Abstract: Let $G = (V, E)$ be a graph. An $r$-dynamic $k$-coloring of $G$ is a function $f$ from $V$ to a set $C$ of colors such that (1) $f$ is a proper coloring, and (2) for all vertices $v$ in $V$, $|f(N(v))| \geq \min \{r, \deg_G(v)\}$. The $r$-dynamic chromatic number of a $G$, denoted by $\chi_r(G)$, is the smallest $k$ such that $f$ is an $r$-dynamic $k$-coloring of $G$.

This study gave the $r$-dynamic chromatic number of paths, cycles, complete graphs, empty graphs, the vertex gluing of graphs and the union of graphs. Some characterizations are also given.

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1. Introduction

As mentioned in [11], the earliest results about graph coloring is on planar graphs motivated by a problem regarding the coloring of maps. While coloring a map of the counties of England, Francis Guthrie presented the four color conjecture which stated that four colors were sufficient to color the map so that no regions sharing a common border received the same color. This idea may have motivated the concept proper vertex coloring. Using graph theoretic
A proper vertex coloring of a graph \( G = (V, E) \) is a function from \( V \) to a finite set \( C \), whose elements are called colors, such that no two adjacent vertices are assigned the same color.

A generalization of the proper vertex coloring is the dynamic coloring of graphs. The dynamic coloring of a graph \( G \) is a proper coloring such that every vertex of \( G \) with degree at least two has at least two neighbors that are colored differently. This concept was introduced by Montgomery in [3]. Since then, the concept was studied by many authors, see for example [1], [2], [4], [5], [6] and [7].

A generalization of the dynamic coloring was also introduced by Montgomery in [3]. The generalized concept is now called the \( r \)-dynamic \( k \)-coloring. The proper vertex coloring and the dynamic coloring are special cases of the concept \( r \)-dynamic \( k \)-coloring, that is, the 1-dynamic \( k \)-coloring is the proper vertex coloring concept, while the 2-dynamic \( k \)-coloring is the dynamic coloring concept.

The general \( r \)-dynamic \( k \)-coloring of planar grids is studied well in [8] and [9].

A vertex coloring of a graph \( G = (V, E) \) is a function \( f \) from \( V \) to a finite set \( C \), whose elements are called color. A \( k \)-coloring is a vertex coloring with at most \( k \)-colors. In this case, we always assume that \( C = \{1, 2, \ldots, k\} \). A \( k \)-coloring may also be viewed as a vertex partition \( V = V_1 \cup V_2 \cup \cdots \cup V_k \) or \( (V_1, V_2, \ldots, V_k) \), where \( V_i = f^{-1}(i) \) are called the color classes. A graph is \( k \)-colorable if it admits a proper vertex coloring with at most \( k \)-colors. The chromatic number of a graph \( G \), denoted by \( \chi(G) \), is the smallest \( k \) such that \( G \) is \( k \)-colorable.

An \( r \)-dynamic \( k \)-coloring of \( G \) is a proper coloring \( f \) such that for all vertices \( v \) in \( V \), \( |f(N(v))| \geq min\{r, \deg_G(v)\} \). The \( r \)-dynamic chromatic number of a graph \( G \), denoted by \( \chi_r(G) \), is the smallest \( k \) such that \( f \) is an \( r \)-dynamic \( k \)-coloring of \( G \).

Let \( G_1, G_2, \ldots, G_t \) be disjoint graphs, each containing a complete subgraph \( K_r \) (\( r \geq 1 \)). Let \( G \) be a graph obtained from the union of \( t \) graphs \( G_i \) by identifying the complete graphs \( K_r \) (from each \( G_i \)) in an arbitrary way. We call \( G \) a \( K_r \)-gluing of \( G_1, G_2, \ldots, G_t \). In particular, when \( r = 1 \) we say that \( G \) is a vertex-gluing.

Hereafter, please refer to [12] for concepts that were used but were not discussed in this paper.
2. r-Dynamic Chromatic Number of Paths and Cycles

In this section we verified the $r$-dynamic chromatic number of paths and cycles. The next Lemma, Lemma 2.1, characterizes $r$-dynamic $k$-coloring in paths.

**Lemma 2.1.** Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n \geq 3$. Then $f : P_n \to \{1, 2, \ldots, k\}$ is an $r$-dynamic $(r > 1)$ $k$-coloring of $P_n$ if and only if for all $i$ with $\deg(v_i) = 2$, $f(v_{i-1})$, $f(v_i)$, $f(v_{i+1})$ are distinct.

**Proof.** Assume that $f$ is an $r$-dynamic $k$-coloring of $P_n$ and there exist $i$ such that at least two of $f(v_{i-1})$, $f(v_i)$, $f(v_{i+1})$ are not distinct. Consider the following cases: (Case 1. $f(v_{i-1}) = f(v_i)$) If $f(v_{i-1}) = f(v_i)$, then $f$ is not proper. This is a contradiction. (Case 2. $f(v_{i-1}) = f(v_{i+1})$) If $f(v_{i-1}) = f(v_{i+1})$, then $f(N(v_i)) = f(N(v_{i+1}))$, that is, $|f(N(v_i))| = |f(v_{i-1})| = 1 < 2 = min\{2, r\}$. This is a contradiction.

