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# CONNECTIVITIES FOR A SYMMETRIC PRETOPOLOGY

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**Abstract:** In this paper, we present properties of connectivities in the case of a symmetric pretopology.

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## 1. Introduction

We already presented how pretopology generalizes both graph theory and topology, see [1], [2], [3]. We also established links between pretopology, matroids and hypergraphs, see [5].

Here, we present results about strong connectivity [3] and connectivity [6] related to a symmetric pretopology.

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## 2. Different Types of Pretopological Spaces, see [1], [2], [3]

**Definition 1.** Let X be a non empty set. P(X) denotes the family of subsets of X. We call pseudoclosure on X any mapping $a$ from P(X) onto P(X) such as:

$$a(\emptyset) = \emptyset,$$
$$\forall\ A\ \subset\ X, \quad A \subset a(A),$$

$(X, a)$ is then called pretopological space.

We can define 4 different types of pretopological spaces.

1. (X, $a$) is a $V$ type pretopological space if and only if

$$\forall\ \mathrm{A} \subset\ \mathrm{X}, \forall\ \mathrm{B} \subset\ \mathrm{X}, \mathrm{A} \subset\ \mathrm{B} \Rightarrow\ a\ (\mathrm{A}) \subset\ a\ (\mathrm{B}).$$

2. (X, $a$) is a $V_D$ type pretopological space if and only if

$$\forall\ \mathrm{A} \subset\ \mathrm{X}, \forall\ \mathrm{B} \subset\ \mathrm{X},\ a\ (\mathrm{A} \cup\ \mathrm{B}) =\ a\ (\mathrm{A}) \cup a\ (\mathrm{B}).$$

3. (X, $a$) is a $V_S$ type pretopological space if and only if

$$\forall\ \mathrm{A} \subset\ \mathrm{X},\ a(A) = \bigcup_{x \in A} a(\{x\}).$$

4. (X, $a$) a $V_D$ type pretopological space, is a topological space if and only if $\forall$ A $\subset$ X, $a$ ($a$ (A)) = $a$ (A).

**Property 2.** *If (X, a) is a $V_S$ space then (X, a) is a $V_D$ space. If (X, a) is a $V_D$ space then (X, a) is a $V$ space.*

**Example 3.** Let X be a non empty set and $R$ be a binary relationship defined on X. The pretopology of ascendant-descendants, noted $a_{ad}$, is defined by:

$$\forall\ A \subset\ X,\ a_{ad}\ (A) = \{\ x \in\ X /\ R^{-1}(x) \cap A \neq \emptyset\ \ and\ R(x) \cap A \neq \emptyset\ \} \cup A.$$

This pretopology is only $V$ one.

## 3. Different Pretopological Spaces Defined from a Space (X,$a$) and Closures, see [1], [2], [4]

**Definition 4.** Let (X, $a$) be a $V$ pretopological space. Let A $\subset$ X. A is a closed subset if and only if $a$ (A) = A.

We note $\forall$ A$\subset$ X, $a^0$ (A) = A and $\forall$ n, n$\geq$ 1, $a^n$(A) = $a(a^{n-1})$(A).

We name closure of A the subset of X, denoted $F_a$(A), which is the smallest closed subset which contains A.

F'$_a$, the inverse of the closure generated by $a$, is defined by:

$$\forall \text{ A} \subset \text{ X}, \text{ F'}_a \text{ (A)} = \{ \text{ x} \in \text{ X/ F}_a \text{ ( \{ x\})} \cap \text{A} \neq \emptyset \} .$$

We note $a$" = F'$_a$F$_a$ ($a$" is the composed of the mapping F'$_a$ and F$_a$) and F"$_a$ the closure according to $a$".

**Remark 5.** F$_a$(A) is the intersect of all closed subsets which contain A. In the case where (X, a) is a "general" pretopological space ( i.e. is not a $V$ space , nor a $V_D$ space, nor a $V_S$ space, nor a topological space) , the closure may not exist.

