K-QUASI CLASS Q AND k-QUASI * CLASS Q COMPOSITION OPERATORS ON WEIGHTED HARDY SPACE

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Abstract: In this paper we discuss the conditions for a composition operator and a weighted composition operator to be k quasi class Q and k quasi * class Q operator and also the characterization of k quasi class Q and k quasi * class Q composition operators on weighted Hardy space.

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Key Words: Hilbert space, quasi * class Q operators, composition operators, Hardy space

1. Introduction

Various properties of composition operators on weighted Hardy spaces have been studied by different authors [3] Cowen and Kriete obtained a nice correlation between hyponormality of composition operator on H^2 . [9] E.A. Nordgeen, studied some results on the hyponormality of composition operators and their adjoints. [13] S. Panayappan D.Senthil kumar and Mohenraj have investigated on M-Quasi hyponormality of composition operators and their adjoints. [15] T. Veluchamy have investigated parahyponormal * paranormal and posinormal operators.

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In this paper, we are interested in k -quasi $*$ class Q Operators, we give a characterization of such operators and other known classes of operators

2. Preliminary Notes

Let f be an analytic map on the open disk D given by the Taylors series

$$f(z) = a_0 + a_1z + a_2z^2 + \dots$$

Let $\beta = \{\beta_n\}_{n=0}^\infty$ be a sequence of positive numbers with $\beta_0 = 1$ and $\frac{\beta_{n+1}}{\beta_n} \rightarrow 1$ as $n \rightarrow \infty$. The set $H^2(\beta)$ of formal complex power series $f(z) = \sum_{n=0}^\infty a_n z^n$ such that $\|f\|_\beta^2 = \sum_{n=0}^\infty |a_n|^2 \beta_n^2 < \infty$ is a Hilbert space of functions analytic in the unit disk with the inner product $\langle f, g \rangle_\beta = \sum_{n=0}^\infty a_n \overline{b_n} \beta_n^2$ for f as above and $g(z) = \sum_{n=0}^\infty b_n z^n$.

Let D be the open unit disk in the complex plane and let $T : D \rightarrow D$ be an analytic self map of the unit disk and consider the corresponding composition operator C acting on $H^2(\beta)$ ie., $Cf = f \circ T$, $f \in H^2(\beta)$.

Let $(x, \sum, \lambda)$ be a sigma-finite measure space and let $T : X \rightarrow X$ be a non singular measurable transformation. A bounded linear operator $Cf = f \circ T$ on $L^2(x, \sum, \lambda)$ is said to be a composition operator induced by T when the measure λT^{-1} is absolutely continuous with respect to the measure λ and the Random-Nikodym derivative $\frac{d\lambda T^{-1}}{d\lambda} = f_0$ is essentially bounded. The random-Nikodym derivative of the measure $\lambda(T^k)^{-1}$ with respect to λ is denoted by $f_0^{(k)}$ where T^k is obtained by composing T - k times. Every essentially bounded complex valued measurable function f_0 induces the bounded operator M_{f_0} on $L^2(\lambda)$ which is defined by $M_{f_0}f = f_0 f$ for every $f \in L^2(\lambda)$. Further $C^*C = M_{f_0}$ and $C^{*2}C^2 = M_{f_0^2}$.

Let u be an essentially bounded function. Then the weighted composition operator $W(= W_{u,T})$ on the space $L^2(\mu)$ induced by u and T is given by $Wf = u \cdot f \circ T$ for each $f \in L^2(\mu)$.

A transformation T is measurable if $T^{-1}(A) \in \mathbf{A}$ for any $A \in \mathbf{A}$. A measurable transformation T is said to be non-singular if $\mu(T^{-1}(A)) = 0$ whenever $\mu(A) = 0$ for $A \in \mathbf{A}$. If T is a measurable transformation then T^n is also a measurable transformation. If T is non-singular, then we say that μT^{-1} is absolutely continuous with respect to μ and hence $\mu(T^{-1})^n$ becomes absolutely continuous with respect to μ . Hence, by Radon-Nikodym theorem there exists a unique non-negative essentially bounded measurable function h_n such that

$$\mu(T^{-1})^n(A) = \int_A h_n d\mu \text{ for every } A \in \mathbf{A}$$

and h_n is called the n th order radon-nikodym derivative and is denoted by $\frac{d\mu(T^{-1})^n}{d\mu}$. It can be seen that $h_n = h.hoT^{-1}.hoT^{-2}...hoT^{-(n-1)}$ and $h_n = h_{n-1}.hoT^{-(n-1)}$. Throughout this paper, we assume that u is non-negative.