Conversely, assume that for all $i$ with $\deg(v_i) = 2$, $f(v_{i-1})$, $f(v_i)$, $f(v_{i+1})$ are distinct. Let $v \in V(P_n)$ and consider the following cases: (Case 1. $v = v_1$ or $v = v_n$) If $v = v_1$, then $\deg(v_1) = 1$. Hence, $|f(N(v))| = |f(N(v_1))| = |\{f(v_1)\}| = 1 = min\{1, r\}$. (Case 2. $v \neq v_1$ and $v \neq v_n$) Without loss of generality, let $v = v_2$. Then $\deg(v) = \deg(v_2) = 2$. Thus, $|f(N(v))| = |f(N(v_2))| = |\{f(v_1), f(v_3)\}| = 2 = min\{2, r\}$. This shows that $f$ is an $r$-dynamic $k$-coloring of $P_n$. □

**Remark 2.2.** Lemma 2.1 says that if $f$ is an $r$-dynamic $k$-coloring in $P_n$, then $k \geq 3$.

An observation in [9] stated that $\chi_r \geq min\{\Delta(G, r)\} + 1$ with equality holding when $G$ is a tree. Thus, when this observation is applied for paths of order $n \geq 3$, we have the following remark.

**Remark 2.3.** Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n \geq 3$. If $r > 1$, then $\chi_r(P_n) = 3$.

The ideas presented in Lemma 2.1 and Remark 2.2 may be used to verify Remark 2.3 as follows. Define $f : V(P_n) \to \{1, 2, 3\}$ as follows

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod{3} \\ 2, & \text{if } i \equiv 2 \pmod{3} \\ 3, & \text{if } i \equiv 0 \pmod{3}. \end{cases}$$

Then by Lemma 2.1 $f$ is an $r$-dynamic 3-coloring of $P_n$. Hence, $\chi_r(P_n) \leq 3$. Now by Remark 2.2 $\chi_r(P_n) = 3$.

The next Lemma 2.4 characterizes an $r$-dynamic $k$-coloring in cycles.
Lemma 2.4. Let \( C_n = [v_1, v_2, \ldots, v_n] \) be a cycle of order \( n \geq 3 \). Then \( f : V(C_n) \to \{1, 2, \ldots, k\} \) is an \( r \)-dynamic \((r > 1)\) \( k \)-coloring of \( C_n \) if and only if for all \( i \), \( f(v_{i-1}), f(v_i), f(v_{i+1}) \) are distinct.

Proof. Assume that \( f \) is an \( r \)-dynamic \( k \)-coloring of \( C_n \) and there exist \( i \) such that at least two of \( f(v_{i-1}), f(v_i), f(v_{i+1}) \) are not distinct. Consider the following cases: (Case 1. \( f(v_{i-1}) = f(v_i) \)) If \( f(v_{i-1}) = f(v_i) \), then \( f \) is not proper. This is a contradiction. (Case 2. \( f(v_{i-1}) = f(v_{i+1}) \)) If \( f(v_{i-1}) = f(v_{i+1}) \), then \( f(N(v_i)) = f([v_{i-1}, v_{i+1}]) = \{f(v_{i-1})\}, \) that is, \( |f(N(v_i))| = |\{f(v_{i-1})\}| = 1 < 2 = \min \{2, r\} \). This is a contradiction.

Conversely, assume that for all \( i \), \( f(v_{i-1}), f(v_i), f(v_{i+1}) \) are distinct. Let \( v \in C_n \). Then \( v = v_i \) for some \( i \) and \( \deg(v) = \deg(v_i) = 2 \). Thus, \( |f(N(v_i))| = |f(N(v_i))| = |\{f(v_{i-1}), f(v_{i+1})\}| = 2 = \min \{2, r\} \). This shows that \( f \) is an \( r \)-dynamic \( k \)-coloring of \( C_n \).

Remark 2.5. Lemma 2.4 says that if \( f \) is an \( r \)-dynamic \( k \)-coloring in \( C_n \), then \( k \geq 3 \).

Lemma 2.6. Let \( G = (V, E) \) be a graph and \( S = \{v_1, v_2, \ldots, v_n\} \subseteq V \) with \( \langle\{v_1, v_2, \ldots, v_n\}\rangle = (v_1, v_2, \ldots, v_n) \). Let \( f : V \to \{1, 2, 3\} \) be a vertex coloring of \( G \). Then for all \( i = 2, 3, \ldots, n - 1 \), \( f(v_{i-1}), f(v_i), f(v_{i+1}) \) are distinct if and only if

\[
f|_S(v_j) = \begin{cases} 
  k_1, & \text{if } j \equiv 0 \pmod{3} \\
  k_2, & \text{if } j \equiv 1 \pmod{3} \\
  k_3, & \text{if } j \equiv 2 \pmod{3}
\end{cases}
\]

where \( k_1k_2k_3 \) is a permutation of 1, 2 and 3.

Proof. Assume that \( f(v_i), f(v_{i+1}) \) are distinct and without loss of generality

\[
f|_S(v_j) \neq \begin{cases} 
  1, & \text{if } j \equiv 0 \pmod{3} \\
  2, & \text{if } j \equiv 1 \pmod{3} \\
  3, & \text{if } j \equiv 2 \pmod{3}.
\end{cases}
\]

Since \( f \) must be proper, there exists \( i \) such that \( f|_S(v_i) = 1, f|_S(v_{i+1}) = 2, f|_S(v_{i+2}) = 3, \) and \( f|_S(v_{i+3}) = 2 \). This is a contradiction.

