ON THE FEATURES OF HEAT TRANSFER IN
ANISOTROPIC REGIONS WITH DISCONTINUOUS
THERMAL-PHYSICAL CHARACTERISTICS

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Abstract: The authors simulate heat transfer in multilayer regions with anisotropic transfer characteristics. Heat conduction of each layer is described with a heat conduction tensor, so that both tensor components and the angles orientating the principal axes of heat conduction tensors become discontinuous at the boundaries of layer mating. It is determined that normal components of the heat flux density vector as well as temperature are continuous at the above boundaries, while tangential components may be discontinuous (i.e., the heat flux density vector is discontinuous at the boundaries between layers). The obtained form of normal component of the heat flux density vector for a free curvilinear boundary of the anisotropic region is suitable for application of economical numerical methods. To illustrate, the economical absolutely stable method of variable directions with extrapolation (developed by the authors) is applied. We discuss the results of numerical simulation that confirm continuity of

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normal components of the heat flux density vectors and discontinuities of the first kind for the tangential components at the interface between two anisotropic bodies.

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1. Introduction

Mathematical simulation of heat transfer in anisotropic regions and at their boundaries (both internal and external) contacting with high-temperature media contains considerable difficulties. They are related with (i) a necessity to simulate the multidimensional problems only, (ii) the fact that heat conduction in such cases is described by tensors of rank two rather than by scalars, which involves presence of mixed derivatives in the equations of heat conduction, and (iii) absence of information about publications on formulation of edge conditions with derivatives at the boundaries of anisotropic media. The development of theory of heat conduction in anisotropic bodies was considered in two monographs [1] and [2] by one of the authors of the present article, as well as in a section of the monograph by Carslaw and Jaeger [3] and in a number of publications (see, e.g., [4], [5] and [6]) by the authors of the present article. However, we do not know the works on the theory of heat conduction in anisotropic regions with discontinuous thermal-physical characteristics (TPC).

Here we simulate heat conduction in anisotropic regions with ideal contacts at the internal TPC discontinuity boundaries (mating boundaries) under heat exchange at the external boundaries. In this case, not only the tensor components but also the angles orientating the principal axes of the heat conduction tensor are discontinuous at the mating boundaries of two anisotropic bodies.

2. Problem Statement

It is known that in anisotropic bodies (contrary to isotropic ones) the heat flux density vectors are not oriented along the normals to isotherms owing to the presence of off-diagonal components of the heat conduction tensor [1]. This indicates that there are normal as well as tangential components of the heat flux density vectors on isotherms in anisotropic space. Especially that fact occurs at the interfaces between two contacting anisotropic bodies.

Let us consider a contact boundary for two anisotropic bodies with TPC $\tilde{n}^I$, ρ^I , K^I and c^{II} , ρ^{II} , K^{II} (heat capacities, densities and heat conduction tensors, respectively). Then transfer of potential (temperature) at the TPC discontinuity boundaries occurs only along the normal to the boundary. According to the first law of thermodynamics, the normal components of the heat flux density vectors, $(K^I \text{grad} T^I, \mathbf{n}^0)$ and $(K^{II} \text{grad} T^{II}, \mathbf{n}^0)$, are continuous at these boundaries, i.e., the boundary conditions of the fourth kind proposed by Lykov are valid [7]:

$$(K^I \text{grad} T^I, \mathbf{n}^0) = (K^{II} \text{grad} T^{II}, \mathbf{n}^0), \quad T^I = T^{II}. \quad (1)$$

Here $\mathbf{n}^0$ is the normal unit vector at a point of the TCP discontinuity boundary; scalar products of the heat flux density vectors and normal unit vectors are in parentheses.

Tangential components of the heat flux density vectors on opposite sides of the TCP discontinuity boundary do not participate in potential transfer through that boundary because they do not cross it. Their values are formed mainly by the prehistory of heat exchange along the mating boundary, so tangential components of the heat flux density vectors are discontinuous at each point, i.e.

$$(K^I \text{grad} T^I, \boldsymbol{\tau}^0) \neq (K^{II} \text{grad} T^{II}, \boldsymbol{\tau}^0),$$

where $\boldsymbol{\tau}^0$ is the unit vector of tangential to the TCP discontinuity boundary. Therefore, the heat flux densities at the TCP discontinuity boundaries are of discontinuous character in anisotropic bodies. At first glance such an analysis leads to a paradoxical conclusion on violation of the first law of thermodynamics, since the input heat flux density at a point of the TCP discontinuity boundary is not equal to the output one. If, however, the total input and output heat fluxes to the whole TCP discontinuity boundary are considered, then continuity of heat fluxes is ensured.