Definition 2.1. [10]

An operator T on a Hilbert space H is said to be k -quasi class Q if

$$T^{*(k+2)}T^{k+2} - 2T^{*(k+1)}T^{k+1} + T^{*k}T^k \geq 0.$$

Definition 2.2. An operator T on a Hilbert space H is said to be $*$ -class Q if

$$T^{*2}T^2 - 2TT^* + I \geq 0.$$

Definition 2.3. An operator T on a Hilbert space H is said to be quasi- $*$ -class Q if

$$T^{*3}T^3 - 2(T^*T)^2 + T^*T \geq 0.$$

Definition 2.4. An operator T on a Hilbert space H is said to be k -quasi- $*$ -class Q if

$$T^{*(k+2)}T^{k+2} - 2T^{*k}TT^*T^k + T^{*k}T^k \geq 0.$$

Proposition 1. Change of variable: Let X be a non-empty set and let $\mathbf{A}$ be a σ -algebra on X . Let μ and μT^{-1} be measures on $\mathbf{A}$ let $h : X \rightarrow [0, \infty]$ be a measurable function. Then the following are equivalent:

(i) μT^{-1} is absolutely continuous with respect to μ and h is Radon-nikodym derivative of μT^{-1} with respect to μ .

(ii) For every measurable function $f : X \rightarrow [0, \infty]$, the equality

$$\int_X f d\mu T^{-1} = \int_X f h d\mu$$

holds.

The conditional expectation operator $E(T^{-1}(\mathbf{A})) = E(f)$ is defined for each non-negative function f in $L^p(1 \leq p < \infty)$ and is uniquely determined by the following set of conditions:

(i) $E(f)$ is $T^{-1}(\mathbf{A})$ measurable.

(ii) If B is any $T^{-1}(\mathbf{A})$ measurable set for which $\int_A f d\mu$ converges then we have

$$\int_A f d\mu = \int_A E(f) d\mu.$$

E is the projection operator onto the closure of the range of the composition operator C on $L^2(\mu)$.

Lemma 1. [14] Let P be the projection of $L^2(X, \mathbf{A}, \mu)$ onto $\overline{R(C)}$. Then:

(i) $C^*Cf = hf$ and $CC^*f = (hoT)Pf$ for all $f \in L^2(\mu)$.

(ii) $\overline{R(C)} = \{f \in L^2(\mu) : f \text{ is } T^{-1}(\mathbf{A}) \text{ measurable}\}.$

(iii) If f is $T^{-1}(\mathbf{A})$ measurable and g and fg belong to $L^2(\mu)$, then $P(fg) = fP(g)$, (f need not be in $L^2(\mu)$).

In [12] Senthikumar has proved the conditions for composition and weighted composition operators to be k -quasi paranormal operator. A. Gupta and N. Bhatia [1] also proved the conditions for composition and weighted composition operators to be (n,k) -quasi paranormal and (n,k) -quasi $*$ -paranormal operators. In this paper we obtain the conditions for composition and weighted composition operators to be k quasi $*$ class Q operator and quasi $*$ class Q operators in terms of expectation operator and Radon-Nikodyn derivative h (or h_n)

3. Composition Operator

Let C be the composition operator and C^* be its adjoint which is given by

$$C^*f = h.E(f)oT^{-1}.$$

Proposition 2. [1] For every $n \in N$:

(i) $(C^*C)^n f = h^n f$.

(ii) $(CC^*)^n f = (hoT)^n P(f)$.

(iii) E is the identity operator on $L^2(\mu)$ iff $T^{-1}(\mathbf{A}) = \mathbf{A}$.