Conversely, assume that

\[
f|_S(v_j) = \begin{cases} 
  k_1, & \text{if } j \equiv 0 \pmod{3} \\
  k_2, & \text{if } j \equiv 1 \pmod{3} \\
  k_3, & \text{if } j \equiv 2 \pmod{3}
\end{cases}
\]
where \( k_1k_2k_3 \) is a permutation of 123, say without loss of generality

\[
f|_S (v_j) = \begin{cases} 
1, & \text{if } j \equiv 0 \pmod{3} \\
2, & \text{if } j \equiv 1 \pmod{3} \\
3, & \text{if } j \equiv 2 \pmod{3}.
\end{cases}
\]

Let \( j \in \{2, 3, \ldots, n - 1\} \), say without loss of generality \( j \equiv 1 \pmod{3} \). Then \( j - 1 \equiv 0 \pmod{3} \) and \( j + 1 \equiv 2 \pmod{3} \). Hence, \( f(v_{j-1}) = 3 \), \( f(v_j) = 1 \), \( f(v_{j+1}) = 2 \) are distinct.

A result in [10] stated that

\[
\chi_2 (C_n) = \begin{cases} 
3, & \text{if } n \equiv 0 \pmod{3} \\
5, & \text{if } n = 5 \\
4, & \text{otherwise}
\end{cases}
\]

and an observation in [8] stated that if \( r \geq \Delta (G) \), then \( \chi_r (G) = \chi_{\Delta(G)} (G) \). By these and since \( \Delta (C_n) = 2 \), we have the following remark.

**Remark 2.7.** Let \( C_n = [v_1, v_2, \ldots, v_n] \) be a cycle of order \( n \geq 3 \). If \( r > 1 \), then

\[
\chi_r (C_n) = \begin{cases} 
3, & \text{if } n \equiv 0 \pmod{3} \\
5, & \text{if } n = 5 \\
4, & \text{otherwise}
\end{cases}
\]

The ideas presented in Lemma 2.4, Remark 2.5 and Lemma 2.6 may be used to verify Remark 2.7 as follows. Let \( C_n = [v_1, v_2, \ldots, v_n] \) be a cycle of order \( n \geq 3 \). Consider the following cases: (Case 1. \( n \equiv 0 \pmod{3} \)) If \( n \equiv 0 \pmod{3} \), then we define \( f : V(C_n) \to \{1, 2, 3\} \) as follows

\[
f (v_i) = \begin{cases} 
1, & \text{if } i \equiv 1 \pmod{3} \\
2, & \text{if } i \equiv 2 \pmod{3} \\
3, & \text{if } i \equiv 0 \pmod{3}.
\end{cases}
\]

Then by Lemma 2.4 \( f \) is an \( r \)-dynamic 3-coloring of \( C_n \). Thus, by Remark 2.5 \( \chi_r (C_n) = 3 \).

(Case 2. \( n \equiv 1 \pmod{3} \)) If \( n \equiv 1 \pmod{3} \), then we define \( f : V(C_n) \to \{1, 2, 3, 4\} \) as follows

\[
f (v_i) = \begin{cases} 
1, & \text{if } i \equiv 1 \pmod{3} \text{ and } j \neq n \\
2, & \text{if } i \equiv 2 \pmod{3} \\
3, & \text{if } i \equiv 0 \pmod{3} \\
4, & \text{if } i = n.
\end{cases}
\]
Then by Lemma 2.4 \( f \) is an \( r \)-dynamic 3-coloring of \( C_n \). Hence, \( \chi_r(C_n) \leq 4 \). Suppose that \( \chi_r(C_n) < 4 \), say without loss of generality \( \chi_r(C_n) = 3 \). Let \( f : V(C_n) \to \{1, 2, 3\} \) be an \( r \)-dynamic 3-coloring of \( C_n \). Then by Lemma 2.4 \( f(v_{i-1}), f(v_i), f(v_{i+1}) \) are distinct for all \( i = 1, 2, \ldots, n \). Let \( S = \{1, 2, \ldots, n - 1\} \). Then \( \langle S \rangle = (1, 2, \ldots, n - 1) \). By Lemma 2.6, without loss of generality

\[
 f(v_j) = \begin{cases} 
 1, & \text{if } j \equiv 0 \pmod{3} \\
 2, & \text{if } j \equiv 1 \pmod{3} \\
 3, & \text{if } j \equiv 2 \pmod{3}.
\end{cases}
\]

for all \( i = 1, 2, \ldots, n - 1 \). Since there are only 3 colors, \( f(v_n) \) is either 1, 2 or 3. Since \( f \) must be a proper vertex coloring, \( f(v_n) = 2 \). This implies that \( f(v_{n-2}), f(v_{n-1}), f(v_n) \) are not distinct. This is a contradiction. Therefore, \( \chi_r(C_n) = 4 \).

(Case 3. \( n \equiv 2 \pmod{3} \) with \( n \neq 5 \)) If \( n \equiv 1 \pmod{3} \) with \( n \neq 5 \), then we define \( f : V(C_n) \to \{1, 2, 3, 4\} \) as follows

\[
 f(v_i) = \begin{cases} 
 1, & \text{if } i \equiv 1 \pmod{4} \\
 2, & \text{if } i \equiv 2 \pmod{4} \\
 3, & \text{if } i \equiv 3 \pmod{4} \\
 4, & \text{if } i \equiv 0 \pmod{4}.
\end{cases}
\]

Then clearly \( f \) is an \( r \)-dynamic 4-coloring of \( C_n \). Hence, \( \chi_r(C_n) \leq 4 \). Suppose that \( \chi_r(C_n) < 4 \), say \( \chi_r(C_n) = 3 \). Using the same arguments in the proof of Case 2, it can be shown also that \( \chi_r(C_n) \) can not be less than 4. Therefore, \( \chi_r(C_n) = 4 \).