In the 2D case, if the smooth TCP discontinuity boundary is described by the function $y = f(x)$, then after expansion Eq. (1) takes the following form:

$$\begin{aligned} (\lambda_{11}^I f'(x) + \lambda_{21}^I) \frac{\partial T^I}{\partial x} + (\lambda_{12}^I f'(x) + \lambda_{22}^I) \frac{\partial T^I}{\partial y} = \\ = (\lambda_{11}^{II} f'(x) + \lambda_{21}^{II}) \frac{\partial T^{II}}{\partial x} + (\lambda_{12}^{II} f'(x) + \lambda_{22}^{II}) \frac{\partial T^{II}}{\partial y}, \end{aligned} \quad (2)$$

where $\lambda_{11}^I, \lambda_{12}^I = \lambda_{21}^I, \lambda_{22}^I$ and $\lambda_{11}^{II}, \lambda_{12}^{II} = \lambda_{21}^{II}, \lambda_{22}^{II}$ are components of the heat conduction tensors K^I and K^{II} , respectively; $f'(x) = tg\alpha$, where α is the angle between the vector $\mathbf{n}^0$ and local axis Oy .

For curvilinear boundaries of anisotropic bodies, it is reasonable to introduce the concept of heat conduction λ_n along the normal to the curvilinear boundary. Then it follows from Eqs. (1) and (2) that

$$\lambda_n^s \frac{\partial T^s}{\partial n} = \left\{ [\lambda_{11}^s f'(x) + \lambda_{21}^s] \frac{\partial T^s}{\partial x} + [\lambda_{21}^s f'(x) + \lambda_{22}^s] \frac{\partial T^s}{\partial y} \right\} \cos \alpha, \quad s = I, 2. \quad (3)$$

After dividing the above expression by and taking into account that

$$\begin{aligned} \frac{\partial T^s}{\partial x} / \frac{\partial T^s}{\partial n} &= \sin \alpha = \frac{tg\alpha}{\sqrt{1 + tg^2\alpha}} = \frac{f'(x)}{\sqrt{1 + f'^2(x)}}, \\ \frac{\partial T^s}{\partial y} / \frac{\partial T^s}{\partial n} &= \cos \alpha = \frac{1}{\sqrt{1 + tg^2\alpha}} = \frac{1}{\sqrt{1 + f'^2(x)}}, \end{aligned}$$

we obtain

$$\lambda_n^s = [\lambda_{11}^s f'^2(x) + 2\lambda_{12}^s f'(x) + \lambda_{22}^s] [1 + f'^2(x)]^{-1}, \quad s = I, 2. \quad (4)$$

Eq. (4) allows for components of the heat conduction tensors and behavior of the boundary $y = f(x)$. For a rectangular plate $y = const (y'(x) = tg\alpha = 0)$ $\lambda_n^s = \lambda_{22}^s$, $s = I, 2$; then the normal component of the heat flux density is

$$\lambda_n^s \frac{\partial T^s}{\partial n} = \lambda_{21}^s \frac{\partial T^s}{\partial x} + \lambda_{22}^s \frac{\partial T^s}{\partial y}, \quad s = I, 2. \quad (5)$$

It involves all the components of temperature gradient and not the only component, as in the case of an isotropic plate.

3. Physico-Mathematical Model

We considered the following problem for a 2D region as a two-layer curvilinear body shown in Figure 1, with TPC discontinuity boundary $w5$:

Figure 1: Mating boundary $w5$ between two anisotropic media with different thermal-physical characteristics.