Theorem 3.1. Let C be a composition operator on $L^2(\mu)$. Then C is k -quasi class Q if and only if $h^{k+2} - 2h^{k+1} + h^k \geq 0$.

Proof. Suppose C is k -quasi class Q operator. Then for every $f \in L^2(\mu)$,

$$\left\langle (C^{*(k+2)}C^{k+2} - 2C^{*(k+1)}C^{k+1} + C^{*k}C^k)f, f \right\rangle \geq 0.$$

Let $f = \chi_A$ with $\mu(A) < \infty$. Therefore

$$\left\langle (C^{*(k+2)}C^{k+2} - 2C^{*(k+1)}C^{k+1} + C^{*k}C^k)\chi_A, \chi_A \right\rangle \geq 0$$

$$\Leftrightarrow \int ((h^{k+2} - 2h^{k+1} + h^k)\chi_A) d\mu \geq 0$$

$$\Leftrightarrow \int (h^{k+2} - 2h^{k+1} + h^k) d\mu \geq 0$$

$$\Leftrightarrow h^{k+2} - 2h^{k+1} + h^k \geq 0. \quad \square$$

Theorem 3.2. Let C be a composition operator on $L^2(\mu)$. Then C is k -quasi * class Q if and only if $h^{k+2} - 2h_k.E(h)oT^{-1} + h^k \geq 0$.

Proof. Consider

$$\begin{aligned} C^{*k}CC^*C^kf &= C^{*k}CC^*(foT^k) \\ &= C^{*k}C(h.E(foT^k)oT^{-1}) \\ &= C^{*k}(hoT^{-1}.foT^k)oT \\ &= h_k.E(h)oT^{-1}.f. \end{aligned}$$

C is k -quasi * class Q if and only if for every $f \in L^2(\mu)$ and $\lambda > 0$,

$$\left\langle (C^{*(k+2)}C^{k+2} - 2C^{*k}CC^*C^k + C^{*k}C^k)f, f \right\rangle \geq 0.$$

$$\Leftrightarrow (h^{k+2} - 2h_k.E(h)oT^{-1} + h^k)f \geq 0$$

$$\Leftrightarrow h^{k+2} - 2h_k.E(h)oT^{-1} + h^k \geq 0. \quad \square$$

Example 3.1. Consider the space $l^2(\omega) = L^2(N, 2^N, \mu)(\omega)$ where $\omega = \langle m_n \rangle_{n=1}^\infty$ is a sequence of positive real numbers. μ is a measure given by $\mu(n) = m_n$. Let $T : N \rightarrow N$ be a non-singular measurable transformation. Then T^n is also a non-singular measurable transformation for $n \in N$. Now,

$$\begin{aligned} h_k(s) &= \frac{1}{m_s} \sum_{j \in T^{-k}(s)} m_j \\ h^k(s) &= \frac{1}{m_s^k} \left(\sum_{j \in T^{-1}(s)} m_j \right)^k \end{aligned}$$

$$E(f)(k) = \frac{\sum_{j \in T^{-1}T(k)} f_j m_j}{\sum_{j \in T^{-1}T(k)} m_j}$$

for all non-negative sequence $f = \langle f_n \rangle_{n=1}^{\infty}$ and $s, k \in N$. by theorem (3.1), C is k-quasi class Q if and only if

$$\begin{aligned} \frac{1}{m_s^{k+2}} \left(\sum_{j \in T^{-1}(s)} m_j \right)^{k+2} - 2 \frac{1}{m_s^{k+1}} \left(\sum_{j \in T^{-1}(s)} m_j \right)^{k+1} \\ + \frac{1}{m_s^k} \left(\sum_{j \in T^{-1}(s)} m_j \right)^k \geq 0. \end{aligned}$$

for all non-negative sequence $f = \langle f_n \rangle_{n=1}^{\infty}$ and $s, k \in N$. by theorem (3.2), C is k-quasi * class Q if and only if

$$\begin{aligned} \frac{1}{m_s^{k+2}} \left(\sum_{j \in T^{-1}(s)} m_j \right)^{k+2} - 2 \frac{1}{m_s} \sum_{j \in T^{-k}(s)} m_j \frac{1}{m_{T^{-k}(s)}} \sum_{j \in T^{-k+1}(s)} m_j \\ + \frac{1}{m_s^k} \left(\sum_{j \in T^{-1}(s)} m_j \right)^k \geq 0. \end{aligned}$$

