ORBIT OF TUPLE OF OPERATORS TENDING TO INFINITY

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Abstract: In this paper we prove that there is a dense set of vectors in $X$ whose orbits under the tuple $T = (T_1, T_2, ..., T_n)$ of commutative bounded linear operators on a infinite dimensional (real, complex) Banach space $X$ tend to infinity.

Key Words: tuple of operators, orbit, spectral radius, spectrum, point spectrum, approximate point spectrum, bounded below operator

1. Introduction

By an $n$-tuple of operators we mean a finite sequence of length $n$ of commuting bounded linear operators on a Banach space $X$.

Throughout, $X$ denotes a infinite dimensional Banach space and $B(X)$ denotes the Banach algebra of all bounded linear operators on $X$. The orbit of a points $x \in X$ under an operator $T \in B(X)$ is the sequence

$$\{T^n x : n = 0, 1, ...\}.$$

For $T \in B(X)$, with $r(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ will denote the spectral radius, the spectrum, the point spectrum and the approximate point spectrum of $T$, respectively. Recall that $\sigma_p(T)$ is the set of all eigenvalues of $T$ while $\sigma_{ap}(T)$ is the set of $\lambda \in \sigma(T)$ for there which there is a sequence of unit vectors
\((x_n)_{n \geq 1}\) such that \(\|Tx_n - \lambda x_n\| \to 0\), as \(n \to \infty\) any such sequence is called a sequence of almost eigenvectors for \(\lambda\).

We are interested in the operators \(T \in B(X)\) for which there is \(x \in X\) whose \(\text{Orb}(T, x)\) tends strongly to infinity, i.e.

\[
\|T^n x\| \to \infty, \text{ as } n \to \infty.
\]

Obviously, if \(\sigma_p(T)\) contains a point \(\lambda\) with \(|\lambda| > 1\), then for every corresponding nonzero vector \(x\) in the eigenspace \(\text{Ker}(T - \lambda)\), the orbit will tend strongly to infinity:

\[
\|T^n x\| = |\lambda^n| \|x\| \to \infty, \text{ as } n \to \infty.
\]

In general, \(\text{Ker}(T - \lambda)\) is not dense in \(X\) (relative to the norm topology). In order to produce a dense set of vectors in \(X\) whose orbits under \(T\) tend strongly to infinity we have to look at the points in the approximate point spectrum which are not eigenvalues.

In [5] S. Mancevska gave a complete proof of the following theorem (originally stated by B. Beauzamy [9, Theorem 2.A.5]): If \(T \in B(X)\) and the circle \(\{\lambda \in \mathbb{C} / |\lambda| = r(T)\}\) contains a point in \(\sigma(T)\) which is not an eigenvalue for \(T\), then for every positive sequence \((\alpha_n)_{n \geq 1}\) with \(\sum_{n \geq 1} \alpha_n < +\infty\), in every open ball in \(X\) with radius strictly larger than \(\sum_{n \geq 1} \alpha_n\), there is \(x \in X\) satisfying

\[
\|T^n x\| \geq \alpha_n \frac{r(T)^n}{2} \text{ for all } n \geq 1.
\]

Note that, if \(r(T) > 1\), then the space will contain a dense set of vectors \(x \in X\) with orbits under \(T\) tending strongly to infinity.

If \(r(T)\) is replaced with \(|\lambda|\), for any \(\lambda \in \sigma_{ap}(T) \setminus \sigma_p(T)\). Thus we have.

**Theorem 1.2.** (see [1], Corollary 3.2.) Let \(X\) be an infinite dimensional reflexive Banach space and \(T \in B(X)\) and \(S \in B(X)\).

If the sets \(\sigma_{ap}(T) \setminus \sigma_p(T)\) and \(\sigma_{ap}(S) \setminus \sigma_p(S)\) both have a non-empty intersection with the domain \(\{\lambda \in \mathbb{C} / |\lambda| > 1\}\) then, there is a dense set of vectors \(x \in X\) such that both the orbits \(\text{Orb}(T; x)\) and \(\text{Orb}(S; x)\) tend strongly to infinity.

