VAGUE VECTOR SPACE AND VAGUE MODULES

K.V. Rama Rao^{1 §}, Amarendra Babu V.²

¹Department of Mathematics

S.R.R. and C.V.R Degree and P.G College

Vijayawada, A.P, INDIA

²Department of Mathematics

Achraya Nagarjuna University

Nagarjuna Nagar, 522510, INDIA

Abstract: We introduce the concept of vague vector spaces by linking the vague set and vector spaces. Also we define vague modules. We discuss some properties both vague vector space and vague module.

AMS Subject Classification: 20N25, 08A72, 03E72

Key Words: vague set, vector space, vague vector space and vague module

1. Introduction

The concept of fuzzy set was introduced by Zadesh since then this idea been applied to other algebraic structures such as groups, rings etc. Meanwhile the deficiency of fuzzy sets is also attract attention. Such as fuzzy set is single function, it does not express the evidence of supporting and opposing. Based on this reason, the concept of vague set introduced by Gau.W.L and Buehrer.D.J[5] in 1993. Vague sets as a extension of fuzzy sets, the idea of vague sets is that the membership of every element can be divided into two aspects including supporting and opposing. The notion of fuzzy groups defined by Rosen Field[7]is

Received: June 29, 2016

Revised: November 18, 2016

Published: December 11, 2016

© 2016 Academic Publications, Ltd.

url: www.acadpubl.eu

[§]Correspondence author

the first application of fuzzy set theory in algebra. Ranjit Biswas[2] initiated the study of vague algebra by studying vague groups. In [3, 4], we introduced the notions of vague additive groups, vague rings and vague fields.

In this paper, as an extension of our mentioned work, we introduce the notions of vague vector space and vague module. Also we discuss some of their properties.

2. Preliminaries

In this section we give some definitions and state some results for later use.

Definition 1. [5] A vague set A in the universe of discourse X is characterized by two membership functions given by

1. A truth membership function $t_A : X \rightarrow [0, 1]$ and
2. A false membership function $f_A : X \rightarrow [0, 1]$,

Where $t_A(x)$ is a lower bound of the grade of membership of x derived from the "evidence for x ", and $f_A(x)$ is a lower bound on the negation of x derived from the "evidence against x " and $t_A(x) + f_A(x) \leq 1$. Thus the grade of membership of x in the vague set A is bounded by subinterval $[t_A(x), 1 - f_A(x)]$ of $[0, 1]$. The vague set A is written as $A = \{ \langle x, [t_A(x), 1 - f_A(x)] \rangle / x \in X \}$. Where the interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of x in the vague set A and denoted by $V_A(x)$.

Definition 2. [5] A vague set A of a universe X with $t_A(x) = 0$ and $f_A(x) = 1$ for all $x \in X$, is called the zero vague set of X .

Definition 3. [5] A vague set A of a universe X with $t_A(x) = 1$ and $f_A(x) = 0$ for all $x \in X$, is called the unit vague set of X .

Definition 4. [9] Let A be a vague set A of a universe X with vague value V_A . Then the image of A , $f[A]$ is the vague set in Y with vague value V_A defined by

$$V_{f[A]}(y) = \begin{cases} \sup_{z \in f^{-1}(y)} & \text{if } f^{-1}(y) \neq \phi, \\ 0 & \text{if otherwise,} \end{cases}$$

for all $y \in Y$, where $f^{-1}(y) = \{x / f(x) = y\}$.

Definition 5. [3] Let $(X, +)$ be a group. A vague set A of X is called a vague additive group (briefly VAG) of X if the following condition is satisfied:

1. $V_A(x + y) \leq \max \{V_A(x), V_A(y)\}$, for all $x, y \in X$;

2. $V_A(-x) \leq V_A(x)$, for all $x \in X$.

Definition 6. [4] Let X be a ring and R be a vague set of X . Then R is called a vague ring of X if the following conditions are satisfies:

1. $V_R(x + y) \leq \max \{V_R(x), V_R(y)\}$, for all $x, y \in X$;
2. $V_R(-x) \leq V_R(x)$, for all $x \in X$;
3. $V_R(xy) \geq \min \{V_R(x), V_R(y)\}$, for all $x, y \in L$.