Proposition 3. If P denote the projection of $L^2(\mu)$ on $\overline{R(C)}$, then $C^*Cf = f_0f$ and $CC^*f = (f_0 \circ T)Pf$. For all $f \in L^2(\mu)$, where P denote the projection of L^2 on $\overline{R(C)}$ and

$$\overline{R(C)} = \left\{ f \in L^2 : f \text{ is } T^{-1} \sum \text{ measurable} \right\}.$$

Theorem 3.3. Let $C \in B(L^2(\mu))$. Then C is of quasi * class Q if and only if

$$f_o^{(3)} - 2(f_0)^2 + f_0 \geq 0.$$

Proof. Let $C \in B(L^2(\mu))$. Then C is of quasi * class Q operator

$$\begin{aligned} \Leftrightarrow C^{*3}C^3 - 2(C^*C)^2 + C^*C \geq 0 \\ \Leftrightarrow \langle (C^{*3}C^3 - 2(C^*C)^2 + C^*C)\chi_E, \chi_E \rangle \geq 0 \end{aligned}$$

for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$.

Since, $C^{*2}C^2 = M_{f_0^{(2)}}$, $C^*C = M_{f_0}$.

$$\Leftrightarrow \langle (M_{f_0^{(3)}} - 2(M_{f_0})^2 + M_{f_0})\chi_E, \chi_E \rangle \geq 0$$

$$\Leftrightarrow \int_E (f_0^{(3)} - 2(f_0)^2 + f_0) d\mu \geq 0$$

for every E in Σ .

Hence C is quasi * class Q operator if and only if $f_o^{(3)} - 2(f_0)^2 + f_0 \geq 0$. $\square$

Corollary 1. Let $C \in B(L^2(\mu))$ with dense range. Then $C \in$ quasi * class Q if and only if $f_o^{(3)} - 2(f_0)^2 + f_0 \geq 0$

Theorem 3.4. Let $C \in B(L^2(\mu))$. Then $C^* \in$ quasi * class Q if and only if

$$(f_0 o T)^3 P_1 - 2((f_0 o T) P_1)^2 + (f_0 o T) P_1 \geq 0$$

where P_1 is the projection of L^2 on to $\overline{R(C)}$.

Proof. Let C^* is of quasi * class Q operator if and only if

$$\begin{aligned} C^3 C^{*3} - 2(CC^*)^2 + CC^* &\geq 0, \\ \langle (C^3 C^{*3} - 2(CC^*)^2 + CC^*) f, f \rangle &\geq 0 \end{aligned}$$

for every $f \in L^2$. $\langle CC^* f, f \rangle = \langle (f_0 o T) P_1 f, f \rangle$ where P_1 is the projections of L^2 on to $\overline{R(C)}$. Thus C^* is of quasi * class Q Operator if and only if

$$\langle (f_0 o T)^3 P_1 f, f \rangle - 2 \langle ((f_0 o T) P_1)^2 f, f \rangle + \langle (f_0 o T) P_1 f, f \rangle \geq 0$$

for every $f \in L^2$,

$$(f_0 o T)^3 P_1 - 2((f_0 o T) P_1)^2 + (f_0 o T) P_1 \geq 0. \quad \square$$

Corollary 2. Let $C \in B(L^2(\mu))$ with dense range. Then $C \in$ quasi * class Q operator if and only if $(f_0 o T)^3 P_1 - 2((f_0 o T) P_1)^2 + (f_0 o T) P_1 \geq 0$

4. Weighted Composition Operators

Let W be the weighted composition operator on $L^2(\mu)$. Let W^* be its adjoint which is given by $W^* f = h.E(u.f) o T^{-1}$ for $f \in L^2(\mu)$. For a positive integer n, $u_n = u.(u o T)^2 \dots (u o T)^{(n-1)}$. For $f \in L^2(\mu)$, $W^n f = u_n.f o T^{-n}$ and $W^{*n} f = h_n.E(u_n.f) o T^{-n}$.