In [2] S. Mancevska and M. Orovrance considered some conditions under which, given a sequence of bounded linear operators \((T_i)_{i \geq 1}\) on an infinite-dimensional complex reflexive Banach space \(X\), and they show that there is a dense set of vectors in \(X\) whose orbits under each \(T_i\) tend strongly to infinity.

**Theorem 1.3.** (see [2], Corollary 10) Let \(X\) an infinite dimensional reflexive Banach space. If \((T_i)_{i \geq 1}\) is a sequence in \(B(X)\) for which there is \(\beta > 0\) such that \(r(T_i) > 1 + \beta\) for all \(i \geq 1\) then, there is a dense set \(D\) in \(X\) such that \(\text{Orb}(T_i; x)\) tend strongly to infinity for every \(x \in D\) and \(i \geq 1\).
If \((T_i)_{i \geq 1}\) is a sequence in \(B(X)\) satisfying the following, weaker condition then the one in Theorem 0.3.

\[
\sigma_{ap}(T_i) \setminus \sigma_a(T_i) \cap \{ \lambda \in \mathbb{C} / |\lambda| \gg 1 \} \neq \emptyset, \quad \text{for all } i \geq 1. \quad (*)
\]

the space may still contain a dense set of vectors with orbits under each \(T_i\), \(i \geq 1\) tending strongly to infinity.

2. Main Results

**Definition 1.1.** Let \(\mathcal{T} = (T_1, T_2, ..., T_n)\) be an \(n\)-tuple of operators acting on an infinite dimensional Banach space \(X\).

Let

\[
F = \left\{ T_1^{k_1} T_2^{k_2} ... T_n^{k_n} : k_i \geq 0, i = 1, ..., n \right\}.
\]

be the semi-group generated by \(\mathcal{T}\). For \(x \in X\), the orbit of \(x\) under the tuple \(\mathcal{T}\) is the set

\[
\text{Orb}(\mathcal{T}, x) = \{ Sx : S \in F \}.
\]

**Definition 1.2.** The orbit of \(x\) under the tuple \(\mathcal{T}\) tending to infinity if:

\[
\left\| T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x \right\| \to \infty
\]

as \(k_i \to \infty\) with \(k_i \geq 0\), for all \(i = 1, ..., n\).

In this paper are considered some conditions under which, The orbit of \(x\) under the tuple \(\mathcal{T}\) tending to infinity. For simplicity we state and prove our results for a pair that is a tuple with \(n = 2\), and the general case follows by a similar method.

**Definition 1.3.** An operator \(T\) is bounded from below if and only there exists a constant \(C > 0\) such that:

\[
\|Tx\| \geq C \|x\|, \text{ for all } x \in X.
\]

**Theorem 1.4.** Let \(X\) an infinite dimensional reflexive Banach space and \(T = (T_1, T_2)\) be the pair of operators \(T_1\) and \(T_2\).

Suppose that the following conditions hold true:

1) \(T_1\) and \(T_2\) are bounded from below.
2) The sets $\sigma_{ap}(T) \setminus \sigma_p(T)$ and $\sigma_{ap}(S) \setminus \sigma_p(S)$ both have a non-empty intersection with the domain $\{ \lambda \in \mathbb{C} / |\lambda| > 1 \}$.

Then there exists $x \in X$ such that the orbit of $x$ under the pair $T$ tends strongly to infinity.

Proof. Using condition 2 we may apply Theorem 1.2. Then, there exists a dense set of vectors $x \in X$ such that both the orbits $\text{Orb}(T_1; x)$ and $\text{Orb}(T_2; x)$ tend strongly to infinity.