**Proposition 6.** *Let (X, a) be a $V$ space. Let A $\subset$ X. If one of the two following conditions is fulfilled:*
*– X is a finite set;*
*– a is of $V_S$ type.*

*Then* $F_a(A) = \bigcup_{n\geq 0} a^n(A)$.

**Remark 7.** If a is of $V$ type then a$^n$, F$_a$, a", F"$_a$ also are of $V$ type and F'$_a$ is of $V_S$ type. If a is of $V_S$ type then a$^n$, F$_a$, a", F"$_a$, F'$_a$ are also of $V_S$ type.

**Definition 8.** Let (X, $a$) be a $V$ pretopological space. Let A $\subset$ X. $a$' inverse of pseudoclosure $a$ is defined as follows:

$$\forall \text{ A} \subset \text{ X}, a\text{'(A)} = \{ \text{ x} \in \text{ X/ } a\text{( \{ x\})} \cap \text{A} \neq \emptyset \}.$$

$a$' is a pseudoclosure defined on P(X). We denote F$_{a'}$ the closure according to

$a$' and F'$_{a'}$ the inverse of the closure according to $a$' (i.e. of F$_{a'}$) . We also denote F"$_{a'}$ the closure according to ($a$')" = F'$_{a'}$F$_{a'}$.

**Remark 9.** (see [7]) Let (X, a) be a pretopological space. If a is of $V$ type then a' is of $V_S$ type.

**Definition 10.** Let (X, $a$) be a $V$ pretopological space. Let A ⊂ X.
We define the induced pretopology on A by $a$, denoted $a_A$, by:

$$\forall \text{ C} \subset \text{ A}, \ a_A \text{ (C)} = a \text{ (C)} \cap \text{A}.$$

(A, $a_A$ ) ( or more simply A ) is said pretopological subspace of (X, $a$) . We denote $\text{F}^{\circ}{}_A$ the closure according to $a_A$.

## 4. Strong Connectivity and Connectivity in (X,$a$), see [1], [2], [7]

**Definition 11.** Let (X, $a$) be a $V$ pretopological space. Let A a non empty subset of X. Let B a non empty subset of X.
There exists a path in (X, $a$) from B to A if and only if B ⊂ $\text{F}_a$(A).

There exists a chain in (X, $a$) from B to A if and only if B ⊂ $\text{F"}_a$(A).
We shall speak about x-connectivity to indicate one of the following types of connectivity:

**Definition 12.** Let (X, $a$) be a $V$ pretopological space.
(X, $a$) is strongly connected if and only if ∀ C ⊂ X, C ≠ ø, $\text{F}_a$(C) = X.
(X, $a$) is connected if and only if ∀C ⊂ X, C≠ ø , $\text{F}_a$(C) = X or $\text{F}_a$(X-$\text{F}_a$(C)) ∩ $\text{F}_a$(C) ≠ ø .

**Proposition 13.** (see [1]) *Let (X, a) be a V pretopological space. (X,a) is strongly connected ⇔ ∀A ⊂ X, A≠ ø, ∀B ⊂ X , B ≠ ø, there exists a path from B to A in (X,a) .*

**Definition 14.** Let (X, $a$) be a $V$ pretopological space. Let A ⊂ X with A non empty.

A is a x-connected subspace of (X,$a$) if and only if (A, $a_A$) , as a pretopological space, is x-connected.
(A, $a_A$) is a greatest x-connected subspace of (X, $a$) if and only if (A, $a_A$) is a x-connected subspace of (X, $a$) and ∀ B, A ⊂ B ⊂ X and A≠ B, (B, $a_B$) is not a x-connected subspace of (X, $a$).

**Definition 15.** Let (X, $a$) be a $V$ pretopological space. Let A ⊂ X with A non empty.

We note $(F_a)_A$ the closing obtained by restriction of closing $F_a$ on A. $(F_a)_A$ is such as $\forall$ C$\subset$ A, $(F_a)_A$ (C) = $F_a$(C) $\cap$A.