$$c^{II}(T)\rho^{II}(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(\lambda_{11}^{II}(T)\frac{\partial T}{\partial x} \right) + 2\frac{\partial}{\partial x} \left(\lambda_{12}^{II}(T)\frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial y} \left(\lambda_{22}^{II}(T)\frac{\partial T}{\partial y} \right), \quad 0 < x < l_1, \quad 0 < y < l_2, \quad t > 0; \quad (6)$$

$$c^I(T)\rho^I(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(\lambda_{11}^I(T)\frac{\partial T}{\partial x} \right) + 2\frac{\partial}{\partial x} \left(\lambda_{12}^I(T)\frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial y} \left(\lambda_{22}^I(T)\frac{\partial T}{\partial y} \right), \quad 0 < x < l_1, \quad l_2 < y < l_2 + l_3, \quad t > 0; \quad (7)$$

$$\begin{aligned} - \left(\lambda_{12}^{II}(T)\frac{\partial T}{\partial x} + \lambda_{22}^{II}(T)\frac{\partial T}{\partial y} \right) \Big|_{w1} + \alpha_1(x, t)(T_{e1}(x, t) - T|_{w1}) = \\ = f_1(x, t), \quad 0 < x < l_1, \quad y = 0, \quad t > 0; \quad (8) \end{aligned}$$

$$\left(\lambda_{12}^I(T)\frac{\partial T}{\partial x} + \lambda_{22}^I(T)\frac{\partial T}{\partial y} \right) \Big|_{w2} + \alpha_2(x, t)(T_{e2}(x, t) - T|_{w2}) =$$

$$= f_2(x, t), \quad 0 < x < l_1, \quad y = l_2 + l_3, \quad t > 0; \quad (9)$$

$$\begin{aligned} - \left(\lambda_{11}^{II}(T) \frac{\partial T}{\partial x} + \lambda_{12}^{II}(T) \frac{\partial T}{\partial y} \right) \Big|_{w3} + \alpha_3(y, t)(T_{e3}(y, t) - T|_{w3}) = \\ = f_3(y, t), \quad x = 0, \quad 0 < y < l_2, \quad t > 0; \quad (10) \end{aligned}$$

$$\begin{aligned} - \left(\lambda_{11}^I(T) \frac{\partial T}{\partial x} + \lambda_{12}^I(T) \frac{\partial T}{\partial y} \right) \Big|_{w3} + \alpha_3(y, t)(T_{e3}(y, t) - T|_{w3}) = \\ = f_3(y, t), \quad x = 0, \quad l_2 < y < l_2 + l_3, \quad t > 0; \quad (11) \end{aligned}$$

$$\begin{aligned} \left(\lambda_{11}^{II}(T) \frac{\partial T}{\partial x} + \lambda_{12}^{II}(T) \frac{\partial T}{\partial y} \right) \Big|_{w4} + \alpha_4(y, t)(T_{e4}(y, t) - T|_{w4}) = \\ = f_3(y, t), \quad x = l_1, \quad 0 < y < l_2, \quad t > 0; \quad (12) \end{aligned}$$

$$\begin{aligned} \left(\lambda_{11}^I(T) \frac{\partial T}{\partial x} + \lambda_{12}^I(T) \frac{\partial T}{\partial y} \right) \Big|_{w4} + \alpha_4(y, t)(T_{e4}(y, t) - T|_{w4}) = \\ = f_4(y, t), \quad x = l_1, \quad l_2 < y < l_2 + l_3, \quad t > 0; \quad (13) \end{aligned}$$

$$\begin{aligned} \left(\lambda_{12}^{II}(T) \frac{\partial T}{\partial x} + \lambda_{22}^{II}(T) \frac{\partial T}{\partial y} \right) \Big|_{y=l_2-0} = \left(\lambda_{12}^I(T) \frac{\partial T}{\partial x} + \lambda_{22}^I(T) \frac{\partial T}{\partial y} \right) \Big|_{y=l_2+0}, \\ 0 < x < l_1, \quad y = l_2, \quad t > 0; \quad (14) \end{aligned}$$

$$T|_{y=l_2-0} = T|_{y=l_2+0}, \quad 0 < x < l_1, \quad y = l_2, \quad t > 0; \quad (15)$$

$$T(x, y, 0) = T_0(x, y), \quad 0 < x < l_1, \quad 0 < y < l_2 + l_3, \quad t = 0. \quad (16)$$

The components K^I, K^{II} of heat conduction tensors are determined from the following equations [1]:

$$\begin{aligned} \lambda_{11}^s = \lambda_{\xi^s}^s \cos^2 \phi^s + \lambda_{\eta^s}^s \sin^2 \phi^s, \quad \lambda_{12}^s = \lambda_{21}^s = (\lambda_{\xi^s}^s - \lambda_{\eta^s}^s) \sin \phi^s \cos \phi^s, \\ \lambda_{22}^s = \lambda_{\xi^s}^s \sin^2 \phi^s + \lambda_{\eta^s}^s \cos^2 \phi^s, \quad s = 1, 2, \quad (17) \end{aligned}$$

where $\lambda_{\xi^s}^s, \lambda_{\eta^s}^s$ are the principal components of tensors K^I, K^{II} and ϕ^s are the orientation angles of principal axes $O\xi^I, O\xi^{II}$ about axis Ox . The components

from Eq. (17) also characterize different orientations of the principal axes of heat conduction tensors at the TPC discontinuity boundaries.

