RICCI AND PROJECTIVE CURVATURE TENSORS ON A TYPE OF PARA-KENMOTSU MANIFOLD

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Abstract: The object of the present paper is to study the curvature properties of Ricci-parallel para-Kenmotsu (briefly, P -Kenmotsu) manifold with the conditions $R(X, \xi).P = P(X, \xi).R$ and $R(X, \xi).P = L[(X \wedge \xi).P]$, ($L \neq -1$), where $R(X, Y)$ is the Riemannian curvature tensor, $P(X, Y)$ is the Weyl-projective curvature tensor, L is some function on M_n and $X \in T(M_n)$. It is shown that the Ricci-parallel P -Kenmotsu manifold with these conditions is of constant curvature -1 and consequently it is locally isomorphic to the hyperbolic space. Further, the para-Kenmotsu manifold with the condition $R(\xi, X).R = 0$ is considered and it is shown that such a manifold is of constant curvature. Finally, it is shown that the para-Kenmotsu manifold with the condition $R(\xi, X).S = 0$ is an Einstein manifold, where $S(X, Y)$ is the Ricci curvature tensor.

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1. Introduction

The notion of an almost para contact Riemannian manifold was introduced by Sato [1], in 1976. After that, Adati and Matsumoto [2] defined and studied P -Sasakian and SP -Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, in 1972, Kenmotsu [3] defined a class of almost contact Riemannian manifold. In 1995, Sinha and Sai Prasad [4] defined a class of almost para contact metric manifold namely para-Kenmotsu (briefly, P -Kenmotsu) and special para-Kenmotsu (briefly, SP -Kenmotsu) manifolds. Later, Satyanarayana and Sai Prasad [5, 6] have studied the curvature properties of semi-symmetric and Ricci-symmetric para-Kenmotsu manifolds.

In this paper, we consider Ricci-parallel para-Kenmotsu manifolds with the conditions $R(X, \xi).P = P(X, \xi).R$ and $R(X, \xi).P = L[(X \wedge \xi).P]$, $L \neq -1$ to obtain some of their curvature properties. Further, the para-Kenmotsu manifolds with the conditions $R(\xi, X).R = 0$ and $R(\xi, X).S = 0$ are also considered and their curvature properties are studied.

Let M_n be an n -dimensional differentiable manifold equipped with structure tensors (Φ, ξ, η) , where Φ is a tensor of type $(1, 1)$, ξ is a vector field, η is a 1-form such that

$$\eta(\xi) = 1 \quad (1.1)$$

$$\Phi^2(X) = X - \eta(X)\xi; \quad \overline{X} = \Phi X. \quad (1.2)$$

Then M_n is called an almost para contact manifold.

Let g be the Riemannian metric satisfying such that, for all vector fields X and Y on M_n ,

$$g(X, \xi) = \eta(X) \quad (1.3)$$

$$\Phi\xi = 0, \quad \eta(\Phi X) = 0, \quad \text{rank } \Phi = n - 1 \quad (1.4)$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (1.5)$$

Then the manifold $M_n[1]$ is said to admit an almost para contact Riemannian structure (Φ, ξ, η, g) .

A manifold M_n of dimension n with Riemannian metric g admitting a tensor field Φ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying (1.1), (1.3) along with

$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0 \quad (1.6)$$

$$(\nabla_X \nabla_Y \eta)Z = [-g(X, Z) + \eta(X)\eta(Z)]\eta(Y) + [-g(X, Y) + \eta(X)\eta(Y)]\eta(Z) \quad (1.7)$$

$$\nabla_X \xi = \Phi^2 X = X - \eta(X)\xi \quad (1.8)$$

$$(\nabla_X \Phi)Y = g(\Phi X, Y)\xi - \eta(Y)\Phi X \quad (1.9)$$

is called a para-Kenmotsu manifold or in brief P -Kenmotsu manifold [4], where ∇ is the covariant differentiation with respect to the metric g .