(Case 4. \( n = 5 \)) If \( n = 5 \), then we define \( f : V(C_5) \to \{1, 2, 3, 4, 5\} \) as follows \( f(v_i) = i \). Then clearly \( f \) is an \( r \)-dynamic 5-coloring of \( C_5 \). Hence, \( \chi_r(C_5) \leq 5 \). Suppose that \( \chi_r(C_5) < 5 \), say \( \chi_r(C_5) = 4 \). Then by Lemma 2.4 (without loss of generality) \( f(v_i) = i \) for \( i = 1, 2, 3, 4 \). Since there are only 4 colors \( f(v_5) \) is either 1 or 2 or 3 or 4. This is a contradiction by Lemma 2.4. Therefore, \( \chi_r(C_5) = 5 \).

### 3.  \( r \)-Dynamic Chromatic Number of Complete Graphs and Empty Graphs

This section presents the \( r \)-dynamic chromatic number of complete graphs and empty graphs. Theorem 3.1 characterizes graphs with \( r \)-dynamic chromatic number equal to its order when \( r \) is less than the minimum degree \( \delta \).
**Theorem 3.1.** Let $G = (V, E)$ be graph of order $n \geq 3$, and $r \in \mathbb{N}$ with $r < \delta$. Then $\chi_r(G) = n$ if and only if $G \cong K_n$.

**Proof.** Assume that $\chi_r(G) = n$ and $G \not\cong K_n$. If $G \not\cong K_n$, then there exist $u, v \in V$ such that $uv \not\in E$. Assume that $V = \{v_1, v_2, \ldots, v_{n-2}, v_{n-1} = u, v_n = v\}$. Define $f : V(K_n) \to \{1, 2, \ldots, n - 1\}$ as follows

$$f(v_i) = \begin{cases} i, & \text{if } i \neq n \\ n - 1, & \text{if } i = n. \end{cases}$$

Let $w \in V(G)$ and consider the following cases: (Case 1. $w = u$ or $w = v$) Without loss of generality, if $w = u$, then $|f(N(w))| = \min\{r, \deg(u)\}$. (Case 2. $w \neq u$ and $w \neq v$) If $w \neq u$ and $w \neq v$, then $|f(N(w))| \geq r = \min\{r, \deg(u)\}$ since $r < \delta$. Thus, $f$ is an $r$-dynamic $k$-coloring of $G$, that is, $\chi_r(G) \leq n - 1$. This is a contradiction.

Conversely, assume that $G = K_n$. Then $n = \chi(G) \leq \chi_r(G) \leq n$. Hence, $\chi_r(G) = n$.

Theorem 3.2 characterizes graphs with $r$-dynamic chromatic number equal to 1.

**Theorem 3.2.** Let $G = (V, E)$ be graph of order $n$, and $r \in \mathbb{N}$. Then $\chi_r(G) = 1$ if and only if $G \cong \overline{K}_n$.

**Proof.** Assume that $\chi_r(G) = 1$ and $G \not\cong \overline{K}_n$. If $G \not\cong \overline{K}_n$, then there exist $u, v \in V$ such that $uv \in E$. Let $f : V(\overline{K}_n) \to \{1\}$ be an $r$-dynamic 1-coloring of $G$. Then $f(u) = 1 = f(v)$. This is a contradiction since $f$ must be proper.

Conversely, assume that $G = \overline{K}_n$. Define $f : V \to \{1\}$ by $f(v) = 1$ for all $v \in V$. Let $u \in V$. Then $|f(N(u))| = |f(\emptyset)| = 0 = \min\{r, 0\}$. Clearly, $f$ is a proper coloring. Hence, $f$ is an $r$-dynamic 1-coloring of $G$. Accordingly, $\chi_r(G) = 1$.

4. 2-Dynamic Chromatic Number of the Vertex Gluing of Graphs

In this section we present the $r$-dynamic chromatic number of the vertex gluing of graphs. Lemma 4.1 characterizes an $r$-dynamic $k$-coloring in a graph.

**Lemma 4.1.** Let $G = (V, E)$ be graph and $f = \{V_1, V_2, \ldots, V_k\}$ be a $k$-coloring of $G$. Then $f$ is an $r$-dynamic $k$-coloring of $G$ if and only if

1. $\{u, v\} \not\in V_i$ for all $i = 1, 2, \ldots, k$ whenever $uv \in E$, and
2. there exist \( \{q_1, q_2, \ldots, q_{\min\{r, \deg(v)\}}\} \subseteq \{1, 2, \ldots, k\} \setminus \{i\} \) such that \( N(v) \cap V_{q_j} \neq \emptyset \) for all \( j = 1, 2, \ldots, \min \{r, \deg(v)\} \) whenever \( v \in V_i \) for some \( i \in \{1, 2, \ldots, k\} \).