Definition 7. [4] Let X be a field and F be a vague set of X . Then F is called a vague feild of X if the following conditions are satisfies:

1. $V_F(x + y) \leq \max \{V_F(x), V_F(y)\}$, for all $x, y \in X$;
2. $V_F(-x) \leq V_F(x)$, for all $x \in X$;
3. $V_F(xy) \geq \min \{V_F(x), V_F(y)\}$, for all $x, y \in L$;
4. $V_F(x^{-1}) \geq V_F(x)$, for all $x \in X$.

3. Vague Vector Space

In this section, we introduce the notion of Vague vector space and investigate some of their properties.

Definition 8. Let V be a vector space over a feild F and A be a vague set of V . Then A is a vague vector space of V if the following properties are hold:

1. $V_A(x + y) \leq \max \{V_A(x), V_A(y)\}$, for all $x, y \in V$;
2. $V_A(ax) \leq V_A(x)$, for all $a \in F$ and $x \in V$;
3. $V_A(0) = 0$, $x \in V$.

Example 9. Let $V = \{\alpha, \beta, \gamma, \delta\}$ be a set of vectors and F be a Galio's field $G(F, 2)$, where $\alpha = (0, 0)$, $\beta = (1, 0)$, $\gamma = (0, 1)$, $\delta = (1, 1)$ and $F = \{0, 1\}$. Vector addition defined on V as folows:

Clearly $(V, +_2)$ is an abelian group. Scalar multiplication defined as follows:

$$c(a_1, a_2) = (ca_1, ca_2),$$

$+_2$	α	β	γ	δ
α	α	β	γ	δ
β	β	α	δ	γ
γ	γ	δ	α	β
δ	δ	γ	β	α

for all $c \in F$ and $a_1, a_2 \in V$. Clearly scalar multiplication is well defined. Then V is a vector space over $G(F, 2)$.

Let A be a vague set of V defined by

$$t_A(x) = \begin{cases} 0 & \text{if } x = \alpha \\ 0.7 & \text{if } x = \beta \\ 0.8 & \text{if } x = \gamma, \delta \end{cases}$$

and

$$f_A(x) = \begin{cases} 1 & \text{if } x = \alpha \\ 0.2 & \text{if } x = \beta \\ 0.1 & \text{if } x = \gamma, \delta \end{cases}$$

Then A is a vague vector space.

Example 10. Let $V = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ be a set of vectors and $F(I_3, +_3, \times_3)$ be a field, where $e_1 = 0$, $e_2 = 1$, $e_3 = 2$, $e_4 = x$, $e_5 = 2x$, $e_6 = x + 1$, $e_7 = x + 2$, $e_8 = 2x + 1$, $e_9 = 2x + 2$ and $I_3 = \{0, 1, 2\}$. Vector addition defined on V as follows:

$+_3$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
e_2	e_2	e_3	e_1	e_6	e_8	e_7	e_4	e_9	e_5
e_3	e_3	e_1	e_2	e_7	e_9	e_4	e_6	e_5	e_8
e_4	e_4	e_6	e_7	e_5	e_1	e_8	e_9	e_2	e_3
e_5	e_5	e_8	e_9	e_1	e_4	e_2	e_3	e_6	e_7
e_6	e_6	e_7	e_4	e_8	e_2	e_9	e_5	e_3	e_1
e_7	e_7	e_4	e_6	e_9	e_3	e_5	e_8	e_1	e_2
e_8	e_8	e_9	e_5	e_2	e_6	e_3	e_1	e_7	e_4
e_9	e_9	e_5	e_8	e_3	e_7	e_1	e_2	e_4	e_6

Clearly $(V, +_3)$ is an abelian group. Then V is a vector space with respect to addition and multiplication of polynomials. Let A be a vague set of V defined

by

$$t_A(x) = \begin{cases} 0 & \text{if } x = e_1 \\ 0.1 & \text{if } x = e_2, e_3 \\ 0.2 & \text{if } x = e_4, e_5, e_6, e_7, e_8, e_9 \end{cases}$$

and

$$f_A(x) = \begin{cases} 1 & \text{if } x = e_1 \\ 0.8 & \text{if } x = e_2, e_3 \\ 0.7 & \text{if } x = e_4, e_5, e_6, e_7, e_8, e_9 \end{cases}$$

Then A is a vague vector space.

