RICCI AND PROJECTIVE CURVATURE TENSORS
ON A TYPE OF PARA-KENMOTSU MANIFOLD

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Abstract: The object of the present paper is to study the curvature properties of Ricci-parallel para-Kenmotsu (briefly, P-Kenmotsu) manifold with the conditions $R(X,\xi).P = P(X,\xi).R$ and $R(X,\xi).P = L[(X \wedge \xi).P]$, (L ≠ -1), where $R(X,Y)$ is the Riemannian curvature tensor, $P(X,Y)$ is the Weyl-projective curvature tensor, L is some function on $M_n$ and $X \in T(M_n)$. It is shown that the Ricci-parallel P-Kenmotsu manifold with these conditions is of constant curvature -1 and consequently it is locally isomorphic to the hyperbolic space. Further, the para-Kenmotsu manifold with the condition $R(\xi,X).R = 0$ is considered and it is shown that such a manifold is of constant curvature. Finally, it is shown that the para-Kenmotsu manifold with the condition $R(\xi,X).S = 0$ is an Einstein manifold, where $S(X,Y)$ is the Ricci curvature tensor.

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1. Introduction

The notion of an almost para contact Riemannian manifold was introduced by Sato [1], in 1976. After that, Adati and Matsumoto [2] defined and studied $P$-Sasakian and $SP$-Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, in 1972, Kenmotsu [3] defined a class of almost contact Riemannian manifolds. In 1995, Sinha and Sai Prasad [4] defined a class of almost para contact metric manifold namely para-Kenmotsu (briefly, $P$-Kenmotsu) and special para-Kenmotsu (briefly, $SP$-Kenmotsu) manifolds. Later, Satyanarayana and Sai Prasad [5, 6] have studied the curvature properties of semi-symmetric and Ricci-symmetric para-Kenmotsu manifolds.

In this paper, we consider Ricci-parallel para-Kenmotsu manifolds with the conditions $R(X, \xi).P = P(X, \xi).R$ and $R(X, \xi).P = L[(X \wedge \xi).P]$, $L \neq -1$ to obtain some of their curvature properties. Further, the para-Kenmotsu manifolds with the conditions $R(\xi, X).R = 0$ and $R(\xi, X).S = 0$ are also considered and their curvature properties are studied.

Let $M_n$ be an $n$-dimensional differentiable manifold equipped with structure tensors $(\Phi, \xi, \eta)$, where $\Phi$ is a tensor of type $(1, 1)$, $\xi$ is a vector field, $\eta$ is a 1-form such that
\begin{align*}
\eta(\xi) &= 1 \quad (1.1) \\
\Phi^2 \xi &= X - \eta(X)\xi; \quad \overline{X} = \Phi X. \quad (1.2)
\end{align*}

Then $M_n$ is called an almost para contact manifold.

Let $g$ be the Riemannian metric satisfying such that, for all vector fields $X$ and $Y$ on $M_n$,
\begin{align*}
g(X, \xi) &= \eta(X) \quad (1.3) \\
\Phi \xi &= 0, \quad \eta(\Phi X) = 0, \quad \text{rank } \Phi = n - 1 \quad (1.4) \\
g(\Phi X, \Phi Y) &= g(X, Y) - \eta(X)\eta(Y). \quad (1.5)
\end{align*}

Then the manifold $M_n[1]$ is said to admit an almost para contact Riemannian structure $(\Phi, \xi, \eta, g)$.

A manifold $M_n$ of dimension $n$ with Riemannian metric $g$ admitting a tensor field $\Phi$ of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying (1.1), (1.3) along with
\begin{align*}
(\nabla_X \eta)Y - (\nabla_Y \eta)X &= 0 \quad (1.6) \\
(\nabla_X \nabla_Y \eta)Z &= [-g(X, \xi) + \eta(X)\eta(Z)]\eta(Y) + [-g(X, Y) + \eta(X)\eta(Y)]\eta(Z). \quad (1.7) \\
\nabla_X \xi &= \Phi^2 X = X - \eta(X)\xi \quad (1.8)
\end{align*}
is called a para-Kenmotsu manifold or in brief $P$-Kenmotsu manifold [4], where $\nabla$ is the covariant differentiation with respect to the metric $g$.