**Proof.** Assume that \( f = \{V_1, V_2, \ldots, V_k\} \) is an \( r \)-dynamic \( k \)-coloring of \( G \). Let \( uv \in E \). Since \( f \) must be proper \( u \in V_s \) and \( v \in V_t \) for some \( s, t \in \{1, 2, \ldots, k\} \) with \( s \neq t \). Since \( \{V_1, V_2, \ldots, V_k\} \) is a pairwise disjoint family, we must have \( \{u, v\} \notin V_i \) for all \( i = 1, 2, \ldots, k \). Next, let \( v \in V_i \) for some \( i \in \{1, 2, \ldots, k\} \). Let \( N(v) = \{v_1, v_2, \ldots, v_m\} \) where \( m \geq \min \{r, \deg(v)\} \). Since \( f \) is an \( r \)-dynamic \( k \)-coloring,

\[
|f(N(v))| = |f(\{v_1, v_2, \ldots, v_m\})| = |\{f(v_1), f(v_2), \ldots, f(v_m)\}| \\
\geq \min \{r, \deg(v)\}.
\]

This implies that there exist \( \{q_1, q_2, \ldots, q_{\min\{r, \deg(v)\}}\} \subseteq \{1, 2, \ldots, k\} \setminus \{i\} \) such that \( v_j \in V_{q_j} \) for all \( j = 1, 2, \ldots, \min \{r, \deg(v)\} \). Hence, there exist

\[
\{q_1, q_2, \ldots, q_{\min\{r, \deg(v)\}}\} \subseteq \{1, 2, \ldots, k\} \setminus \{i\}
\]

such that \( N(v) \cap V_{q_j} \neq \emptyset \) for all \( j = 1, 2, \ldots, \min \{r, \deg(v)\} \).

Conversely, assume that conditions (1) and (2) hold. Let \( uv \in E \). Then by (1), \( \{u, v\} \notin V_i \) for all \( i = 1, 2, \ldots, k \). Since \( \{V_1, V_2, \ldots, V_k\} \) is a partition of \( V \), \( u \in V_s \) and \( v \in V_t \) for some \( s, t \in \{1, 2, \ldots, k\} \) with \( s \neq t \), that is, \( f(u) \neq f(v) \). This shows that \( f \) is proper. Next, let \( v \in V_i \). Then for some \( i \in \{1, 2, \ldots, k\} \). By (2), there exist \( \{q_1, q_2, \ldots, q_{\min\{r, \deg(v)\}}\} \subseteq \{1, 2, \ldots, k\} \setminus \{i\} \) such that \( N(v) \cap V_{q_j} \neq \emptyset \) for all \( j = 1, 2, \ldots, \min \{r, \deg(v)\} \). Thus, there exist \( \{q_1, q_2, \ldots, q_{\min\{r, \deg(v)\}}\} \subseteq \{1, 2, \ldots, k\} \setminus \{i\} \) such that \( v_j \in V_{q_j} \) for all \( j = 1, 2, \ldots, \min \{r, \deg(v)\} \). This implies that if \( N(v) = \{v_1, v_2, \ldots, v_m\} \) where \( m \geq \min \{r, \deg(v)\} \), then \( |f(N(v))| = |f(\{v_1, v_2, \ldots, v_m\})| = |\{f(v_1), f(v_2), \ldots, f(v_m)\}| \geq \min \{r, \deg(v)\} \). \[\square\]

Lemma 4.1, says that the colors of the color classes may be interchanged and the new coloring will still be an \( r \)-dynamic \( k \)-coloring.

**Corollary 4.2.** Let \( G = (V, E) \) be a graph and \( (V_1, V_2, \ldots, V_k) \) be an \( r \)-dynamic \( k \)-coloring of \( G \). If \( \{i_1, i_2, \ldots, i_k\} \) is a permutation of \( \{1, 2, \ldots, k\} \), then \( (V_1, V_2, \ldots, V_k) \) is also an \( r \)-dynamic \( k \)-coloring of \( G \).

Next, we have Lemma 4.3. This Lemma says that if \( f \) is a proper coloring of \( G \), then the restriction of \( f \) to any subgraphs of \( G \) is also a proper coloring.

**Lemma 4.3.** Let \( G = (V, E) \) be a graph and \( H \) be a subgraph of \( G \). If \( f \) is a proper coloring of \( G \), then \( f|_{V(H)} \) is a proper coloring of \( H \).
Proof. Let \( f : V \to C \) be a proper coloring of \( G \). If \( f : V \to C \) be a proper coloring of \( G \), then \( f(u) \neq f(v) \) for all \( uv \in E \). Let \( ab \in E(H) \) and consider \( f|_{V(H)} \). If \( ab \in E(H) \), then \( ab \in E \). Hence, \( f(a) \neq f(b) \), that is, \( f|_{V(H)}(a) \neq f|_{V(H)}(b) \). This shows that \( f|_{V(H)} \) is a proper coloring of \( H \). \( \square \)

Next, we have Lemma 4.3. The Lemma says that if \( f \) is an \( r \)-dynamic \( k \)-coloring of \( G \), then the restriction of \( f \) to the vertex set of any subgraph \( H \) of \( G \) is an \( r \)-dynamic \( k \)-coloring of \( H \).

**Lemma 4.4.** Let \( G = (V, E) \) be graph and \( H \) be a subgraph of \( G \). If \( f \) is an \( r \)-dynamic \( k \)-coloring of \( G \), then \( f|_{V(H)} \) is an \( r \)-dynamic \( k \)-coloring of \( H \).