Theorem 11. *If A is a vague vector space of a vector space V over a field F . Then $V_A(\lambda x) = V_A(x)$ for all $0 \neq \lambda \in F$.*

Proof. For all $x \in V$, we have $V_A(\lambda x) \leq V_A(x)$, for some $0 \neq \lambda \in F$.

Also for all $x \in V$, we have $V_A(x) = V_A(\lambda^{-1}\lambda x) \leq V_A(\lambda x)$. That is $V_A(x) \leq V_A(\lambda x)$.

Hence $V_A(\lambda x) = V_A(x)$. □

Theorem 12. *Let V be a vector space over the field F and A is a vague subset of V . Then A is a vague vector space of V if and only if*

$$V_A(ax + by) \leq \max \{V_A(x), V_A(y)\},$$

for all $x, y \in V$ and $a, b \in F$.

Proof. Suppose that A is a vague vector space of V , we have

$$V_A(ax) \leq V_A(x)$$

and

$$V_A(by) \leq V_A(y),$$

for all $x, y \in V$ and $a, b \in F$.

Hence $V_A(ax + by) \leq \max \{V_A(x), V_A(y)\}$.

Conversely suppose that if $a = b = 1$. We have

$$V_A(x + y) \leq \max \{V_A(x), V_A(y)\}.$$

Also if $b = 0$, we have $V_A(ax) = V_A(ax + 0.x) \leq \max \{V_A(x), V_A(0)\}$.

Hence $V_A(ax) \leq V_A(x)$ for all $x \in V$ and $a \in F$. □

Theorem 13. *The intersection of a family of vague vector spaces is a vague vector space.*

Proof. Let $\{A_i/i \in I\}$ be a family of vague vector spaces of a vector space V_F .

If $x, y \in F$ and $\lambda \in F$, then

$$\begin{aligned} V_A(x+y) &= \text{Sup}_{i \in I} V_A(x+y) \\ &\leq \text{Sup}_{i \in I} \{\max V_A(x), V_A(y)\} \\ &= \max \{\text{sup}_{i \in I} V_A(x), \text{sup}_{i \in I} V_A(y)\} \end{aligned}$$

also

$$\begin{aligned} V_A(\lambda x) &= \text{Sup}_{i \in I} V_A(\lambda x) \\ &\leq \text{Sup}_{i \in I} \{\max V_A(x)\} \\ &\leq \max \{\text{Sup}_{i \in I} V_A(x)\} \\ &= V_A(x). \end{aligned}$$

□

Theorem 14. *Let V_1 and V_2 be two vector spaces over a field F and f be a linear transformation of V_1 in to V_2 . Let A be a vague vector space of V_2 . Then the inverse image $f^{-1}(A)$ of A is a vague vector space of V_1 .*

Proof. Let $x, y \in V_1$ and $a, b \in F$, then

$$\begin{aligned} V_{f^{-1}(A)}(ax+by) &= V_A(f(ax+by)) \\ &= V_A[af(x)+bf(y)] \\ &\leq \max \{V_A f(x), V_A f(y)\} \\ &= \max \{V_{f^{-1}(A)}(x), V_{f^{-1}(A)}(y)\}. \end{aligned}$$

So $f^{-1}(A)$ is a vague vector space of V_1 .

□

Theorem 15. *Let V_1 and V_2 be two vector spaces over a field F and f be a linear transformation of V_1 in to V_2 . Let A be a vague vector space of V_1 that has the sup property then the image $f(A)$ of A is a vague vector space of V_2 .*

Proof. Let $\alpha, \beta \in V_2$. If either $f^{-1}(\alpha)$ or $f^{-1}(\beta)$ is empty, then the inequality theorem

$$V_A(ax+by) \leq \max \{V_A(x), V_A(y)\}$$

is satisfied, for all $x, y \in V$ and $a, b \in F$.