One can see from the edge conditions Eqs. (8)-(14) that conductive heat fluxes at the boundaries involve all temperature gradient components. It follows from Eqs. (14) and (17) that the tensors K^I, K^{II} (and consequently tangential components of the heat flux densities) are discontinuous at the boundary $w5$, even if the principal components of tensors K^I, K^{II} both equal each other (i.e., $\lambda_{\xi^I}^I = \lambda_{\xi^{II}}^{II}$ and $\lambda_{\eta^I}^I = \lambda_{\eta^{II}}^{II}$) and the orientation angles of principal axes, ϕ^I and ϕ^{II} , are not equal to each other.

4. Solution Method

Equations (6)-(17) are solved numerically on the mesh

$$\omega_{h_1, h_2^I, h_2^{II}, \tau} = \left\{ x_i = ih_1, \ i = \overline{0, I}; \ y_j = jh_2^{II}, \ j = \overline{0, J_1}; \ y_j = jh_2^I, \right. \\ \left. j = \overline{J_1 + 1, J_2}; \ t^n = n\tau, \ n = 0, 1, 2, \dots \right\} \quad (18)$$

by using the alternating directions method with time extrapolation [2]. The integro-interpolation method is applied for boundary nodes to retain the second order [8].

For regular nodes on the mesh given by Eq. (18), the above method gives

$$c\rho \frac{T^{n+1/2} - T^n}{\tau/2} = \Lambda_{11}T^{n+1/2} + 2\Lambda_{21}\tilde{T}^{n+1/2} + \Lambda_{22}\tilde{T}^{n+1/2}, \quad (19)$$

where

$$\Lambda_{11}T^{n+1/2} = \frac{\lambda_{11}}{h_1^2}(T_{i+1,j}^{n+1/2} - 2T_{i,j}^{n+1/2} + T_{i-1,j}^{n+1/2}), \\ \Lambda_{12}\tilde{T}^{n+1/2} = \frac{\lambda_{12}}{4h_1h_2}(\tilde{T}_{i+1,j+1}^{n+1/2} - T_{i+1,j-1}^{n+1/2} - \tilde{T}_{i-1,j+1}^{n+1/2} + T_{i-1,j-1}^{n+1/2}), \\ \Lambda_{22}\tilde{T}^{n+1/2} = \frac{\lambda_{22}}{h_2^2}(\tilde{T}_{i,j+1}^{n+1/2} - 2T_{i,j}^{n+1/2} + T_{i,j-1}^{n+1/2}), \\ \tilde{T}_{m,j+1}^{n+1/2} = 2T_{m,j+1}^n - T_{m,j+1}^{n-1/2}, \ m = i-1, i, i+1; \\ \frac{T^{n+1} - T^{n+1/2}}{\tau/2} = \Lambda_{11}\tilde{T}^{n+1} + 2\Lambda_{12}\tilde{T}^{n+1} + \Lambda_{22}T^{n+1}. \quad (20)$$

Here

$$\begin{aligned}\Lambda_{11}\tilde{T}^{n+1} &= \frac{\lambda_{11}}{h_1^2}(\tilde{T}_{i+1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i-1,j}^{n+1}), \\ \Lambda_{12}\tilde{T}^{n+1} &= \frac{\lambda_{12}}{4h_1h_2}(\tilde{T}_{i+1,j+1}^{n+1} - \tilde{T}_{i+1,j-1}^{n+1} - T_{i-1,j+1}^{n+1} + T_{i-1,j-1}^{n+1}), \\ \Lambda_{22}T^{n+1} &= \frac{\lambda_{22}}{h_2^2}(T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1}), \\ \tilde{T}_{i+1,m}^{n+1} &= 2T_{i+1,m}^{n+1/2} - T_{i+1,m}^n, \quad m = j-1, j, j+1.\end{aligned}$$