Hence $T_1$ and $T_2$ are bounded from below or,

$$\left\| T_1^{k_1} T_2^{k_2} x \right\| = \left\| T_1^{k_1} (T_2^{k_2} x) \right\| \geq C_1 \left\| T_2^{k_2} x \right\| \to \infty, \text{ as } k_2 \to \infty,$$

and

$$\left\| T_1^{k_1} T_2^{k_2} x \right\| = \left\| T_2^{k_2} T_1^{k_1} x \right\| = \left\| T_2^{k_2} (T_1^{k_1} x) \right\| \geq C_2 \left\| T_1^{k_1} x \right\| \to \infty, \text{ as } k_1 \to \infty.$$

Then

$$\left\| T_1^{k_1} T_2^{k_2} x \right\| \to \infty \text{ as } k_1 \to \infty, \text{ and } k_1 \to \infty,$$

i.e. $\text{Orb}(T, x)$ tend strongly to infinity.

Corollary 1.1. Let $X$ an infinite dimensional reflexive Banach space and $T = (T_1, T_2, ... , T_n)$ be the $n$-tuple of operators in $B(X)$ bounded below for all $i \geq 1$.

If there is $x \in X$ such that the orbit of $x$ under $T_i$ for all $i \geq 1$ tend strongly to infinity then the orbit of $x$ under the tuple $T$ tend strongly to infinity.

Remark 1.1. The converse is also true, i.e. If there is $x \in X$ such that the orbit of $x$ under the tuple $T$ tend strongly to infinity then the orbit of $x$ under $T_i$ for all $i \geq 1$ tend strongly to infinity.

Proof. $T_i$ are the commuting bounded linear operators then,

$$\left\| T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x \right\| \leq \prod_{i=1, i \neq j}^{n} \left\| T_i^{k_i} \right\| \left\| T_j^{k_j} x \right\|,$$

or

$$\left\| T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x \right\| \to \infty \text{ as } k_j \to \infty \text{ for all } j \geq 1.$$
Therefore, \( \text{Orb}(T_j, x) \to \infty \) for all \( j \geq 1 \). \qed

**Example 1.1.** Let \( S \) be the unilateral forward shift on \( \ell^2(\mathbb{N}) \):

\[
Se_n = e_{n+1}, \quad i \geq 1,
\]

where \( \{e_n : n \in \mathbb{N}\} \) is the standard orthonormal basis for \( \ell^2(\mathbb{N}) \).

Given a sequence of positive numbers \((a_i)_{i \geq 1}\) so that \( a_i \to 1 \) as \( i \to \infty \) and \( a_i \geq 1 \) for all \( i \geq 1 \) and let

\[
T_i = a_i S, \quad i = 1, \ldots, n.
\]

\( T_i \) is unilateral injective forward weighted shift and hence (see [10], Theorem 6):

\[
\sigma_p(T_i) = \emptyset \text{ and } \sigma_{ap}(T_i) = \{\lambda \in \mathbb{C} : |\lambda| = a_i\}.
\]

Then the set \( \sigma_{ap}(T_i) \setminus \sigma_p(T_i) \) have a non-empty intersection with the domain \( \lambda \in \mathbb{C} : |\lambda| \geq 1 \).

Obviously, \( (T_i)_{i \geq 1} \) satisfies the weaker condition (*) and there exists a dense set of vectors in \( \ell^2(\mathbb{N}) \) with orbits in each \( T_i \) tending strongly to infinity.

Actually

\[
\|T_i^n x\| = \|(a_i S)^n x\| = a_i^n \|x\| \to \infty, \quad n \to \infty, \quad \text{for all } x \neq 0 \text{ and } i \geq 1.
\]

Moreover, \( T_i \) is bounded from below. Indeed:

\[
\|T_i x\| = \|a_i S x\| \geq \|S x\| = \|x\|.
\]

Therefore, by the use of Corollary 1.1, the orbit of \( x \) in the tuple \( (T_1, T_2, \ldots, T_3) \) tends strongly to infinity.

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