Suppose neither $f^{-1}(\alpha)$ nor $f^{-1}(\beta)$ is empty. Let $r_0 \in f^{-1}(\alpha)$, $s_0 \in f^{-1}(\beta)$ and

$$V_A(r_0) = \sup_{x \in f^{-1}(\alpha)} V_A(x),$$

$$V_A(s_0) = \sup_{y \in f^{-1}(\beta)} V_A(y).$$

Then

$$\begin{aligned} V_{f(A)}(a\alpha + b\beta) &= \sup_{r \in f^{-1}(a\alpha + b\beta)} V_A(r) \\ &\leq \max \{V_A(r_0), V_A(s_0)\} \\ &= \max \{V_{f(A)}(\alpha), V_{f(A)}(\beta)\}. \end{aligned}$$

So $f(A)$ is a vague vector space of V_2 . □

4. Vague Module

In this section, we introduce the notion of Vague module and investigate some of their properties.

Definition 16. Let R be a ring and M be a R module. Then the vague set A of M is a vague module of M if the following properties are hold:

1. $V_A(x + y) \leq \max \{V_A(x), V_A(y)\}$, for all $x, y \in M$;
2. $V_A(rx) \leq V_A(x)$, for all $r \in R$ and $x \in M$;
3. $V_A(0) = 0$, $0 \in M$.

Example 17. Let $Z_3[i] = \{0, 1, 2, i, 1 + i, 2 + i, 2i, 1 + 2i, 2 + 2i\}$ and $(Z_3, +_3, \times_3)$ is a ring. Addition modulo 3 table on $Z_3[i]$ as follows:

Clearly $(Z_3, +_3)$ is an abelian group.

Multiplication modulo 3 table on $Z_3[i]$ as follows:

Clearly $(Z_3, +_3, \times_3)$ is a R - module.

Let A be a vague set of Z_3 defined by

$$t_A(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0.1 & \text{if } x = 1, 2 \\ 0.2 & \text{if } x = i, 1 + i, 2 + i, 2i, 1 + 2i, 2 + 2i \end{cases}$$

$+_3$	0	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
0	0	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
1	1	2	0	$1+i$	$2+i$	i	$1+2i$	$2+2i$	$2i$
2	2	0	1	$2+i$	i	$1+i$	$2+2i$	$2i$	$1+2i$
i	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$	0	1	2
$1+i$	$1+i$	$2+i$	i	$1+2i$	$2+2i$	$2i$	1	2	0
$2+i$	$2+i$	i	$1+i$	$2+2i$	$2i$	$1+2i$	2	0	1
$2i$	$2i$	$1+2i$	$2+2i$	0	1	2	i	$1+i$	$2+i$
$1+2i$	$1+2i$	$2+2i$	$2i$	1	2	0	$1+i$	$2+i$	i
$2+2i$	$2+2i$	$2i$	$1+2i$	2	0	1	$2+i$	i	$1+i$

$\times_3$	0	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
2	0	2	1	$2i$	$2+2i$	$1+2i$	i	$2+i$	$1+i$
i	0	i	$2i$	2	$2+i$	$2+2i$	1	$1+i$	$1+2i$
$1+i$	0	$1+i$	$2+2i$	$2+i$	$2i$	1	$1+2i$	2	i
$2+i$	0	$2+i$	$1+2i$	$2+2i$	1	i	$1+i$	$2i$	2
$2i$	0	$2i$	i	1	$1+2i$	$1+i$	2	$2+2i$	$2+i$
$1+2i$	0	$1+2i$	$2+i$	$1+i$	2	$2i$	$2+2i$	i	1
$2+2i$	0	$2+2i$	$1+i$	$1+2i$	i	2	$2+i$	1	$2i$

and

$$f_A(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.8 & \text{if } x = 1, 2 \\ 0.7 & \text{if } x = i, 1+i, 2+i, 2i, 1+2i, 2+2i \end{cases}$$

Then A is a vague R - module.

Theorem 18. The intersection of a family of modules is a vague module.

Proof. Let $\{A_i/i \in I\}$ be a family of vague modules and $A = \bigcap_{i \in I} A_i$.