Proposition 4. [2] For $u \geq 0$:

- (i) $W^*Wf = hE[(u^2)]oT^{-1}f.$
- (ii) $WW^*f = u(hoT)E(uf).$

Theorem 4.1. Let W be a weighted composition operator on $L^2(\mu)$. Then W is k -quasi class Q operator if and only if

$$h_{k+2}.E(u_{k+2}^2)oT^{-(2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-1} + h_k.E(u_k^2) \geq 0.$$

Proof. Suppose W is k -quasi class Q operator. Then for $f \in L^2(\mu)$,

$$\left\langle (W^{*k}W^{*2}W^2W^k - 2W^{*k}W^*WW^k + W^{*k}W^k)f, f \right\rangle \geq 0.$$

Let $f = \chi_A$ with $\mu(A) < \infty$. Then

$$\begin{aligned} & \left\langle (W^{*k}W^{*2}W^2W^k - 2W^{*k}W^*WW^k + W^{*k}W^k)\chi_A, \chi_A \right\rangle \geq 0, \\ \Leftrightarrow & \left\langle (h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-(k+1)} \right. \\ & \quad \left. + h_k.E(u_k^2)oT^{-k})\chi_A, \chi_A \right\rangle \geq 0, \\ \Leftrightarrow & \int (h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-(k+1)} \\ & \quad + h_k.E(u_k^2)oT^{-k})\chi_A d\mu \geq 0 \\ \Leftrightarrow & \int h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-(k+1)} \\ & \quad + h_k.E(u_k^2)oT^{-k} d\mu \geq 0 \\ \Leftrightarrow & h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-(k+1)} + h_k.E(u_k^2)oT^{-k} \geq 0. \square \end{aligned}$$

Corollary 3. If W be a weighted composition operator on $L^2(\mu)$ and $T^{-1}(A) = A$. Then W is k -quasi class Q operator if and only if

$$h_{k+2}.u_{k+2}^2oT^{-2} - 2h_{k+1}.u_{k+1}^2oT^{-1} + h_k.u_k^2 \geq 0$$

Theorem 4.2. Let W be a weighted composition operator on $L^2(\mu)$. Then W is k -quasi $*$ class Q operator if and only if

$$h_{k+2}.E(u_{k+2}^2)oT^{-2} - 2h_k.hoT^{-(k-1)}.E(u_{k+1}^2) + h_k.E(u_k^2) \geq 0.$$

Proof. Suppose W is k -quasi $*$ class Q operator. Then for $f \in L^2(\mu)$.

$$\left\langle (W^{*k}W^{*2}W^2W^k - 2W^{*k}WW^*W^k + W^{*k}W^k)f, f \right\rangle \geq 0.$$

Let $f = \chi_A$ with $\mu(A) < \infty$. Then

$$\begin{aligned}
 & \left\langle (W^{*k}W^{*2}W^2W^k - 2W^{*k}WW^*W^k + W^{*k}W^k)\chi_A, \chi_A \right\rangle \geq 0. \\
 & \Leftrightarrow \left\langle (h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_k.E(u_{k+1}(hoT)E(u_{k+1}))oT^{-k} \right. \\
 & \quad \left. + h_k.E(u_k^2)oT^{-k})\chi_A, \chi_A \right\rangle \geq 0 \\
 & \Leftrightarrow \int (h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_k.E(u_{k+1}(hoT)E(u_{k+1}))oT^{-k} \\
 & \quad + h_k.E(u_k^2)oT^{-k})\chi_A d\mu \geq 0 \\
 & \Leftrightarrow \int_A h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_k.E(u_{k+1}(hoT)E(u_{k+1}))oT^{-k} \\
 & \quad + h_k.E(u_k^2)oT^{-k} d\mu \geq 0 \\
 & \Leftrightarrow h_{k+2}.E(u_{k+2}^2)oT^{-(2)} - 2h_k.hoT^{-(k-1)}.E(u_{k+1}^2) + h_k.E(u_k^2) \geq 0. \quad \square
 \end{aligned}$$