A is a x-connected subset of (X, $a$) if and only if A endowed with $(F_a)_A$ is x-connected.

A is a x-connected component of (X, $a$) if and only if A is a x-connected subset of (X, $a$) and $\forall$ B, A $\subset$ B $\subset$ X with A$\neq$ B, B is not a x-connected subset of (X, $a$).

## 5. Symmetric Pretopology

**Definition 16.** Let (X, $a$) be a pretopological space. $a$ is symmetric if and only if $\forall$ x $\in$ X, $\forall$ y $\in$ X, y $\in$ $a$( { x}) $\Leftrightarrow$ x $\in$ $a$( { y}) .

**Remark 17.** The pretopology of ascendant-descendants is symmetric.

*Proof.* We have

$$\begin{aligned} y \in a_{ad}(\{x\}) &\Leftrightarrow (R^{-1}(y) \cap \{x\} \neq \emptyset \text{ and } R(y) \cap \{x\} \neq \emptyset) or (x = y) \\ &\Leftrightarrow (x \in R^{-1}\ (y) \text{ and } x \in R(y)) or (x = y) \\ &\Leftrightarrow (y \in R(x) \text{ and } y \in R^{-1}(x)) or (x = y) \\ &\Leftrightarrow (R^{-1}(x) \cap \{y\} \neq \emptyset \text{ and } R(x) \cap \{y\} \neq \emptyset) or (x = y) \\ &\Leftrightarrow x \in a_{ad}(\{y\}). \end{aligned}$$

**Proposition 18.** *Let (X, a) be a V pretopological space.*

$$a \textit{ is symmetric } \Leftrightarrow \forall x \in X, a'(\{x\}) = a(\{x\}).$$

*Proof.* If $a$ is symmetric

$$\begin{aligned} a\text{'}(\{x\}) &= \{y \in X/a(\{y\}) \cap \{x\} \neq \emptyset\} \\ &= \{y \in X/x \in a(\{y\})\} \\ &= \{y \in X/y \in a(\{x\})\} (a \text{ symmetric}) \\ &= a(\{x\}). \end{aligned}$$

Conversely, if $\forall$ x $\in$ X, $a$'( { x}) = $a$( { x}), then y $\in$ $a$'( { x}) $\Leftrightarrow$ y $\in$ $a$( { x}). So $a$( { y}) $\cap$ { x}$\neq$ ∅ $\Leftrightarrow$ y $\in$ $a$( { x}), and x $\in$ $a$( { y}) $\Leftrightarrow$ y $\in$ $a$( { x}).

Then $a$ is symmetric.

**Proposition 19.** *Let (X, a) be a pretopological space with a symmetric.*
*(i) If a is of $V$ type,*

$$\forall A \subset X, a'(A) = (a')'(A) \subset a(A).$$

*(ii) If a is of $V_S$ type, a' = a.*

*Proof.* (i) We have

$$\begin{aligned} a'(A) =& \{x \in X / a(\{x\}) \cap A \neq \emptyset\} \\ =& \{x \in X / a'(\{x\}) \cap A \neq \emptyset\} \text{ (see Proposition 18)} \\ =& (a')'(A) \\ \subset& a(A) \text{ (see [7])} . \end{aligned}$$

(ii) If $a$ is of $V_S$ type, $(a')' = a$ ([7]) and the result according to (i).