Let $(M_n, g)$ be an $n$-dimensional Riemannian manifold admitting a tensor field $\Phi$ of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying

$$ (\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y) $$

(1.10)

$$ g(X, \xi) = \eta(X), \text{ where } g(QX, Y) = S(X, Y) $$

(1.12)

$$ g[R(X, Y)Z, \xi] = \eta[R(X, Y, Z)] = g(X, Z)\eta(Y) - g(Y, Z)\eta(X) $$

(1.13)

$$ R(X, \xi) = -1 $$

(1.14)

$$ R(\xi, X)\xi = X - \eta(X)\xi $$

(1.15)

$$ R(X, \xi, X) = \xi $$

(1.16)

$$ R(X, Y, \xi) = \eta(X)Y - \eta(Y)X; \text{ when } X \text{ is orthogonal to } \xi $$

(1.17)

$$ R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi; $$

(1.18)

where $Q$ is the Ricci operator.

An almost para-contact Riemannian manifold $M_n$ is said to be an $\eta$-Einstein manifold [7] if the Ricci curvature tensor $S$ is of the form

$$ S = aI_d + b\eta \otimes \xi, $$

(1.19)

where $a$ and $b$ are smooth functions on $M_n$. In particular, if $b = 0$ then it is said to be an Einstein manifold [7].

Moreover, it is also known that if a P-Kenmotsu manifold is projectively flat then it is an Einstein manifold and the scalar curvature has a negative constant value $-n(n-1)$. Especially, if a P-Kenmotsu manifold is of constant curvature, the scalar curvature has a negative constant value $-n(n-1)$ [4] and in this case

$$ S(Y, Z) = -(n-1)g(Y, Z); $$

(1.20)

$$ S(\Phi Y, \Phi Z) = S(Y, Z) + (n-1)\eta(Y)\eta(Z) \text{ and } $$

(1.21)

$$ R(X, Y, Z, W) = \frac{1}{(n-1)}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)]. $$

(1.22)
2. P-Kenmotsu Manifold with the condition $R(\xi, X).R = 0$

The endomorphisms $(X \wedge Y)$, $(X \wedge_s Y)$, and the homeomorphism $R(\xi, X).R$ are respectively defined as:

\[(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y;\]  \hspace{1cm} (2.1)

\[(X \wedge_s Y)Z = S(Y, Z)X - S(X, Z)Y and \]  \hspace{1cm} (2.2)

\[(R(X, \xi) \cdot R)(U, Z)W = R(X, \xi)R(U, Z)W - R(R(X, \xi)U, Z)W - R(U, R(X, \xi)Z)W - R(U, Z)R(X, \xi)W,\]  \hspace{1cm} (2.3)

where $R(X, Y)$ is the Riemannian curvature tensor.

If we consider $(R(X, \xi) \cdot R)(U, Z)W = 0$ and by putting $U = \xi$ in (2.3), we have

\[R(X, \xi)R(\xi, \xi)R(\xi, \xi)ZW - R(\xi, R(X, \xi)\xi,Z)W - R(\xi, Z)R(X, \xi)W = 0.\]  \hspace{1cm} (2.4)

Now, by using (1.15) and (1.18), equation (2.4) becomes

\[R(X, Z, W) = g(X, W)Z - g(Z, W)X.\]  \hspace{1cm} (2.5)

By a suitable contraction of (2.5), we get $r = -n(n - 1)$, where $r$ is the scalar curvature of $M_n$. Thus we state the following theorem.