The scheme of Eqs. (19) and (20) completely approximates the differential Eqs. (6) and (7) in the case of constant components of the heat conduction tensor, with $O(\tau + |h|^2 + \tau(h_1 + h_2))$ (where $|h| = \sqrt{h_1^2 + h_2^2}$), and is absolutely stable [2]. In Eqs. (19) and (20) $h_2 = \{h_2^I, h_2^{II}\}$, $t^n = n\tau$, $t^{n+1/2} = t^n + \tau/2$, $t^{n+1} = t^n + \tau$; $y_{i,j}^n$, $y_{i,j}^{n+1/2}$, $y_{i,j}^{n+1}$ are mesh functions on the corresponding time layers $t^n, t^{n+1/2}, t^{n+1}$. The scheme of Eqs. (19) and (20) is economical because it is realized by scalar sweeps along the coordinate axes Ox and Oy only. To retain the order of approximation for regular nodes, let us discuss approximation of boundary conditions with derivatives using the *integro-interpolation* method [8].

Our consideration will be performed for nodes at the layer mating boundary (an approach to other irregular nodes is similar). Let us write down Eqs. (6) and (7) as

$$c^s(T)\rho^s(T)\frac{\partial T}{\partial t} = \frac{\partial q_x^s}{\partial x} + \frac{\partial q_y^s}{\partial y}, \quad s = I, 2 \quad (21)$$

where q_x^s, q_y^s are components of the heat flux density vector. For a plane TPC discontinuity boundary

$$\begin{aligned}q_x^s &= \lambda_{11}^s(T)\frac{\partial T}{\partial x} + \lambda_{12}^s(T)\frac{\partial T}{\partial y}, \quad s = I, 2, \\ q_y^s &= \lambda_{21}^s(T)\frac{\partial T}{\partial x} + \lambda_{22}^s(T)\frac{\partial T}{\partial y}, \quad s = I, 2,\end{aligned} \quad (22)$$

and the edge mating condition Eq. (14) at the TPC discontinuity boundary is

$$q_y^I \Big|_{y=l_2-0} = q_y^{II} \Big|_{y=l_2+0}. \quad (23)$$

Let us choose an arbitrary node $(x_i, y_j = l_2)$ at the boundary $w5$ and integrate Eqs. (21) on an interval $x \in [x_{i-1/2}, x_{i+1/2}]$ with respect to variable x and then

with respect to variable y (at first on an interval $y \in [y_{j-1/2}, y_j]$ and then on an interval $y \in [y_j, y_{j+1/2}]$). After this, by applying the quadrature formulae for rectangles with the second degree of accuracy on the steps h_1, h_2 we obtain

$$c^s(T)\rho^s(T)\frac{\partial T}{\partial t}h_1\frac{h_2^s}{2} = (q_x^s)_{i+1/2,j}\frac{h_2^s}{2} - (q_x^s)_{i-1/2,j}\frac{h_2^s}{2} + \\ + (q_y^s)_{i,j-\frac{1-s}{2}}h_1 - (q_y^s)_{i,j-\frac{2-s}{2}}h_1 + O(h_1^2 + h_2^2), \quad s = 1, 2, \quad i = \overline{0, I}. \quad (24)$$

By adding Eqs. (24) for $s = 1$ and $s = 2$ and using Eqs. (21), we obtain the expressions that approximate the initially stated problem at the nodes at the mating boundary w_5 with components q_x^s, q_y^s of the heat flux density vector accurate to the second degree:

$$\left[c^I(T)\rho^I(T)h_2^I + c^{II}(T)\rho^{II}(T)h_2^{II} \right] \frac{h_1}{2} \frac{\partial T}{\partial t} \Big|_{i,j} = \\ = [(q_x^{II})_{i+1/2,j} - (q_x^{II})_{i-1/2,j}] \frac{h_2^{II}}{2} + [(q_y^I)_{i,j+1/2} - (q_y^{II})_{i,j-1/2}] h_1 + \\ + [(q_x^I)_{i+1/2,j} - (q_x^I)_{i-1/2,j}] \frac{h_2^I}{2} + \\ + O(h_1^2 + (h_2^I)^2 + (h_2^{II})^2), \quad i = \overline{0, I}. \quad (25)$$

The terms $(q_y^I)_{i,j}h_1$ and $(q_y^{II})_{i,j}h_1$ in Eq. (25) cancel because of Eq. (23), so the mating conditions are automatically obeyed.