**Proposition 20.** *Let (X, a) be a pretopological space with a symmetric. Let n be an integer.*
*(i) If a is of $V$ type,*

$$\forall A \subset X, (a')^n(A) \subset a^n(A).$$

*(ii) If a is of $V_S$ type , $(a')^n = a^n = (a^n)'$.*

*Proof.* (i) By using recurrence:
– This is true for n=1 (Proposition 19(i)).
– Let us suppose it is true for n. We receive:

$$\begin{aligned} (a')^n(A) &\subset a^n(A) \\ (a')^{n+1}(A) &= (a')(a')^n(A) \\ &\subset a'(a^n(A)) \text{ (because the property is true for n)} \\ &\subset a(a^n(A)) \text{ (Proposition 19(i))} \\ &\subset a^{n+1}(A). \end{aligned}$$

(ii) Obvious because $a = a'$ and because $(a^n)' = (a')^n$ (Proposition 19(ii) and [7]).

**Remark 21.** Generally speaking, if a is of $V$ type, $a^n(A) \neq (a^n)'(A)$.

**Exemple 22.** Let (X, a) be a $V$ pretopological space with X = { a, b, c, d, e, f } and a the pretopology of ascendant-descendants defined according to data in the following table:

| x | R(x) |
|---|---|
| a | {b,c,e} |
| b | {a} |
| c | {b,d} |
| d | {a,e} |
| e | {f} |
| f | {d,e} |

Let A = { a }.

We get $a^3(A) = a^3$ ( { a }) = $a^2$ ( { a, b }) = a( { a, b, c }) = { a, b, c, d } and

$$\begin{aligned}(a^3)'(A) =& \{x \in \ X/a^3(\{x\}) \cap A \neq \varnothing\} \\ =& \{x \in \ X/a \in a^3(\{x\})\} \\ =& \{a, b, e, f\},\end{aligned}$$

and then $a^3$ (A) $\neq$ $(a^3)$' (A).

**Proposition 23.** *Let (X, a) be a pretopological space with a symmetric.*
*(i) If a is of $V$ type, $\forall$ $A \subset \ X$, $F_{a'}(A) \subset \ F_a(A)$.*
*(ii) If a is of $V_S$ type, $F_{a'} = F_a = F'_a$.*

*Proof.* (i) We receive

$$\begin{aligned}F_{a'}(A) =& \bigcup_{n\geq 0}(a')^n(A) \ (a' \text{ is of } V_S \text{ type}) \\ \subset& \bigcup_{n\geq 0} a^n(A) \ (\text{Proposition 20(i)}) \\ \subset& F_a(A).\end{aligned}$$

(ii) $F_{a'} = F_a$ because *a' = a* (see Proposition 19(ii)).
Moreover, $F'_a = F_{a'}$ (see [7]).

**Proposition 24.** *Let (X, a) be a $V$ pretopological space. Let $A \subset \ X$ with A non empty.*

*If a is symmetric then $a_A$ is symmetric.*

*Proof.* Let x ∈ A and y ∈ A. Then

$$\begin{aligned} y \in a_A(\{x\}) &\Leftrightarrow y \in (a(\{x\}) \cap A) \\ &\Leftrightarrow y \in a(\{x\}) \text{ and } y \in A \\ &\Leftrightarrow x \in a(\{y\})(a \text{ is symmetric}) \\ &\Leftrightarrow x \in (a(\{y\}) \cap A) \\ &\Leftrightarrow x \in a_A(\{y\}). \end{aligned}$$

## 6. Decomposition of (X,*a*) with *a* Symmetric

**Proposition 25.** *Let (X, a) be a V pretopological space with a symmetric. Let x ∈ X and let y ∈ X.*

*There exists a sequence $x_0,\ldots, x_n$ of elements of X such as $x_0$= x, $x_n$= y with ∀j = 0,..., n-1,$x_{j+1}$ ∈ a ({$x_j$})*

⇔ *there exists a sequence $x_0,\ldots, x_n$ of elements of X such as $x_0$= x, $x_n$= y with ∀j = 0,..., n-1, $x_{j+1}$ ∈ a ({$x_j$}) or $x_j$ ∈ a ({$x_{j+1}$}).*

*Proof.* Obvious according to the definition of symmetry.