**Theorem 2.1.** A P-Kenmotsu manifold with the condition $R(\xi, X).R = 0$ is of constant curvature.

3. P-Kenmotsu Manifold with the Condition $R(\xi, X).S = 0$

Let us suppose that $(R(\xi, X).S)(Y, Z) = 0$, where $S(X, Y)$ is the Ricci tensor. Then,

\[S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0.\]  \hspace{1cm} (3.1)

By putting $Z = \xi$ in (3.1) and on using the equations (1.15) and (1.18), we get

\[S[\eta(Y)X - g(X, Y)\xi, \xi] + S[Y, X - \xi(X)\xi] = 0.\]  \hspace{1cm} (3.2)

Again, on using (1.12), we get

\[S(X, Y) = -(n - 1)g(X, Y) \hspace{1cm} (or)\]  \hspace{1cm} (3.3)
\[ S(X, Y) = \frac{r}{n} g(X, Y), \text{ as the scalar curvature is } r = -n(n-1). \quad (3.4) \]

This proves that the manifold is an Einstein one and hence we state the following theorem.

**Theorem 3.1.** A P-Kenmotsu manifold with the condition \( R(\xi, X).S = 0 \) is an Einstein manifold.

**Corollary 3.1.** If a P-Kenmotsu manifold is an Einstein manifold then its scalar curvature \( r \) is constant and is equal to \(-n(n-1)\), and hence it is an SP-Kenmotsu manifold.

### 4. Weyl-Projective Curvature Tensor on Ricci-Parallel P-Kenmotsu Manifold

In this section, first we consider a Ricci-parallel P-Kenmotsu manifold with the condition \( R(\xi, X).P = P(\xi, X).R \), where the Weyl-projective curvature tensor \( P(X, Y) \) of a Riemannian manifold, of type (1, 3), is defined as \[ P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1} [S(Y, Z)X - S(X, Z)Y]. \quad (4.1) \]

**Definition 4.1.** A P-Kenmotsu manifold is called Ricci-parallel if \( (\nabla_X S) = 0 \), where \( S(X, Y) \) is the Ricci tensor and \( \nabla \) is the covariant derivative.

The above condition implies that \( S(X, Y) = -(n-1)g(X, Y) \).

Therefore, the equation (4.1) reduces to \[ P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1} [S(Y, Z)X - S(X, Z)Y]. \quad (4.2) \]

By putting \( X = \xi, Z = \xi \) in (4.2), and on using the equations (1.3) and (1.15), we get \[ P(\xi, Y)\xi = P(Y, \xi)\xi = 0, \text{ for any vector field } Y. \quad (4.3) \]

The homeomorphisms \( R(\xi, \xi).P \) and \( P(X, \xi).R \) are respectively defined as:

\[ (R(\xi, \xi) \cdot P)(U, Z)W = R(U, \xi)P(U, Z)W - P(R(\xi, \xi)U, Z)W - P(U, R(\xi, \xi)Z)W - P(U, Z)R(\xi, \xi)W, \quad (4.4) \]

and

\[ (P(\xi, \xi) \cdot R)(U, Z)W = P(U, \xi)R(U, Z)W - R(P(\xi, \xi)U, Z)W - R(U, P(\xi, \xi)Z)W - R(U, Z)P(\xi, \xi)W. \quad (4.5) \]
By putting $U = W = \xi$ in (4.4), and on using (4.3), we get
\[(R(X, \xi) \cdot P)(\xi, Z)\xi = -P(R(X, \xi)\xi, Z)\xi - P(\xi, Z)R(X, \xi)\xi. \tag{4.6}\]

By using the equations (4.3) and (1.15), we get
\[P(R(X, \xi)\xi, Z)\xi = -P(X, Z)\xi \quad \text{and} \quad P(\xi, Z)R(X, \xi)\xi = -P(\xi, Z)X, \tag{4.7}\]

which on substituting in (4.6) gives
\[(R(X, \xi) \cdot P)(\xi, Z)\xi = P(X, Z)\xi + P(\xi, Z)X. \tag{4.8}\]

Similarly, by putting $U = W = \xi$ in (4.5), we get
\[(P(X, \xi) \cdot R)(\xi, Z)\xi = P(X, \xi)R(\xi, Z)\xi - R(P(X, \xi)\xi, Z)\xi - R(\xi, P(X, \xi)Z)\xi - R(\xi, Z)P(X, \xi)\xi. \tag{4.9}\]