**Proof.** Let \( f : V \to C \) be an \( r \)-dynamic \( k \)-coloring. If \( f : V \to C \) is an \( r \)-dynamic \( k \)-coloring of \( G \), then \( f \) is a proper coloring, and for all vertices \( v \) in \( V \), \( |f(N(v))| \geq \min \{r, deg_G(v)\} \). By Lemma 4.3 \( f|_{V(H)} \) is a proper vertex coloring of \( H \). Now, let \( v \in V(H) \). Then

\[
|f(N_G(v))| - |f(N_H(v))| \leq |N_G(v)| - |N_H(v)|
\]

\[
\Rightarrow |f(N_G(v))| - |f(N_H(v))| \leq deg_G(v) - deg_H(v)
\]

\[
\Rightarrow |f(N_G(v))| - deg_G(v) \leq |f(N_H(v))| - deg_H(v)
\]

\[
\Rightarrow 0 \leq |f(N_H(v))| - deg_H(v)
\]

\[
\Rightarrow min \{r, deg_H(v)\} \leq |f(N_H(v))|
\]

This shows that \( f|_{V(H)} \) is an \( r \)-dynamic \( k \)-coloring of \( H \). \( \square \)

We call Corollary 4.5 the *monotonicity* property.

**Corollary 4.5.** Let \( G = (V, E) \) be graph and \( H \) be a subgraph of \( G \). Then \( \chi_r(G) \geq \chi_r(H) \).

**Proof.** Let \( \mathcal{F} = \{f : f \) is an \( r \)-dynamic \( k \)-coloring of \( G \} \) and \( \mathcal{G} = \{g : g \) is an \( r \)-dynamic \( k \)-coloring of \( H \} \). Then by Lemma 4.4, \( \mathcal{F} = \{f|_{V(H)} : f \in \mathcal{F}\} \) is a subset of \( \mathcal{G} \). Next, let \( \mathcal{F} = \{f|_{V(H)}(V(H)) : f|_{V(H)} \in \mathcal{F}\} \) and \( \mathcal{B} = \{|g(V(H))| : g \in \mathcal{G}\} \). Then, \( \mathcal{F} \subseteq \mathcal{B} \). Thus, \( \chi_r(G) \geq \chi_r(H) \) since \( |f(V(H))| \leq |f(V)| \) for all \( f \in \mathcal{F} \). \( \square \)

**Theorem 4.6.** Let \( G_1 \) and \( G_2 \) be connected graphs with \( \chi_r(G_1) \geq 3 \). Let \( G \) be the graph obtained by identifying a vertex in \( G_1 \) and a vertex in \( G_2 \), that is \( G \) is a vertex gluing of \( G_1 \) and \( G_2 \). If \( r = 2 \), then \( \chi_r(G) = \max \{\chi_r(G_1), \chi_r(G_2)\} \).

**Proof.** Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be a graphs. Let \( G \) be the graph obtained by identifying \( u \in V_1 \) and \( v \in V_2 \), that is \( G \) is a vertex gluing
of $G_1$ and $G_2$. Let $f_1 = \left( V_1^{(1)}, V_2^{(1)}, \ldots, V_k^{(1)} \right)$ and $f_2 = \left( V_1^{(2)}, V_2^{(2)}, \ldots, V_k^{(2)} \right)$ be $r$-dynamic $k_1$-coloring of $G_1$ and $r$-dynamic $k_2$-coloring of $G_2$, respectively, such that $k_1 = \chi_r(G_1)$ and $k_2 = \chi_r(G_2)$. Consider the following cases: (Case 1. $\text{deg}_{G_1}(u) = 1$ or $\text{deg}_{G_2}(v) = 1$) Without loss of generality assume that $\text{deg}_{G_1}(u) = 1$. Let $w$ be a neighborhood of $u$ in $G_1$ and $z$ be a neighborhood of $v$ in $G_2$. Without loss of generality, assume that $u \in V_1^{(1)}$, $w \in V_2^{(1)}$, $v \in V_s$ and $z \in V_t^{(2)}$. Then by Corollary 4.2, $f_3 = \left( V_1^{r(1)}, V_2^{r(2)}, \ldots, V_{k_2}^{r(2)} \right)$ with

$$i_m = \begin{cases} 1 & \text{if } m = s \\ s & \text{if } m = 1 \\ 3 & \text{if } m = t \\ t & \text{if } m = 3 \\ m & \text{otherwise} \end{cases}$$

is also an $r$-dynamic $k$-coloring of $G_2$. Now, let $f : V(G) \to \{1, 2, \ldots, \max \{\chi_2(G_1), \chi_2(G_2)\}\}$ be given by

$$f(w) = \begin{cases} f_1(w) & \text{if } w \in V_1 \\ f_3(w) & \text{if } w \in V_2, \end{cases}$$

that is,

$$f = \left( V_{j_1}, V_{j_2}, V_{j_3}, \ldots, V_{j_{k_1}} \right)$$

$$= \left( V_1^{(1)} \cup V_{i_s}^{(2)}, V_2^{(1)} \cup V_{i_2}^{(2)}, V_3^{(1)} \cup V_{i_t}^{(2)}, \ldots, V_{k_2}^{(1)} \cup V_{i_{k_2}}^{(2)}, V_{k_2+1}^{(1)} \right)$$