Corollary 4. If W be a weighted composition operator on $L^2(\mu)$ and $T^{-1}(\mathbf{A}) = \mathbf{A}$. Then W is k -quasi * class Q operator if and only if

$$h_{k+2}.u_{k+2}^2oT^{-2} - 2h_k.hoT^{-(k-1)}.u_{k+1}^2 + h_k.u_k^2 \geq 0.$$

Theorem 4.3. Let $T^{-1}\sum = \sum$, $W \in B(L^2(\mu))$. Then W is of quasi * class Q if and only if

$$f_0^{(3)}E(\pi_3^2)oT^{-3} - 2(f_0^{(1)}E(\pi_1^2)oT^{-1})^2 + f_0^{(1)}E(\pi_1^2)oT^{-1} \geq 0$$

Proof. Since W is weighted composition operator with weight $\pi = (\frac{f_0}{f_0oT})^{\frac{s}{2}}$ it follows that W is of quasi * class Q operator if and only if $f_0^{(3)}E(\pi_3^2)oT^{-3} - 2(f_0^{(1)}E(\pi_1^2)oT^{-1})^2 + f_0^{(1)}E(\pi_1^2)oT^{-1} \geq 0$. $\square$

The second Aluthge transformation of T described by B.P.Duggal is given by

$$\tilde{T} = \left| \tilde{T} \right|^{\frac{1}{2}} V \left| \tilde{T} \right|^{\frac{1}{2}},$$

where $\tilde{T} = V \left| \tilde{T} \right|$ is the polar decomposition of $\tilde{T}$, $\tilde{C} = |W|^{\frac{1}{2}}$, where $W = V |W|$ is the polar decomposition of the generalized Aluthge transformation $W : 0 < s < 1$ is a weighted composition operator with weight

$$\omega' = J^{\frac{1}{4}} \pi \left(\frac{\chi_{supJ}}{J^{\frac{1}{4}}} oT \right),$$

where $J = f_0E(\pi^2)oT^{-1}$.

Corollary 5. Let $T^{-1}\sum = \sum$, $W \in B(L^2(\mu))$. Then W is of quasi $*$ class Q if and only if $f_0^{(3)}E(\omega_3'^2)oT^{-3} - 2(f_0^{(1)}E(\omega_1'^2)oT^{-1})^2 + f_0^{(1)}E(\omega_1'^2)oT^{-1} \geq 0$

5. k-quasi $*$ -class Q Composition Operator on Weighted Hardy Space

The operator C_T are not necessarily defined on all of $H^2(\beta)$. They are ever where defined in some special cases in the classical Hardy spaces H^2 (the case when $\beta_n = 1$ for all n).

Let ω be a point on the open disk. Define $k_\omega^\beta(z) = \sum_{n=0}^{\infty} \frac{z^n \bar{\omega}^n}{\beta_n^2}$. Then the function k_ω^β is a point evaluation for $H^2(\beta)$. Then k_ω^β is in $H^2(\beta)$ and $\|k_\omega^\beta\|^2 = \sum_{n=0}^{\infty} \frac{|\omega|^{2n}}{\beta_n^2}$. Thus, $\|k_\omega\|$ is an increasing function of $|\omega|$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then $\langle f, k_\omega^\beta \rangle_\beta = f(\omega)$ for all f and k_ω^β is known as the point evaluation kernel at ω . It can be easily shown that $C_T^* k_\omega^\beta = k_{T(\omega)}^\beta$ and $k_0^\beta = 1$ (the function identically equal to 1).

Now we introduce the class of k -quasi $*$ -class Q operators, which is a common generalization of $*$ class Q and quasi $*$ -class Q operators, defined as follows: An operator $T \in B(H)$ is said to be k -quasi- $*$ -class Q operator if $T^{*k}(T^{*2}T^2 - 2TT^* + I)T^k \geq 0$, where k is a natural number.