Also, from the equations (1.13), (1.15) and (4.3), we get
\[P(X, \xi)R(\xi, Z)\xi = P(X, \xi)Z; \quad R(P(X, \xi)\xi, Z)\xi = 0; \quad R(\xi, P(X, \xi)Z)\xi = P(X, \xi)Z \quad \text{and} \quad R(\xi, Z)P(X, \xi)\xi = 0. \tag{4.10}\]

Therefore, from equations (4.9) and (4.10), we get
\[(P(X, \xi) \cdot R)(\xi, Z)\xi = 0. \tag{4.11}\]

Since $R(X, \xi).P = P(X, \xi).R$, we also have
\[(R(X, \xi) \cdot P)(\xi, Z)\xi = 0, \tag{4.12}\]

and hence from equations (4.8) and (4.12), we get
\[P(X, Z)\xi + P(\xi, Z)X = 0. \tag{4.13}\]

Using the equations (4.2), (1.3) and (1.17), we also have
\[P(X, Z)\xi = 0 \quad \text{and} \quad P(\xi, Z)X = R(\xi, Z)X + g(Z, X)\xi - \eta(X)Z. \tag{4.14}\]

Therefore, from (4.13) and (4.14), we get $R(\xi, Z)X = \eta(X)Z - g(Z, X)\xi$ and it implies that the manifold is of constant curvature $-1$. Thus we state the following theorem.

**Theorem 4.1.** A Ricci-parallel P-Kenmotsu manifold with the condition $R(X, \xi).P = P(X, \xi).R$ is of constant curvature $-1$ and consequently it is locally isomorphic to the hyperbolic space.
Further, in this section, we consider the Ricci-parallel P-Kenmotsu manifold with the condition $R(X, \xi).P = L[(X \wedge \xi).P]$, $L \neq -1$, where $L$ is some function on $M_n$. That is

$$ (R(X, \xi).P)(\xi, Z)\xi = L[(X \wedge \xi).P](\xi, Z)\xi], \quad (4.15) $$

where

$$ L[(X \wedge \xi).P](\xi, Z)\xi = L[(X \wedge \xi)P(\xi, Z)\xi - P((X \wedge \xi)\xi, Z)\xi - P(\xi, (X \wedge \xi)Z)\xi - P(\xi, Z)(X \wedge \xi)\xi]. \quad (4.16) $$

Using (4.3) and (2.1), we get

$$ (X \wedge \xi)P(\xi, Z)\xi = 0; \quad P((X \wedge \xi)\xi, Z)\xi = P(X, Z)\xi; $$

$$ P(\xi, (X \wedge \xi)Z)\xi = 0 \text{ and } P(\xi, Z)(X \wedge \xi)\xi = P(\xi, Z)X, \quad (4.17) $$

and hence the equation (4.16) becomes

$$ L[(X \wedge \xi).P)(\xi, Z)\xi] = L[P(X, Z)\xi + P(\xi, Z)X]. \quad (4.18) $$

By using (4.8) and (4.18), equation (4.15) reduces to

$$ (1 + L)[P(X, Z)\xi + P(\xi, Z)X] = 0. \quad (4.19) $$

Then on using equations (4.2), (1.3) and (1.17), we get from (4.19) that

$$ (1 + L)[R(\xi, Z)X - \eta(X)Z + g(X, Z)\xi] = 0. \quad (4.20) $$

Since $L \neq -1$, we have $R(\xi, Z)X = \eta(X)Z - g(X, Z)\xi$ and it implies that the manifold $M_n$ is of constant curvature $-1$. Thus we state the following result.

**Theorem 4.2.** A Ricci-parallel P-Kenmotsu manifold with the condition

$$ R(X, \xi).P = L[(X \wedge \xi).P] \quad (L \neq -1) $$

is of constant curvature $-1$ and consequently it is locally isomorphic to the hyperbolic space.

5. Conclusion

In this paper, we obtain certain curvature properties of Ricci-parallel para-Kenmotsu manifolds with the conditions

$$ R(X, \xi).P = P(X, \xi).R \text{ and } R(X, \xi).P = L[(X \wedge \xi).P], $$

$L \neq -1$. Some of the results obtained in this paper are in similar to the results reported earlier in the case of para-Sasakian manifolds [9, 10].
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