(without loss of generality we assume here that $k_1 \geq k_2$). Let $xy \in E(G)$. Then either $xy \in E_1$ or $xy \in E_2$. Without loss of generality we assume that $xy \in E_1$. Since $f_1 = \left( V_1^{(1)}, V_2^{(1)}, \ldots, V_{k_1}^{(1)} \right)$ is an $r$-dynamic $k$-coloring, by Lemma 4.1, \{x, y\} \not\subseteq V_i^{(1)} for all $i = 1, 2, \ldots, k_1$. Since $xy \in E_1$, $x, y \in V_1$. Hence, there exists $s, t \in \{1, 2, \ldots, k_1\}$ with $s \neq t$ such that $x \in V_s$ and $y \in V_t$, that is, $x \in V_{j_s}$ and $y \in V_{j_t}$. Since is $\{V_{j_1}, V_{j_2}, V_{j_3}, \ldots, V_{j_{k_1}}\}$ a pairwise disjoint family, we have \{x, y\} \not\subseteq V_i^{(1)} for all $i = 1, 2, \ldots, k_1$.

Next, let $y \in V(G)$. Consider the following subcases: (Subcase 1. either $y \in V_1 \setminus V_2$ or $y \in V_2 \setminus V_1$). Without loss of generality, assume that $y \in V_1 \setminus V_2$, say $y \in V_{j_r}$. Then since $f_1$ is an $r$-dynamic $k$-coloring, by Lemma 4.1 there exist $s, t \in \{1, 2, \ldots, k_1\} \setminus \{r\}$ such that $V_{j_s}^{(1)} \cap N_{G_1}(y) \neq \emptyset$ and $V_{j_t}^{(1)} \cap N_{G_1}(y) \neq \emptyset$. This implies that there exist $s, t \in \{1, 2, \ldots, k_1\} \setminus \{r\}$ such that $V_{j_s} \cap N_{G}(y) \neq \emptyset$ and $V_{j_t} \cap N_{G}(y) \neq \emptyset$. 
(Subcase 2. $y \in V_1 \cap V_2$) If $y \in V_1 \cap V_2$, then $y = v = u$. Since $w \in V_2^{(1)} \subseteq V_{j_2}$ and $z \in V_3^{(2)}$, we have $V_{j_2} \cap N_G(y) \neq \emptyset$ and $V_{j_3} \cap N_G(y) \neq \emptyset$.

(Case 2. $deg_{G_1}(u) > 1$ or $deg_{G_2}(v) > 1$) Without loss of generality, assume that $u \in V_1^{(1)}$ and $v \in V_2^{(2)}$. Then by Corollary 4.2, $f_3 = \left\{ V_{i_1}^{(2)}, V_{i_2}^{(2)}, \ldots, V_{i_{k_2}}^{(2)} \right\}$ with

$$i_m = \begin{cases} 1 & \text{if } m = 2 \\ 2 & \text{if } m = 1 \\ m & \text{otherwise} \end{cases}$$

is also an 2-dynamic $k$-coloring of $G_2$. Now, let $f : V(G) \to \{1, 2, \ldots, \max\{\chi_2(G_1), \chi_2(G_2)\}\}$ be given by

$$f(w) = \begin{cases} f_1(w) & \text{if } w \in V_1 \\ f_3(w) & \text{if } w \in V_2, \end{cases}$$

that is,

$$f = \left( V_{j_1}, V_{j_2}, V_{j_3}, \ldots, V_{j_{k_1}} \right)$$

$$= \left( V_1^{(1)} \cup V_{i_1}^{(2)}, V_2^{(1)} \cup V_{i_2}^{(2)}, \ldots, V_{k_2}^{(1)} \cup V_{i_{k_2}}^{(2)} ; V_{k_2+1}^{(1)}, \ldots, V_{k_1}^{(1)} \right)$$

(without loss of generality we assume here that $k_1 \geq k_2$). Let $xy \in E(G)$. Then either $xy \in E_1$ or $xy \in E_2$. Without loss of generality we assume that $xy \in E_1$. Since $f_1 = \left( V_1^{(1)}, V_2^{(1)}, \ldots, V_{i_1}^{(1)} \right)$ is an $r$-dynamic $k$-coloring, by Lemma 4.1, \(\{x, y\} \not\subseteq V_i^{(1)}\) for all $i = 1, 2, \ldots, k_1$. Since $xy \in E_1$, $x, y \in V_1$. Hence, there exists $s, t \in \{1, 2, \ldots, k_1\}$ with $s \neq t$ such that $x \in V_s$ and $y \in V_t$, that is, $x \in V_{j_s}$ and $y \in V_{j_t}$. Since is \(\{V_{j_1}, V_{j_2}, V_{j_3}, \ldots, V_{j_{k_1}}\}\) a pairwise disjoint family, we have \(\{x, y\} \not\subseteq V_{j_i}\) for all $i = 1, 2, \ldots, k_1$.

Next, let $y \in V(G)$. Without loss of generality, assume that $y \in V_1$, say $y \in V_{r_1}^{(1)}$, that is $y \in V_{r_1}$. Then since $f_1$ is an $r$-dynamic $k$-coloring, by Lemma 4.1 there exist $s, t \in \{1, 2, \ldots, k_1\} \setminus \{r\}$ such that $V_s^{(1)} \cap N_G(y) \neq \emptyset$ and $V_t^{(1)} \cap N_G(y) \neq \emptyset$. This implies that there exist $s, t \in \{1, 2, \ldots, k_1\} \setminus \{r\}$ such that $V_{j_s} \cap N_G(y) \neq \emptyset$ and $V_{j_t} \cap N_G(y) \neq \emptyset$.