Theorem 5.1. If C_T is a k -quasi $*$ class Q operator in $H^2(\beta)$ then

$$C_T^{*(k+2)}C_T^{k+2} - 2C_T^{*k}C_TC_T^*C_T^k + C_T^{*k}C_T^k = 0.$$

Proof. An operator C_T is k -quasi $*$ class Q

$$\begin{aligned} & C_T^{*(k+2)}C_T^{k+2} - 2C_T^{*k}C_TC_T^*C_T^k + C_T^{*k}C_T^k \geq 0 \\ & \left\langle (C_T^{*(k+2)}C_T^{k+2} - 2C_T^{*k}C_TC_T^*C_T^k + C_T^{*k}C_T^k)f, f \right\rangle \geq 0 \\ & \left\langle (C_T^{*(k+2)}C_T^{k+2})f, f \right\rangle - 2\left\langle (C_T^{*k}C_TC_T^*C_T^k)f, f \right\rangle \\ & \quad + \left\langle (C_T^{*k}C_T^k)f, f \right\rangle \geq 0 \\ & \left\| C_T^{k+2}f \right\|^2 - 2\left\| C_T^kC_T^*f \right\|^2 + \left\| C_T^k f \right\|^2 \geq 0 \\ & \left\| C_T^{k+1}(C_T f) \right\|^2 - 2\left\| C_T^k(C_T^* f) \right\|^2 \\ & \quad + \left\| C_T^{k-1}(C_T f) \right\|^2 \geq 0 \end{aligned}$$

let $f = k_0^\beta$ then

$$\begin{aligned} & \left\| C_T^{k+1}(C_T k_0^\beta) \right\|^2 - 2 \left\| C_T^k(C_T^* k_0^\beta) \right\|^2 + \left\| C_T^{k-1}(C_T k_0^\beta) \right\|^2 \geq 0 \\ & \left\| C_T^{k+1}(k_0^\beta) \right\|^2 - 2 \left\| C_T^k(k_{T(0)}^\beta) \right\|^2 + \left\| C_T^{k-1}(k_0^\beta) \right\|^2 \geq 0 \end{aligned}$$

Repeating the steps for k times and $T(0) = 0$ we get

$$\begin{aligned} & \left\| k_0^\beta \right\|^2 - 2 \left\| k_0^\beta \right\|^2 + \left\| k_0^\beta \right\|^2 \\ & = 0. \end{aligned}$$

□

Theorem 5.2. If C_T^* is a k-quasi * class Q operator in $H^2(\beta)$ then

$$C_T^{k+2}C_T^{*(k+2)} - 2C_T^kC_T^*C_TC_T^{*k} + C_T^kC_T^{*k} = 0.$$

Proof. An operator C_T^* is k-quasi * class Q

$$\begin{aligned} & C_T^{k+2}C_T^{*(k+2)} - 2C_T^kC_T^*C_TC_T^{*k} + C_T^kC_T^{*k} \geq 0 \\ & \left\langle (C_T^{k+2}C_T^{*(k+2)} - 2C_T^kC_T^*C_TC_T^{*k} + C_T^kC_T^{*k})f, f \right\rangle \geq 0 \\ & \left\langle (C_T^{k+2}C_T^{*(k+2)})f, f \right\rangle - 2 \left\langle (C_T^kC_T^*C_TC_T^{*k})f, f \right\rangle + \left\langle (C_T^kC_T^{*k})f, f \right\rangle \geq 0 \\ & \left\| C_T^{*(k+2)}f \right\|^2 - 2 \left\| C_T^{*k}C_Tf \right\|^2 + \left\| C_T^{*k}f \right\|^2 \geq 0 \\ & \left\| C_T^{*(k+1)}(C_T^*f) \right\|^2 - 2 \left\| C_T^{*k}(C_Tf) \right\|^2 + \left\| C_T^{*(k-1)}(C_T^*f) \right\|^2 \geq 0 \end{aligned}$$

let $f = k_0^\beta$ then

$$\begin{aligned} & \left\| C_T^{*(k+1)}(C_T^*k_0^\beta) \right\|^2 - 2 \left\| C_T^{*k}(C_Tk_0^\beta) \right\|^2 + \left\| C_T^{*(k-1)}(C_T^*k_0^\beta) \right\|^2 \geq 0 \\ & \left\| C_T^{*(k+1)}(k_{T(0)}^\beta) \right\|^2 - 2 \left\| C_T^{*k}(k_0^\beta) \right\|^2 + \left\| C_T^{*(k-1)}(k_{T(0)}^\beta) \right\|^2 \geq 0. \end{aligned}$$