**Corollary 26.** *Let (X, a) be a V pretopological space with a symmetric.*

*Let { $C_i$ , i ∈ I } the family of non empty subsets of X such as:*

*(1)* $\bigcup_{i\in I} C_i$= X.

(2) ∀ i ∈ I , ∀ x ∈ $C_i$ , ∀ y ∈ $C_i$, there exists a sequence $x_0,\ldots, x_n$ of elements of X such as $x_0$= x, $x_n$= y with ∀j = 0,..., n-1, $x_{j+1}$ ∈ a ({$x_j$}).

(3) ∀ i ∈ I, ∀ k ∈ I, i ≠ k, ∀ x ∈ $C_i$, ∀ y ∈ $C_k$, there does not exist a sequence $x_0 \ldots x_n$ of elements of X such as $x_0$= x, $x_n$= y with ∀j = 0,...., n-1, $x_{j+1}$ ∈ a({$x_j$}) or there does not exist a sequence $x_0 \ldots x_n$ of elements of X such as $x_0$= y, $x_n$= x with ∀j = 0,..., n-1, $x_{j+1}$ ∈ a ( { $x_j$})

Let { $S_j$, j ∈ J } the family of non empty subsets of X such as:

*1-* $\bigcup_{j\in J} S_j$= X

*2- ∀ j ∈ J, ∀ x ∈ $S_j$, ∀ y ∈ $S_j$, there exists a sequence $x_0 \ldots x_n$ of elements of X such as $x_0$= x, $x_n$= y with ∀i = 0,..., n-1, $x_{i+1}$ ∈ a ({$x_i$}) or $x_i$ ∈ a ({$x_{i+1}$})*

*3- $\forall$ $i \in J$, $\forall$ $k \in J$, $i \neq k$, $\forall$ $x \in S_i$, $\forall$ $y \in S_k$, there does not exist a sequence $x_0 \ldots x_n$ of elements of X such as $x_0 = x$, $x_n = y$ with $\forall j = 0, \ldots, n-1$, $x_{j+1} \in a(\{x_j\})$ or $x_j \in a(\{x_{j+1}\})$*

*We have $\forall$ $i \in I$, there exists $j \in J$, $C_i = S_j$.*

*Proof.* Obvious according to the definitions of $C_i$ and $S_j$ and the Proposition 25.

**Corollary 27.** *Let (X, a) be a $V_S$ pretopological space with a symmetric.*

*Let $A \subset X$ with A non empty.*

*A is a strongly connected subspace of (X,a) $\Leftrightarrow$ A is a connected subspace of (X,a) .*

*Proof.* A is a strongly connected subspace of (X,$a$)

$\Leftrightarrow$ $\forall$ x $\in$ A and $\forall$ y $\in$ A, there exists a path from { y } to { x } in (A, $a_A$) ( [5])

$\Leftrightarrow$ $\forall$ x $\in$ A and $\forall$ y $\in$ A, there exists a chain from { y } to { x } in (A, $a_A$) ([5], Proposition 25)

$\Leftrightarrow$ A is a connected subspace of (X,$a$) ([5]).

**Corollary 28.** *Let (X, a) be a $V_S$ pretopological space with a symmetric.*

*Let $A \subset X$ with A non empty.*

*A is a greatest strongly connected subspace of (X,a) $\Leftrightarrow$ A is a greatest connected subspace of (X,a).*

*Proof.* Obvious according to Corollary 27.

**Remark 29.** Generally speaking, if (X, a) is a $V$ pretopological space with a symmetric, A is a greatest strongly connected subspace of (X,a) is not equivalent to A is a greatest connected subspace of (X,a).

**Example 30.** Let (X, a) be a pretopological space with X = { a, b, c, d, e, f } and a the pretopology of ascendant-descendants defined according to data in the following table:

| x | R(x) |
|---|---|
| a | {b,c} |
| b | {a} |
| c | {b,d} |
| d | {a,c,e} |
| e | {c,f} |
| f | {e} |

$S_1 = \{\ a,\ b\ \}$ *and* $F_a\ (S_1) = X = F"_a\ (S_1)$,
$S_2 = \{\ c,\ d\ \}$ *and* $F_a\ (S_2) = X = F"_a\ (S_2)$,
$S_3 = \{\ e,\ f\ \}$ *and* $F_a\ (S_3) = \{\ e,\ f\ \}$.