Accordingly, by Lemma 4.1 $f$ is a 2-dynamic $k$-coloring. Therefore, $\chi_2(G) \leq k_1 = \max\{\chi_r(G_1), \chi_r(G_2)\}$.

Suppose that $\chi_2(G) < k_1 = \max\{\chi_r(G_1), \chi_r(G_2)\}$. Note that $G_1$ is a subgraph of $G$. Thus by Corollary 4.5, $\chi_2(G_1) \leq \chi_2(G)$. This is a contradiction.

\[\square\]

Theorem 4.6 may be extended to the vertex gluing of a finite number of graphs.
Corollary 4.7. Let $n \in \mathbb{N}$ and $G_1, G_2, \ldots, G_n$ be graphs. Let $G$ be the vertex gluing of $G_1, G_2, \ldots, G_n$. If $r = 2$, then

$$
\chi_2(G) = \max \{\chi_r(G_1), \chi_r(G_2), \ldots, \chi_r(G_n)\}
$$

Proof. This follows from Theorem 4.6 by induction.

5. $r$-Dynamic Chromatic Number of Forest

This section presents the $r$-dynamic chromatic number of forest. We note that a forest is a finite union of trees, and a tree is a finite vertex gluing of paths. The next remark gives the 2-dynamic chromatic number of trees. An observation in [8] states that $\chi_r(G) \geq \min \{\Delta(G), r\} + 1$ where equality holds when $G$ is a tree. The next remark verified this idea for $r = 2$.

Remark 5.1. Let $T$ be a tree of order $n \geq 3$. If $r = 2$, then $\chi_r(T) = 3$.

Proof. Assume that $T$ is the vertex gluing of $n$ paths $P_1, P_2, \ldots, P_n$. Then by Corollary 4.7,

$$
\chi_2(T) = \max \{\chi_r(P_1), \chi_r(P_2), \ldots, \chi_r(P_n)\} = 3.
$$

The next result gives the $r$-dynamic chromatic number of the union of two graphs.

Theorem 5.2. Let $G$ and $H$ be graphs. Then

$$
\chi_r(G \cup H) = \max \{\chi_r(G), \chi_r(H)\}.
$$

Proof. Let

$$
f_1 : V(G) \to \{1, 2, \ldots, \chi_r(G)\} \text{ and } f_1 : V(G) \to \{1, 2, \ldots, \chi_r(H)\}
$$

be $r$-dynamic $k$-colorings of $G$ and $H$, respectively. Define $f : V(G \cup H) \to \{1, 2, \ldots, \max \{\chi_r(G), \chi_r(H)\}\}$ by

$$
f(v) = \begin{cases} f_1(v) & \text{if } v \in V(G) \\ f_2(v) & \text{if } v \in V(H). \end{cases}
$$
(Claim 1. $f$ is a proper vertex coloring) Let $uv \in E(G \cup H)$. Then either $uv \in E(G)$ or $uv \in E(H)$. Without loss of generality, assume that $uv \in E(G)$. If $uv \in E(G)$, then since $f_1$ is proper, $f_1(u) \neq f_1(v)$, that is, $f(u) \neq f(v)$. This shows the claim.

(Claim 2. $|f(N(v))| \geq \min\{r, \deg_{G \cup H}(v)\}$ for all $v \in V(G \cup H)$) Let $v \in V(G \cup H)$. Then either $v \in V(G)$ or $v \in V(H)$. Without loss of generality, assume that $v \in V(G)$. If $v \in V(G)$, then since $f_1$ is an $r$-dynamic $k$-coloring, $|f_1(N(v))| \geq \min\{r, \deg_{G}(v)\}$, that is, $|f(N(v))| \geq \min\{r, \deg_{G \cup H}(v)\}$. This shows the claim.

By Claims 1 and 2, $\chi_r(G \cup H) \leq \max\{\chi_r(G), \chi_r(H)\}$. And by Corollary 4.5, $\chi_r(G \cup H) = \max\{\chi_r(G), \chi_r(H)\}$. □

Theorem 5.2 may be extended to finite union of graphs.

**Corollary 5.3.** Let $n \in \mathbb{N}$ and $G_1, G_2, \ldots, G_n$ be graphs. Then

$$\chi_r(G_1 \cup G_2 \cup \cdots \cup G_n) = \max\{\chi_r(G_1), \chi_r(G_2), \ldots, \chi_r(G_n)\}.$$ 

We revisit an observation in [8] which states that $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$ where equality holds when $G$ is a tree. Since a forest is a finite union of trees, the next result gives the $r$-dynamic chromatic number of forest.

**Corollary 5.4.** Let $F$ be a forest with a component of order $n \geq 3$. Then $\chi_r(F) = \min\{\Delta(G), r\} + 1$.

**Proof.** Assume that $F$ is the union of $n$ trees $T_1, T_2, \ldots, T_n$. Then by Corollary 5.3,

$$\chi_r(F) = \max\{\chi_r(T_1), \chi_r(T_2), \ldots, \chi_r(T_n)\} = \min\{\Delta(G), r\} + 1.$$ □

**References**