Repeating the steps for k times and $T(0) = 0$ we get

$$\begin{aligned} & \left\| k_0^\beta \right\|^2 - 2 \left\| k_0^\beta \right\|^2 + \left\| k_0^\beta \right\|^2 \\ & = 0. \end{aligned}$$

□

Example 5.1. Let $f = \sum_{n=0}^{\infty} f_n z^n \in H^2(\beta)$ and $\phi : D \rightarrow D$ be defined by $\phi(z) = \frac{e^{i\theta}}{2}z$, where $0 \leq \theta \leq 2\pi$ is fixed. Then

$$(C_\phi^* f)(\omega) = \langle C_\phi^* f, k_\omega \rangle$$

$$\begin{aligned}
&= \langle f, k_{\omega o \phi} \rangle \\
&= \langle f, k_{\varphi(\omega)} \rangle \quad \text{where } \varphi(\omega) = \frac{e^{-i\theta}}{2}\omega \\
&= f(\varphi(\omega)) \\
&= f\left(\frac{e^{-i\theta}}{2}\omega\right)
\end{aligned}$$

Now

$$\begin{aligned}
(C_\phi^* C_\phi f)(z) &= (C_\phi f)\left(\frac{e^{-i\theta}}{2}z\right) \\
&= f\left(\frac{e^{i\theta}}{2} \cdot \frac{e^{-i\theta}}{2}z\right) \\
&= f\left(\frac{1}{4}z\right)
\end{aligned}$$

and

$$\begin{aligned}
(C_\phi C_\phi^* f)(z) &= (C_\phi^* f)\left(\frac{e^{i\theta}}{2}z\right) \\
&= f\left(\frac{e^{-i\theta}}{2} \cdot \frac{e^{i\theta}}{2}z\right) \\
&= f\left(\frac{1}{4}z\right)
\end{aligned}$$

$$\begin{aligned}
&(C_\phi^{*(k+2)} C_\phi^{k+2} - 2C_\phi^{*k} C_\phi C_\phi^* C_\phi^k + C_\phi^{*k} C_\phi^k f)(z) \geq 0 \\
&(C_\phi^{*(k+2)} C_\phi^{k+2} f)(z) - 2(C_\phi^{*k} C_\phi C_\phi^* C_\phi^k f)(z) + (C_\phi^{*k} C_\phi^k f)(z) \geq 0 \\
&(C_\phi^{k+2} f)\left(\frac{e^{-i(k+2)\theta}}{2^{k+2}}z\right) - 2(C_\phi C_\phi^* C_\phi^k f)\left(\frac{e^{-i(k)\theta}}{2^k}z\right) + (C_\phi^k f)\left(\frac{e^{-i(k)\theta}}{2^k}z\right) \geq 0 \\
&f\left(\frac{e^{i(k+2)\theta}}{2^{k+2}} \frac{e^{-i(k+2)\theta}}{2^{k+2}}z\right) - 2f\left(\frac{e^{i(k)\theta}}{2^k} \frac{e^{-i\theta}}{2} \frac{e^{i\theta}}{2} \frac{e^{-i(k)\theta}}{2^k}z\right) + f\left(\frac{e^{i(k)\theta}}{2^k} \frac{e^{-i(k)\theta}}{2^k}z\right) \geq 0 \\
&f\left(\frac{1}{2^{2k+4}}z\right) - 2f\left(\frac{1}{2^{2k+2}}z\right) + f\left(\frac{1}{2^{2k}}z\right) \geq 0.
\end{aligned}$$

for every $f \in H^2(\beta)$ and $z \in D$. Hence C_ϕ is k quasi $*$ class Q operator.

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