Then $F'_a\ F_a\ (S_3) = F'_a\ (\ \{\ e,f\ \}) = X$, so $F"_a\ (S_3) = X$.

We get, for each j, $F"_a\ (S_j) = X$, then $\forall$ x $\in$ X and $\forall$ y $\in$ X, x $\in$ $F"_a$ ({ y }) and X is a greatest connected subspace of (X,a) (see [5]).

Moreover, $C_1 = S_1$ ; $C_2 = S_2$ ; $C_3 = S_3$ (because a is symmetric) with $F_a\ (C_1) = X$ ; $F_a\ (C_2) = X$ and $F_a\ (C_3) = C_3$.

So A= $C_1 \cup C_2$ is a strongly connected component of (X,a) and $C_3$ is also a strongly connected component of (X,a) (see [3]).

In addition, $F^\circ_{C3}(C_3) = C_3$ so $C_3$ is a greatest strongly connected subspace of (X,a )([3]).

Also, $F^\circ_A(\ C_1) = F^\circ_A(\ C_2) = C_1 \cup C_2$ so we get A= $C_1 \cup C_2$ is a greatest strongly connected subspace of (X,a)([3]).

Consequence: Decomposing a pretopological space (X,$a$) of $V_S$ type , with $a$ symmetric, into greatest connected subspaces is equivalent to decomposing the pretopological space (X,$a$) into greatest strong connected subspaces.

**Example 31.** Let X a non empty set and $R$ a symmetric binary relationship defined on X. The pretopology of descendants, noted $a_d$, is defined by the following pseudoclosure:

$$\forall A \subset X, a_d(A) = \{x \in\ X/R(x) \cap A \neq \emptyset\} \cup A \text{ with} R(x) = \{y \in\ X/xRy\}.$$

$a_d$ is of $V_S$ type and $a_d$ is symmetric then decomposing (X, $a_d$) into greatest connected subspaces is equivalent to decomposing the pretopological space (X, $a_d$) into greatest strong connected subspaces.

## 7. Example of a Particular Symmetric Pretopology

**Proposition 32.** *Let (X, a) be a pretopological space with as defined by the following pseudoclosure:*

$$\forall A \subset X,\ as(A) = \bigcup_{x \in A} \{y \in X / x \in a(\{y\}) \text{ or } y \in a(\{x\})\}.$$

*i- as is of $V_S$ type.*

*ii- ∀ x ∈ X and ∀ y ∈ X, there exists a sequence $x_0,\ldots,x_n$ of elements of X such as $x_0$= x, $x_n$= y with ∀j = 0,..., n-1, $x_{j+1} \in a(\{x_j\})$ or $x_j \in a(\{x_{j+1}\})$*

*⇔ there exists a sequence $x_0,\ldots,x_n$ of elements of X such as $x_0$= x, $x_n$= y with ∀j = 0,..., n-1, $x_{j+1} \in as(\{x_j\})$.*

*iii- Let A ⊂ X with A non empty.*

*If a is of $V_S$ type then A is a strongly connected component of (X, as) ⇔ A is a connected component of (X,a).*

*Proof.* i- ∀ x ∈ X, as ( { x}) = {y ∈ X/ x ∈ a ( {y}) or y ∈ a ( {x}) } then as (A) = $\bigcup_{x \in A} a_s(\{x\})$.

ii- as ( { $x_j$ }) = {y ∈ X/ $x_j$ ∈ a ( {y}) or y ∈ a ( {$x_j$}) }

then $x_{j+1} \in a(\{x_j\})$ or $x_j \in a(\{x_{j+1}\})$

⇔ $x_{j+1}$ ∈ as ( { $x_j$ }) by definition.

iii- If A is a strongly connected component of (X, as)

then ∀ x ∈ A and ∀ y ∈ A, there exists a sequence $x_0,\ldots,x_n$ of elements of A such as $x_0$= x, $x_n$= y with ∀j = 0,..., n-1, $x_{j+1} \in as(\{x_j\})$ ([3]).