Let us apply the scheme Eq. (19) of the alternating directions method with extrapolation, by approximating the derivatives in Eq. (25) with using Eq. (22) towards axis Ox on the $n + 1/2$ -th temporal half layer at the nodes $y = l_2, x = x_i, i = \overline{1, I-1}$,

$$\frac{\partial T}{\partial t} \approx \frac{\mathbf{T}_{i,j}^{n+1/2} - \mathbf{T}_{i,j}^n}{\tau/2}; \quad (26)$$

$$(q_x^s)_{i+1/2,j} = \left(\lambda_{11}^s(T) \frac{\partial T}{\partial x} \right)_{i+1/2,j} + \left(\lambda_{12}^s(T) \frac{\partial T}{\partial y} \right)_{i+1/2,j} \approx \\ \approx \left(\tilde{\lambda}_{11}^s \right)_{i+1/2,j}^{n+1/2} \frac{\mathbf{T}_{i+1,j}^{n+1/2} - \mathbf{T}_{i,j}^{n+1/2}}{h_1} + \frac{1}{2} \left[\left(\tilde{\lambda}_{12}^s \right)_{i+1/2,j}^{n+1/2} + \left(\tilde{\lambda}_{12}^s \right)_{i+1/2,j+2s-3}^{n+1/2} \right] \times \\ \times \frac{1}{h_2^s} \left[\frac{1}{2} \left(\mathbf{T}_{i+1,j}^{n+1/2} + \mathbf{T}_{i,j}^{n+1/2} \right) - \frac{1}{2} \left(\mathbf{T}_{i,j-1}^{n+1/2} + \mathbf{T}_{i+1,j-1}^{n+1/2} \right) \right], \quad s = I, 2; \quad (27)$$

$$\begin{aligned}
(q_x^s)_{i-1/2,j} &= \left(\lambda_{11}^s(T) \frac{\partial T}{\partial x} \right)_{i-1/2,j} + \left(\lambda_{12}^s(T) \frac{\partial T}{\partial y} \right)_{i-1/2,j} \approx \\
&\approx (\tilde{\lambda}_{11}^s)_{i-1/2,j}^{n+1/2} \frac{\mathbf{T}_{i,j}^{n+1/2} - \mathbf{T}_{i-1,j}^{n+1/2}}{h_1} + \frac{1}{2} \left[\left(\tilde{\lambda}_{12}^s \right)_{i-1/2,j+2s-3}^{n+1/2} + \left(\tilde{\lambda}_{12}^s \right)_{i-1/2,j}^{n+1/2} \right] \times \\
&\times \frac{1}{h_2^s} \left[\frac{1}{2} \left(\mathbf{T}_{i-1,j}^{n+1/2} + \mathbf{T}_{i,j}^{n+1/2} \right) - \frac{1}{2} \left(\mathbf{T}_{i-1,j-1}^{n+1/2} + \mathbf{T}_{i,j-1}^{n+1/2} \right) \right], \quad s = I, 2; \quad (28)
\end{aligned}$$