Let (M_n, g) be an n -dimensional Riemannian manifold admitting a tensor field Φ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying

$$(\nabla_X \eta)Y = g(X, Y)\xi - \eta(X)\eta(Y) \quad (1.10)$$

$$g(X, \xi) = \eta(X) \text{ and } (\nabla_X \eta)Y = \varphi(\bar{X}, Y), \text{ where } \varphi \text{ is an associate of } \Phi. \quad (1.11)$$

Then M_n is called special para-Kenmotsu manifold or in brief SP -Kenmotsu manifold [4].

It is known that, in a P -Kenmotsu manifold the following relations hold good [4]:

$$S(X, \xi) = -(n-1)\eta(X), \text{ where } g(QX, Y) = S(X, Y) \quad (1.12)$$

$$g[R(X, Y)Z, \xi] = \eta[R(X, Y, Z)] = g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \quad (1.13)$$

$$R(X, \xi) = -1 \quad (1.14)$$

$$R(\xi, X)\xi = X - \eta(X)\xi \quad (1.15)$$

$$R(X, \xi, X) = \xi \quad (1.16)$$

$$R(X, Y, \xi) = \eta(X)Y - \eta(Y)X; \text{ when } X \text{ is orthogonal to } \xi \quad (1.17)$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi; \quad (1.18)$$

where Q is the Ricci operator.

An almost para-contact Riemannian manifold M_n is said to be an η -Einstein manifold [7] if the Ricci curvature tensor S is of the form

$$S = aI_d + b\eta \otimes \xi, \quad (1.19)$$

where a and b are smooth functions on M_n . In particular, if $b = 0$ then it is said to be an Einstein manifold [7].

Moreover, it is also known that if a P -Kenmotsu manifold is projectively flat then it is an Einstein manifold and the scalar curvature has a negative constant value $-n(n-1)$. Especially, if a P -Kenmotsu manifold is of constant curvature, the scalar curvature has a negative constant value $-n(n-1)$ [4] and in this case

$$S(Y, Z) = -(n-1)g(Y, Z); \quad (1.20)$$

$$S(\Phi Y, \Phi Z) = S(Y, Z) + (n-1)\eta(Y)\eta(Z) \text{ and} \quad (1.21)$$

$$R(X, Y, Z, W) = \frac{1}{(n-1)}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)]. \quad (1.22)$$

2. P-Kenmotsu Manifold with the condition $R(\xi, X).R = 0$

The endomorphisms $(X \wedge Y)$, $(X \wedge_s Y)$, and the homeomorphism $R(\xi, X).R$ are respectively defined as:

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y; \quad (2.1)$$

$$(X \wedge_s Y)Z = S(Y, Z)X - S(X, Z)Y \text{ and} \quad (2.2)$$

$$\begin{aligned} (R(X, \xi) \cdot R)(U, Z)W &= R(X, \xi)R(U, Z)W - R(R(X, \xi)U, Z)W \\ &\quad - R(U, R(X, \xi)Z)W - R(U, Z)R(X, \xi)W, \end{aligned} \quad (2.3)$$

where $R(X, Y)$ is the Riemannian curvature tensor.

If we consider $(R(X, \xi) \cdot R)(U, Z)W = 0$ and by putting $U = \xi$ in (2.3), we have

$$\begin{aligned} R(X, \xi)R(\xi, Z)W - R(R(X, \xi)\xi, Z)W - R(\xi, R(X, \xi)Z)W \\ - R(\xi, Z)R(X, \xi)W = 0. \end{aligned} \quad (2.4)$$

Now, by using (1.15) and (1.18), equation (2.4) becomes

$$R(X, Z, W) = g(X, W)Z - g(Z, W)X. \quad (2.5)$$

By a suitable contraction of (2.5), we get $r = -n(n-1)$, where r is the scalar curvature of M_n . Thus we state the following theorem.