So ∀ x ∈ A and ∀ y ∈ A, there exists a sequence $x_0,\ldots,x_n$ of elements of A such as $x_0$= x, $x_n$= y with ∀j = 0,..., n-1, $x_{j+1} \in a(\{x_j\})$ or $x_j \in a(\{x_{j+1}\})$ (by definition of as).

Then ∀ x ∈ A,∀ y ∈ A, , there exists a chain from { y } to { x } in (A, $a_A$) ([5]), and then A is a connected subset of (X,$a$) ([5][3]).

If A is not a connected component of (X,$a$), then there exists B, A⊂ B⊂ X with A≠B, B connected component of (X,$a$).

So there exists B, A⊂ B⊂ X with A≠B, ∀ x ∈ B,∀ y ∈ B, there exists a sequence $x_0,\ldots,x_n$ of elements of B such as $x_0$= x, $x_n$= y with ∀j = 0,..., n-1, $x_{j+1} \in a(\{x_j\})$ or $x_j \in a(\{x_{j+1}\})$ ([5][3])

Then there exists B, A⊂ B⊂ X with A≠B, ∀ x ∈ B,∀ y ∈ B, there exists a sequence $x_0,\ldots,x_n$ of elements of B such as $x_0$= x, $x_n$= y with ∀j = 0,..., n-1, $x_{j+1} \in as(\{x_j\})$ (by definition of as).

So there exists B, A⊂ B⊂ X with A≠B, ∀ x ∈ B,∀ y ∈ B, there exists a path from { y} to { x} in (X, $a$s) ([5]).

We get B is a strong connected subset of (X, $a$s) ([3]), which contradicts that A is a strongly connected component of (X, $a$s).

So A is a connected component of (X,$a$).

Conversely, if A is a connected component of (X,$a$), then ∀ x ∈ A,∀ y ∈ A, there exists a sequence $x_0$,. . . , $x_n$ of elements of A such as $x_0$= x, $x_n$= y with ∀j = 0,. . . , n-1, $x_{j+1}$ ∈ $a$ ({$x_j$}) or $x_j$ ∈ $a$ ({$x_{j+1}$}) ([5][3]).

So ∀ x ∈ A,∀ y ∈ A, there exists a sequence $x_0$,. . . , $x_n$ of elements of A such as $x_0$= x, $x_n$= y with ∀j = 0,. . . , n-1, $x_{j+1}$ ∈ $a$s ({$x_j$}) (by definition of $a$s).

We get A is a strong connected subset of (X, $a$s) ([5][3]).

If A is not a strong connected component of (X, $a$s).

Then there exists B, A⊂ B⊂ X with A≠B, B strong connected component of (X, $a$s).

Hence there exists B, A⊂ B⊂ X with A≠B, ∀ x ∈ B,∀ y ∈ B, there exists a sequence $x_0$,. . . , $x_n$ of elements of B such as $x_0$= x, $x_n$= y with ∀j = 0,. . . , n-1, $x_{j+1}$ ∈ $a$s ({$x_j$}) ([3]).

Then, there exists B, A⊂ B⊂ X with A≠B, ∀ x ∈ B,∀ y ∈ B, there exists a sequence $x_0$,. . . , $x_n$ of elements of B such as $x_0$= x, $x_n$= y with ∀j = 0,. . . , n-1, $x_{j+1}$ ∈ $a$ ({$x_j$}) or $x_j$ ∈ $a$ ({$x_{j+1}$})(by definition of $a$s).