$$\begin{aligned}
(q_y^I)_{i,j-1/2} &= \left(\lambda_{12}^I(T) \frac{\partial T}{\partial x} \right)_{i,j-1/2} + \left(\lambda_{22}^I(T) \frac{\partial T}{\partial y} \right)_{i,j-1/2} \approx \\
&\approx \frac{1}{2} \left[\left(\tilde{\lambda}_{12}^I \right)_{i+1,j-1/2}^{n+1/2} + \left(\tilde{\lambda}_{12}^I \right)_{i-1,j-1/2}^{n+1/2} \right] \times \\
&\times \frac{1}{2h_1} \left[\frac{1}{2} \left(\mathbf{T}_{i+1,j}^{n+1/2} + \mathbf{T}_{i+1,j-1}^{n+1/2} \right) - \frac{1}{2} \left(\mathbf{T}_{i-1,j}^{n+1/2} + \mathbf{T}_{i-1,j-1}^{n+1/2} \right) \right] + \\
&+ \left(\tilde{\lambda}_{22}^I \right)_{i,j-1/2}^{n+1/2} \frac{\mathbf{T}_{i,j}^{n+1/2} - \mathbf{T}_{i,j-1}^{n+1/2}}{h_2^I}; \quad (29)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2} \left[\left(\tilde{\lambda}_{12}^{II} \right)_{i+1,j+1/2}^{n+1/2} + \left(\tilde{\lambda}_{12}^I \right)_{i-1,j+1/2}^{n+1/2} \right] \times \\
&\times \frac{1}{2h_1} \left[\frac{1}{2} \left(\tilde{\mathbf{T}}_{i+1,j+1}^{n+1/2} + \mathbf{T}_{i+1,j}^{n+1/2} \right) - \frac{1}{2} \left(\tilde{\mathbf{T}}_{i-1,j+1}^{n+1/2} + \mathbf{T}_{i-1,j}^{n+1/2} \right) \right] + \\
&+ \left(\tilde{\lambda}_{22}^{II} \right)_{i,j+1/2}^{n+1/2} \frac{\tilde{\mathbf{T}}_{i,j+1}^{n+1/2} - \mathbf{T}_{i,j}^{n+1/2}}{h_2^{II}}; \quad (30)
\end{aligned}$$

$$\tilde{T}_{m,j+1}^{n+1/2} = 2\mathbf{T}_{m,j+1}^n - \mathbf{T}_{m,j+1}^{n-1/2}; \quad m = i-1, i, i+1 \quad (31)$$

(the desired quantities are in bold type).

Here

$$\begin{aligned}
\left(\tilde{\lambda}_{11}^s \right)_{i\pm 1/2,j}^{n+1/2} &= \frac{1}{2} \left[\lambda_{11}^s \left(\tilde{T}_{i\pm 1,j}^{n+1/2} \right) + \lambda_{11}^s \left(\tilde{T}_{i,j}^{n+1/2} \right) \right], \\
\left(\tilde{\lambda}_{22}^s \right)_{i,j\mp 1/2}^{n+1/2} &= \frac{1}{2} \left[\lambda_{22}^s \left(\tilde{T}_{i,j\mp 1}^{n+1/2} \right) + \lambda_{22}^s \left(\tilde{T}_{i,j}^{n+1/2} \right) \right], \\
\left(\tilde{\lambda}_{12}^s \right)_{i\pm 1/2,j}^{n+1/2} &= \lambda_{12}^s \left(\frac{\tilde{T}_{i\pm 1,j}^{n+1/2} + \tilde{T}_{i,j}^{n+1/2}}{2} \right),
\end{aligned}$$

$$\begin{aligned}
\left(\tilde{\lambda}_{12}^s\right)_{i\pm 1/2, j+2s-3}^{n+1/2} &= \lambda_{12}^s \left(\frac{\tilde{T}_{i, j+2s-3}^{n+1/2} + \tilde{T}_{i\pm 1, j+2s-3}^{n+1/2}}{2} \right), \\
\left(\tilde{\lambda}_{12}^s\right)_{i\pm 1, j-1/2}^{n+1/2} &= \lambda_{12}^s \left(\frac{\tilde{T}_{i\pm 1, j-1}^{n+1/2} + \tilde{T}_{i\pm 1, j}^{n+1/2}}{2} \right), \\
\left(\tilde{\lambda}_{12}^s\right)_{i\pm 1, j+1/2}^{n+1/2} &= \lambda_{12}^s \left(\frac{\tilde{T}_{i\pm 1, j+1}^{n+1/2} + \tilde{T}_{i\pm 1, j}^{n+1/2}}{2} \right).
\end{aligned}$$

Substituting Eqs. (26)-(31) in Eq. (25), we get a set of linear algebraic equations with a tridiagonal matrix relative to $T_{i-1, j}^{n+1/2}$, $T_{i, j}^{n+1/2}$, $T_{i+1, j}^{n+1/2}$, $i = \overline{1, I-1}$, the degree of approximation being $O(h_1^2 + (h_2^I)^2 + (h_2^{II})^2)$. A procedure of the integro-interpolation method with usage of boundary conditions Eqs. (10)-(13) is applied separately at the nodes at $i = 0$ and $i = I$ to close the system.

After solving the set of linear algebraic equations by the scalar sweep method, we get distribution of temperature $T_{i, j}^{n+1/2}$ at the boundary $y = l_2$. The systems of algebraic equations using the sub-scheme Eq. (20) relative to $T_{i, j}^{n+1}$ are obtained similarly.