Theorem 2.1. *A P-Kenmotsu manifold with the condition $R(\xi, X).R = 0$ is of constant curvature.*

3. P-Kenmotsu Manifold with the Condition $R(\xi, X).S = 0$

Let us suppose that $(R(\xi, X).S)(Y, Z) = 0$, where $S(X, Y)$ is the Ricci tensor. Then,

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0. \quad (3.1)$$

By putting $Z = \xi$ in (3.1) and on using the equations (1.15) and (1.18), we get

$$S[\eta(Y)X - g(X, Y)\xi, \xi] + S[Y, X - \eta(X)\xi] = 0. \quad (3.2)$$

Again, on using (1.12), we get

$$S(X, Y) = -(n-1)g(X, Y) \quad (or) \quad (3.3)$$

$$S(X, Y) = \frac{r}{n}g(X, Y), \text{ as the scalar curvature is } r = -n(n-1). \quad (3.4)$$

This proves that the manifold is an Einstein one and hence we state the following theorem.

Theorem 3.1. *A P-Kenmotsu manifold with the condition $R(\xi, X).S = 0$ is an Einstein manifold.*

Corollary 3.1. *If a P-Kenmotsu manifold is an Einstein manifold then its scalar curvature r is constant and is equal to $-n(n-1)$, and hence it is an SP-Kenmotsu manifold.*

4. Weyl-Projective Curvature Tensor on Ricci-Parallel P-Kenmotsu Manifold

In this section, first we consider a Ricci-parallel P-Kenmotsu manifold with the condition $R(X, \xi).P = P(X, \xi).R$, where the Weyl-projective curvature tensor $P(X, Y)$ of a Riemannian manifold, of type (1, 3), is defined as [8]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1} [S(Y, Z)X - S(X, Z)Y]. \quad (4.1)$$

Definition 4.1. A P-Kenmotsu manifold is called Ricci-parallel if $(\nabla_X S) = 0$, where $S(X, Y)$ is the Ricci tensor and ∇ is the covariant derivative.

The above condition implies that $S(X, Y) = -(n-1)g(X, Y)$.

Therefore, the equation (4.1) reduces to

$$P(X, Y)Z = R(X, Y)Z + [g(Y, Z)X - g(X, Z)Y]. \quad (4.2)$$

By putting $X = \xi$, $Z = \xi$ in (4.2), and on using the equations (1.3) and (1.15), we get

$$P(\xi, Y)\xi = P(Y, \xi)\xi = 0, \text{ for any vector field } Y. \quad (4.3)$$

The homeomorphisms $R(X, \xi).P$ and $P(X, \xi).R$ are respectively defined as:

$$\begin{aligned} (R(X, \xi) \cdot P)(U, Z)W &= R(X, \xi)P(U, Z)W - P(R(X, \xi)U, Z)W \\ &\quad - P(U, R(X, \xi)Z)W - P(U, Z)R(X, \xi)W, \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} (P(X, \xi) \cdot R)(U, Z)W &= P(X, \xi)R(U, Z)W - R(P(X, \xi)U, Z)W \\ &\quad - R(U, P(X, \xi)Z)W - R(U, Z)P(X, \xi)W. \end{aligned} \quad (4.5)$$

By putting $U = W = \xi$ in (4.4), and on using (4.3), we get

$$(R(X, \xi) \cdot P)(\xi, Z)\xi = -P(R(X, \xi)\xi, Z)\xi - P(\xi, Z)R(X, \xi)\xi. \quad (4.6)$$

By using the equations (4.3) and (1.15), we get

$$P(R(X, \xi)\xi, Z)\xi = -P(X, Z)\xi \text{ and } P(\xi, Z)R(X, \xi)\xi = -P(\xi, Z)X, \quad (4.7)$$

which on substituting in (4.6) gives

$$(R(X, \xi) \cdot P)(\xi, Z)\xi = P(X, Z)\xi + P(\xi, Z)X. \quad (4.8)$$

Similarly, by putting $U = W = \xi$ in (4.5), we get

$$\begin{aligned} (P(X, \xi) \cdot R)(\xi, Z)\xi &= P(X, \xi)R(\xi, Z)\xi - R(P(X, \xi)\xi, Z)\xi \\ &\quad - R(\xi, P(X, \xi)Z)\xi - R(\xi, Z)P(X, \xi)\xi. \end{aligned} \quad (4.9)$$

