\} \left. \right\}.
 \end{aligned}$$

Letting $n \rightarrow \infty$, we get

$$\begin{aligned}
 &\lim_n M(x_{2n}, z) \\
 &= \max \left\{ d(z, z), d(Tz, Tz), d(Tz, z), d(Tz, z), \right. \\
 &\quad \left. \min \left\{ \frac{1}{d(z, Tz)}, d(z, Tz), 1 \right\} \right\} \\
 &= d(Tz, z).
 \end{aligned}$$

This implies that $Tz = z$ and hence $z = Az = Bz = Sz = Tz$. Hence z is a common fixed point of A, S, B and T .

Finally, suppose that z and w are two common fixed points of A, B, S and T .

Now putting $x = z$ and $y = w$ in (C_2)

$$d(z, w) = d^\lambda(Sz, Tw) \leq M^\lambda(z, w),$$

where

$$\begin{aligned} M(z, w) &= \max \left\{ d(Az, Sz), d(Bw, Tw), d(Bw, Az), \right. \\ &\quad (d(Az, Tw) \cdot d(Bw, Sz))^{1/2}, \min \left\{ \frac{d(Az, Sz) \cdot d(Bw, Tw)}{d(Az, Bw)}, \right. \\ &\quad \left. \frac{d(Az, Tw) \cdot d(Bw, Sz)}{d(Az, Bw)}, \frac{d(Az, Tw) \cdot d(Bw, Sz)}{d(Sz, Tw)} \right\} \left. \right\} \\ &= \max \left\{ 1, 1, d(z, w), d(z, w), \min \left\{ \frac{1}{d(z, w)}, d(z, w), d(z, w) \right\} \right\} \\ &= d(z, w), \end{aligned}$$

which implies that $z = w$. Therefore, A, B, S and T have a unique common fixed point in X . This completes the proof. $\square$

Finally, we give the following theorem for compatible mappings of type (E) .

Theorem 2.3. *Let A, B, S and T be mappings of a complete multiplicative metric space (X, d) into itself satisfying the conditions (C_1) , (C_2) and (C_3) .*

Assume that the pairs A, S and B, T are compatible of type (E) , Then A, B, S and T have a unique common fixed point in X .

Proof. From the proof of Theorem 2.2, $\{y_n\}$ is a multiplicative Cauchy sequence in X and hence it converges to some point $z \in X$. Further the subsequence $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$, $\{Bx_{2n+1}\}$ and $\{Ax_{2n}\}$ of $\{y_n\}$ converges to z .

Now suppose that A and S are compatible of type (E) and one of mappings A and S is continuous. Then by Proposition 1.8 we have $Az = Sz$. Since $SX \subset BX$ so there exists a point $u \in X$ such that $Sz = Bu$.

Now, putting $x = z$ and $y = u$ in (C_2) , we have

$$d(Sz, Tu) \leq M^\lambda(z, u),$$

where

$$\begin{aligned}
 M(z, u) &= \max \left\{ d(Az, Sz), d(Bu, Tu), d(Bu, Az), \right. \\
 &\quad (d(Az, Tu) \cdot d(Bu, Sz))^{1/2}, \min \left\{ \frac{d(Az, Sz) \cdot d(Bu, Tu)}{d(Az, Bu)}, \right. \\
 &\quad \left. \frac{d(Az, Tu) \cdot d(Bu, Sz)}{d(Az, Bu)}, \frac{d(Az, Tu) \cdot d(Bu, Sz)}{d(Sz, Tu)} \right\} \left. \right\} \\
 &= d(Sz, Tu).
 \end{aligned}$$

Hence we have

$$d(Sz, Tu) \leq d^\lambda(Sz, Tu),$$

which implies that $Sz = Tu$ and hence we have $Az = Sz = Tu = Bu$.

Next putting $x = z$ and $y = x_{2n+1}$ in (C_2) , we have

$$d(Sz, Tx_{2n+1}) \leq M^\lambda(z, x_{2n+1}),$$

where

$$\begin{aligned}
 M(z, x_{2n+1}) &= \max \left\{ d(Az, Sz), d(Bx_{2n+1}, Tx_{2n+1}), d(Bx_{2n+1}, Az), \right. \\
 &\quad (d(Az, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sz))^{1/2}, \min \left\{ \frac{d(Az, Sz) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{d(Az, Bx_{2n+1})}, \right. \\
 &\quad \left. \frac{d(Az, Ty) \cdot d(Bx_{2n+1}, Sz)}{d(Az, Bx_{2n+1})}, \frac{d(Az, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sz)}{d(Sz, Tx_{2n+1})} \right\} \left. \right\}.
 \end{aligned}$$

Taking $n \rightarrow \infty$, we get

$$\begin{aligned}
 \lim_n M(z, x_{2n+1}) &= \max \left\{ d(Sz, Sz), d(z, z), d(z, Sz), (d(Sz, z) \cdot d(z, Sz))^{1/2}, \right. \\
 &\quad \left. \min \left\{ \frac{1}{d(Sz, z)}, d(Sz, z), d(Sz, z) \right\} \right\} \\
 &= d(Sz, z).
 \end{aligned}$$

This implies that $Sz = z$. Hence, z is a common fixed point of A and S .

Again, suppose that B and T are compatible of type (E) and one of mappings B and T is continuous we get $Bu = Tu = Az = z$. By Proposition 1.8 we have $BBu = BTu = TBu = TTu$, that is, $Bz = Tz$.

Next putting $x = x_{2n}$ and $y = z$ in inequality (C_2) , we have

$$d(Sx_{2n}, Tz) \leq M^\lambda(x_{2n}, z),$$

where

$$\begin{aligned} & M(x_{2n}, z) \\ &= \max \left\{ d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), d(Bz, Ax_{2n}), \right. \\ & \quad (d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n}))^{1/2}, \min \left\{ \frac{d(Ax_{2n}, Sx_{2n}) \cdot d(Bz, Tz)}{d(Ax_{2n}, Bz)}, \right. \\ & \quad \left. \frac{d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n})}{d(Ax_{2n}, Bz)}, \frac{d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n})}{d(Sx_{2n}, Tz)} \right\} \left. \right\}. \end{aligned}$$

Letting $n \rightarrow \infty$, we get

$$\begin{aligned} & \lim_n M(x_{2n}, z) \\ &= \max \left\{ 1, 1, d(Tz, z), d(Tz, z), \min \left\{ \frac{1}{d(z, Tz)}, d(z, Tz), d(Tz, z) \right\} \right\} \\ &= d(Tz, z). \end{aligned}$$

This implies that $Tz = z$ and hence we have $Tz = Bz = z$. Hence, z is a common fixed point of B and T . Thus, z is a common fixed point of A, B, S and T .

Uniqueness follows easily. Therefore, A, B, S and T have a unique common fixed point in X . This completes the proof. $\square$

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