Then

$$\begin{aligned} V_A(x+y) &= \inf_{i \in I} V_A(a+b) \\ &\leq \inf_{i \in I} \{\max V_A(a), V_A(b)\} \\ &= \max \{V_A(x), V_A(y)\} \end{aligned}$$

also

$$\begin{aligned}
 V_A(ra) &= \sup_{i \in I} V_A(rx) \\
 &\leq \sup_{i \in I} \{ \max V_A(a) \} \\
 &\leq \max \{ \sup_{i \in I} V_A(a) \} \\
 &= V_A(a)
 \end{aligned}$$

□

Theorem 19. Let M_1 and M_2 be two R - modules over the ring R and f be a linear transformation of M_1 in to M_2 . Let A be a vague module of M_2 . Then the inverse image $f^{-1}(A)$ is a vague module of M_1 .

Proof. Let $x, y \in M_1$ and $a, b \in R$, then

$$\begin{aligned}
 V_{f^{-1}(A)}(ax + by) &= V_A(f(ax + by)) \\
 &= V_A[af(x) + bf(y)] \\
 &\leq \max \{ V_A f(x), V_A f(y) \} \\
 &= \max \{ V_{f^{-1}(A)}(x), V_{f^{-1}(A)}(y) \}.
 \end{aligned}$$

So $f^{-1}(A)$ is a vague module of M_1 .

□

Theorem 20. Let M_1 and M_2 be two R - modules over a ring R and f be a linear transformation of M_1 in to M_2 . Let A be a vague module of M_1 then the image $f(A)$ of A is a vague module of M_2 .

Proof. Let $\alpha, \beta \in M_2$. If either $f^{-1}(\alpha)$ or $f^{-1}(\beta)$ is empty, then the inequality theorem

$$V_A(ax + by) \leq \max \{ V_A(x), V_A(y) \}$$

is satisfied, for all $x, y \in V$ and $a, b \in F$.

Suppose neither $f^{-1}(\alpha)$ or $f^{-1}(\beta)$ is empty. Let $r_0 \in f^{-1}(\alpha)$, $s_0 \in f^{-1}(\beta)$ and

$$\begin{aligned}
 V_A(r_0) &= \sup_{x \in f^{-1}(\alpha)} V_A(x), \\
 V_A(s_0) &= \sup_{y \in f^{-1}(\beta)} V_A(y).
 \end{aligned}$$

Then

$$\begin{aligned}
 V_{f(A)}(a\alpha + b\beta) &= \sup_{r \in f^{-1}(a\alpha + b\beta)} V_A(r) \\
 &\leq \max \{ V_A(r_0), V_A(s_0) \} \\
 &= \max \{ V_{f(A)}(\alpha), V_{f(A)}(\beta) \}.
 \end{aligned}$$

So $f(A)$ is a vague module of M_2 .

□

5. Conclusion

In this paper, the concept of vague vector space and vague modules are introduced and some of properties are studied. It is hoped that these concepts will rise to the notations like vague normed linear spaces and vague inner product spaces, continuous mappings etc.

References

- [1] L.A. Zadeh, Fuzzy sets, *Information and Control*, **8**, (1965), 338-353.
- [2] Rangit Biswas, Vague groups, *Int. J. Comput. Congition*, **4**, No.2, (2006), 20 - 23.
- [3] V. Amarendra Babu and K.V. Rama Rao, Vague additiv groups, *Universal Journal of Applied Mathematics*, **3**, No. 3 (2015), 50 - 52.
- [4] V. Amarendra Babu and K.V. Rama Rao, Vague rings and vague fields , *Journal of Multidisciplinary Engineering Science and Technology*, **2**, No. 7, (2015), 1943 - 1945.
- [5] W.L. Gau and D.J. Buehrer, Vague sets, *IEEE Transactions on Systems, Man and Cybernetics*, **23**, No.20, (1993), 610-614.
- [6] Hakimuddin Khan, Ahmad and Rangit Biswas, On Vague groups, *International Journal of Computational Cognition*, **5**, No. 1,(2007), 27 - 30.
- [7] Rosen Field, A fuzzy groups, *Journal of Math. Anal and Appl.*, **35**, (1971), 512 - 517.
- [8] M. Derirci, Vague groups, *Journal of Math. and Appl.*, **230**, (1999), 142 - 156.
- [9] T.Eswarlal and N. Rama Krishna, Vague fields and vague vector spaces, *International Journal of Pure and Applied Mathematics*, **94**,No. 3, (2014), 295 - 305.