Also, from the equations (1.13), (1.15) and (4.3), we get

$$\begin{aligned} P(X, \xi)R(\xi, Z)\xi &= P(X, \xi)Z; \quad R(P(X, \xi)\xi, Z)\xi = 0; \\ R(\xi, P(X, \xi)Z)\xi &= P(X, \xi)Z \text{ and } R(\xi, Z)P(X, \xi)\xi = 0. \end{aligned} \quad (4.10)$$

Therefore, from equations (4.9) and (4.10), we get

$$(P(X, \xi) \cdot R)(\xi, Z)\xi = 0. \quad (4.11)$$

Since $R(X, \xi).P = P(X, \xi).R$, we also have

$$(R(X, \xi) \cdot P)(\xi, Z)\xi = 0, \quad (4.12)$$

and hence from equations (4.8) and (4.12), we get

$$P(X, Z)\xi + P(\xi, Z)X = 0. \quad (4.13)$$

Using the equations (4.2), (1.3) and (1.17), we also have

$$P(X, Z)\xi = 0 \text{ and } P(\xi, Z)X = R(\xi, Z)X + g(Z, X)\xi - \eta(X)Z. \quad (4.14)$$

Therefore, from (4.13) and (4.14), we get $R(\xi, Z)X = \eta(X)Z - g(Z, X)\xi$ and it implies that the manifold is of constant curvature -1 . Thus we state the following theorem.

Theorem 4.1. *A Ricci-parallel P-Kenmotsu manifold with the condition $R(X, \xi).P = P(X, \xi).R$ is of constant curvature -1 and consequently it is locally isomorphic to the hyperbolic space.*

Further, in this section, we consider the Ricci-parallel P-Kenmotsu manifold with the condition $R(X, \xi).P = L[(X \wedge \xi).P]$, $L \neq -1$, where L is some function on M_n . That is

$$(R(X, \xi).P)(\xi, Z)\xi = L[(X \wedge \xi).P](\xi, Z)\xi, \quad (4.15)$$

where

$$\begin{aligned} L[(X \wedge \xi).P](\xi, Z)\xi &= L[(X \wedge \xi)P(\xi, Z)\xi - P((X \wedge \xi)\xi, Z)\xi \\ &\quad - P(\xi, (X \wedge \xi)Z)\xi - P(\xi, Z)(X \wedge \xi)\xi]. \end{aligned} \quad (4.16)$$

Using (4.3) and (2.1), we get

$$\begin{aligned} (X \wedge \xi)P(\xi, Z)\xi &= 0; P((X \wedge \xi)\xi, Z)\xi = P(X, Z)\xi; \\ P(\xi, (X \wedge \xi)Z)\xi &= 0 \text{ and } P(\xi, Z)(X \wedge \xi)\xi = P(\xi, Z)X, \end{aligned} \quad (4.17)$$

and hence the equation (4.16) becomes

$$L[(X \wedge \xi).P](\xi, Z)\xi = L[P(X, Z)\xi + P(\xi, Z)X]. \quad (4.18)$$

By using (4.8) and (4.18), equation (4.15) reduces to

$$(1 + L)[P(X, Z)\xi + P(\xi, Z)X] = 0. \quad (4.19)$$

Then on using equations (4.2), (1.3) and (1.17), we get from (4.19) that

$$(1 + L)[R(\xi, Z)X - \eta(X)Z + g(X, Z)\xi] = 0. \quad (4.20)$$

Since $L \neq -1$, we have $R(\xi, Z)X = \eta(X)Z - g(X, Z)\xi$ and it implies that the manifold M_n is of constant curvature -1 . Thus we state the following result.

Theorem 4.2. *A Ricci-parallel P-Kenmotsu manifold with the condition*

$$R(X, \xi).P = L[(X \wedge \xi).P] \quad (L \neq -1)$$

is of constant curvature -1 and consequently it is locally isomorphic to the hyperbolic space.

5. Conclusion

In this paper, we obtain certain curvature properties of Ricci-parallel para-Kenmotsu manifolds with the conditions

$$R(X, \xi).P = P(X, \xi).R \text{ and } R(X, \xi).P = L[(X \wedge \xi).P],$$

$L \neq -1$. Some of the results obtained in this paper are in similar to the results reported earlier in the case of para-Sasakian manifolds [9, 10].

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