We get B is a connected subset of (X, $a$) ([5][3]), which contradicts that A is a connected component of (X, $a$).

So A is a strong connected component of (X,$a$s).

**Remark 33.** as is symmetric.

*Proof.* Let x ∈ X and y ∈ X.
$a$s ( {x}) = { y ∈ X/ x ∈ $a$( {y}) or y ∈ $a$( {x}) } (by definition).
Hence y ∈ $a$s ( {x})⇔ x ∈ $a$( {y}) or y ∈ $a$( {x}).
⇔ x ∈ $a$s ( {y}).

**Proposition 34.** *Let (X, a) be a V pretopological space. Let A ⊂ X, with A non empty.*

*Then (as)$_A$ = (a$_A$)s.*

*Proof.* ∀ C⊂ A,

$$(a_A)s(C) = \bigcup_{x\in C} \{y \in A/x \in a_A(\{y\}) \text{ or } y \in a_A(\{x\})\}$$

$$
\begin{aligned}
&= \bigcup_{x\in C} \{y \in A/x \in a(\{y\}) \text{ or } y \in a(\{x\})\}(x \in A \text{ and } y \in A) \\
&=(\bigcup_{x\in C} \{y \in X/x \in a(\{y\}) \text{ or } y \in a(\{x\})\}) \cap A \\
&=as(C) \cap A \\
&=(as)_A(C).
\end{aligned}
$$

**Proposition 35.** *Let (X, a) be a pretopological space.*
*Let $A \subset X$ with A non empty.*

*(i) If (X,a) is of V type then A strongly connected subspace of (X, as) implies A connected subspace of (X,a).*

*(ii) If (X,a) is of $V_S$ type then A strongly connected subspace of (X, as) $\Leftrightarrow$ A connected subspace of (X,a).*

*Proof.* (i) A strongly connected subspace of (X,$a$s)

$\Leftrightarrow$ $\forall$ x $\in$ A, $\forall$ y $\in$ A, there exists a sequence $x_0,\ldots, x_n$ of elements of A such as $x_0$= x, $x_n$= y with $\forall$j = 0,..., n-1, $x_{j+1} \in (as)_A (\{x_j\})$ (see [5])

$\Leftrightarrow$ $\forall$ x $\in$ A, $\forall$ y $\in$ A, there exists a sequence $x_0,\ldots, x_n$ of elements of A such as $x_0$= x, $x_n$= y with $\forall$j = 0,..., n-1, $x_{j+1} \in (a_A)$s $(\{x_j\})$ (see Proposition 34)

$\Leftrightarrow$ $\forall$ x $\in$ A, $\forall$ y $\in$ A, there exists a sequence $x_0,\ldots, x_n$ of elements of A such as $x_0$= x, $x_n$= y with $\forall$j = 0,..., n-1, $x_{j+1} \in a_A (\{x_j\})$ or $x_j \in a_A (\{x_{j+1}\})$ (see Proposition 32 (ii)).

So $\forall$ x $\in$ A, $\forall$ y $\in$ A, there exists a chain from { y } to { x} in (A, $a_A$) (see [5])

Therefore A is connected subspace of (X,$a$) (see [5]).

(ii) We get equivalences if $a$ is of $V_S$ type (see [5]).

**Corollary 36.** *Let (X, a) be a $V_S$ pretopological space.*
*Let $A \subset X$ with A non empty. The following assertions are equivalent:*

*(i) A is a greatest strong connected subspace of (X, as);*
*(ii) A is a greatest connected subspace of (X, as);*
*(iii) A is a greatest connected subspace of (X,a).*

*Proof.* (i) and (ii) are equivalent according to the Corollary 28.

(i) and (iii) are equivalent according to the Proposition 35-ii.

**Consequence.** Decomposing a pretopological space (X,$a$) of $V_S$ type into greatest connected subspaces is equivalent to decomposing the pretopological space ( X,$a$s) into greatest strong connected subspaces.

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