5. Results of Numerical Experiments

Using the above numerical methods, we developed a program complex that realizes the mathematical model Eqs. (6)-(17). This program complex makes it possible to calculate temperature fields and components of heat flux density vector, in particular, at the TPC discontinuity boundary.

Shown in Figure 2 are temperature fields in a moment of $t = 200$ s in an anisotropic two-layer plate with sizes of $l_1 = 0.05m$, $l_2 = 0.02m$ and $l_3 = 0.02$.

Figure 2: Temperature distribution in an anisotropic two-layer body with discontinuous characteristics of heat transfer.

The principal components and orientation angles of principal axes of the heat conduction tensors K^I and K^{II} for different versions are as follows:

a) $\lambda_\xi^I = \lambda_\xi^{II} = 100W/(m \cdot K)$, $\lambda_\eta^I = \lambda_\eta^{II} = 10W/(m \cdot K)$, $\phi^I = -45^\circ$, $\phi^{II} = 45^\circ$;

b) $\lambda_\xi^I = \lambda_\eta^{II} = 30W/(m \cdot K)$, $\lambda_\xi^I = 100W/(m \cdot K)$, $\lambda_\eta^{II} = 10W/(m \cdot K)$, $\phi^I = 0^\circ$, $\phi^{II} = 45^\circ$;

c) $\lambda_\xi^I = \lambda_\eta^{II} = 10W/(m \cdot K)$, $\lambda_\eta^I = \lambda_\xi^{II} = 100W/(m \cdot K)$, $\phi^I = 0^\circ$, $\phi^{II} = 45^\circ$;

d) $\lambda_\xi^I = \lambda_\eta^{II} = 10W/(m \cdot K)$, $\lambda_\eta^I = \lambda_\xi^{II} = 100W/(m \cdot K)$, $\phi^I = \phi^{II} = 0^\circ$;

Non-zero fluxes at the external boundaries were set in the following way:

$f_1(x) = f_2(x) = 3 \cdot \frac{10^5 W}{m^2}$, in the interval $x \in [0.02; 0.03]$ and are zero beyond this interval. The other input data were as follows: $c^I = c^{II} = 1000J/(kg \cdot K)$; $\rho^I = \rho^{II} = 1500kg/m^3$; $T_0 = 300K$; $\alpha_i(x, t) = 0$; $Te_i(x, t) = 0$, $i = \overline{1, 4}$. The boundary lengths w_1, w_2 and w_5 are $l_1 = 0.005m$, $l_2 = 0.02m$ and $l_3 = 0.02m$, respectively.

One can see from Figure 2 that the temperature field in the body remains continuous at the mating boundary, while the temperature derivatives and tangential components of the heat flux densities have discontinuities of the first kind at this boundary. This can be seen more clearly in Figure 3 that presents (with the same input data) the distributions of normal (q_y) and tangential (q_x) components of the heat flux density vector along the variable y , at fixed $x = 0.004$.

Figure 3: Behavior of components q_x and q_y of the heat flux density vectors at the mating boundary of anisotropic bodies.

One can clearly see in Figure 3 that normal component q_y of the heat flux density along axis Oy remains continuous, while tangential component q_x has discontinuities of the first kind at the TPC discontinuity boundary.

6. Conclusion

1. In this article, we developed a mathematical model of mating heat transfer between two anisotropic bodies with an ideal contact at the mating boundary (i.e., the discontinuity boundary for components and orientation angles of the principal axes of heat conduction tensors).

2. We modified the previous economical absolutely stable alternating directions method with extrapolation for numerical solution of parabolic equations with mixed derivatives. The modification deals with application of integro-interpolation method proposed by Samarskii for obtaining homogeneous approximation at the nodes at anisotropic body boundaries, particularly at the TPC discontinuity boundaries.

3. To use the decomposition methods for numerical solution of anisotropic heat conduction problems, a special form of boundary conditions at the TPC discontinuity boundaries has been derived. It uses all components and orientation angles of the principal axes of heat conduction tensors, as well as a function that describes behavior of the mating boundary.

4. By application of numerical experiments, we confirmed the hypothesis about continuity of normal components of the heat flux density vector at the anisotropic TPC discontinuity boundaries, as well as about occurrence of discontinuities of the first kind for tangential components.

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