MAPPING PROPERTIES OF λ -SPIRALLIKE
AND λ -ROBERTSON FUNCTIONS OF
COMPLEX ORDER UNDER THE GENERALIZED
INTEGRAL OPERATOR

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Abstract: In this paper, we define the some mapping properties of two subclasses of analytic function classes $\mathcal{S}_\alpha^\lambda(b)$ and $\mathcal{C}_\alpha^\lambda(b)$ under generalized integral operator.

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1. Introduction

Let $\mathcal{A}$ denote the family of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization conditions $f(0) = 0$ and $f'(0) = 1$.

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A function $f \in \mathcal{A}$ is said to be λ - Spirallike function of complex order b and type α ($0 \leq \alpha < 1$), denoted by $\mathcal{S}_\alpha^\lambda(b)$ (see [2]) if and only if

$$\operatorname{Re} \left\{ 1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{zf(z)}{f(z)} - 1 \right) \right\} > \alpha \quad (z \in \mathbb{U}), \quad (2)$$

for some real numbers $\lambda \in \mathbb{R}$ with $|\lambda| < \frac{\pi}{2}$ and $b \in \mathbb{C} \setminus \{0\}$.

A function $f \in \mathcal{A}$ is said to be λ - Robertson functions of convex of complex order b and type α ($0 \leq \alpha < 1$), denoted by $\mathcal{C}_\alpha^\lambda(b)$ (see [14]) if and only if

$$\operatorname{Re} \left\{ 1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{zf(z)}{f(z)} \right) \right\} > \alpha \quad (z \in \mathbb{U}), \quad (3)$$

for some real numbers $\lambda \in \mathbb{R}$ with $|\lambda| < \frac{\pi}{2}$ and $b \in \mathbb{C} \setminus \{0\}$.

For $b=1$ and $\lambda = 0$, $S_\lambda(\rho, 1) = S(\rho)$ and $C_\lambda(\rho, 1) = C(\rho)$ the classes of functions that are starlike of order ρ and convex of order ρ in $\mathbb{U}$, respectively.

$$C(\rho) \subset S(\rho) \quad (0 \leq \rho < 1).$$

Notice that $S_0(b) = S(b)$ and $C_0(b) = C(b)$ the classes considered earlier by Nasr and Aouf [9] and Wiatrowski [15].

Let $P_k^\lambda(\rho)$ denote the class of functions $p : \mathbb{U} \rightarrow \mathbb{C}$, analytic in $\mathbb{U}$ satisfying the following properties

1. $p(0) = 1$ and

2.

$$\int_0^{2\pi} \left| \frac{\mathcal{R}\{p(z) - \rho\}}{1 - \rho} \right| d\theta \leq k\pi, \quad z = re^{i\theta} \in \mathbb{U},$$

where $k \geq 2$, $0 \leq \rho < 1$, λ is real with $|\lambda| < \frac{\pi}{2}$. For $\lambda = 0$, this class was introduced in [17] and for $\rho = 0$, see [13]. For $k = 2$, $\lambda = 0$ and $\rho = 0$, the class $P_k^\lambda(\rho)$ reduces to the class P of functions with positive real part and the class $k = 2$ gives us the class $P(\rho)$ of functions with positive real part greater than ρ . For $k > 2$, the functions in P_k may not have positive real part.

Definition 1. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}_\alpha^\lambda(b)$, ($b \in \mathbb{C} \setminus \{0\}$) if and only if

$$1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{zf(z)}{f(z)} - 1 \right) \in P_k^\lambda(\rho) \quad (k > 2, \quad 0 \leq \rho < 1). \quad (4)$$

Notice that $\mathcal{S}_\alpha^0(1) = \mathcal{S}_\alpha$, which is the class of functions with bounded radius rotation.

Definition 2. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{C}_\alpha^\lambda(b)$ ($b \in \mathbb{C} \setminus \{0\}$) if and only if

$$1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{zf'(z)}{f(z)} \right) \in P_k^\lambda(\rho) \quad (k > 2, \quad 0 \leq \rho < 1). \quad (5)$$

Notice that $\mathcal{C}_\alpha^0(1) = \mathcal{C}_\alpha$, which is the class of functions with bounded radius rotation.

First discussed by Paatero [3]. It is clear that

$$f(z) \in \mathcal{C}_\alpha^\lambda(b) \iff zf'(z) \in \mathcal{S}_\alpha^\lambda(b). \quad (6)$$

Recently, Frasin[8] introduced the following generalized integral operators:

Let $\alpha_i, \beta_i \in \mathbb{C}$ for all $i = 1, 2, 3, \dots, n, n \in \mathbb{N}, \gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$.

Let $\mathcal{I}_\gamma^{\alpha_i, \beta_i} : \mathcal{A}^n \longrightarrow \mathcal{A}$ be the integral operator defined by

$$\begin{aligned} & \mathcal{I}_\gamma^{\alpha_i, \beta_i}(f_1, f_2, \dots, f_n)(z) \\ &= \left\{ \int_0^z \gamma t^{\gamma-1} (f_1(t))^{\alpha_1} \left(\frac{f_1(t)}{t} \right)^{\beta_1} \dots (f_n(t))^{\alpha_n} \left(\frac{f_n(t)}{t} \right)^{\beta_n} dt \right\}^{\frac{1}{\gamma}}, \end{aligned} \quad (7)$$

where the power is taken as the principal one.

Notice that, the integral operator $\mathcal{I}_\gamma^{\alpha_i, \beta_i}(f_1, f_2, \dots, f_n)(z)$ generalizes several previously studied operators as follows:

1. For $\alpha_i = 0$ for all $i = 1, 2, 3, \dots, n$, the integral operator

$$\mathcal{I}_\gamma^{0, \beta_i}(f_1, f_2, \dots, f_n)(z) = \mathcal{I}_\gamma(f_1, f_2, \dots, f_n)(z),$$

where

$$\begin{aligned} & \mathcal{I}_\gamma(f_1, f_2, \dots, f_n)(z) \\ &= \left\{ \int_0^z \gamma t^{\gamma-1} \left(\frac{f_1(t)}{t} \right)^{\beta_1} \dots \left(\frac{f_n(t)}{t} \right)^{\beta_n} dt \right\}^{\frac{1}{\gamma}}, \end{aligned} \quad (8)$$

is introduced and studied by Breaz and Breaz [4].

2. For $\alpha_i = 0$ for all $i = 1, 2, 3, \dots, n$ and $\gamma = 1$, the integral operator $\mathcal{I}_1^{0, \beta_i}(f_1, f_2, \dots, f_n)(z) = \mathcal{F}_n(z)$, where

$$\mathcal{F}_n(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\beta_1} \dots \left(\frac{f_n(t)}{t} \right)^{\beta_n} dt \quad (9)$$

is introduced and studied by Breaz and Breaz [4].

3. For $\beta_i = 0$ for all $i = 1, 2, 3, \dots, n$ and $\gamma = 1$, the integral operator $\mathcal{I}_1^{\alpha_i, 0}(f_1, f_2, \dots, f_n)(z) = \mathcal{F}_{\alpha_1, \alpha_2, \dots, \alpha_n}(z)$, where

$$\mathcal{F}_{\alpha_1, \alpha_2, \dots, \alpha_n}(z) = \int_0^z (f_1(t))^{\alpha_1} \dots (f_n(t))^{\alpha_n} dt \quad (10)$$

is introduced and studied by Breaz and Breaz et al [6].

Recently, Breaz and Güney [5] considered the integral operators $\mathcal{F}_n(z)$ and $\mathcal{F}_{\alpha_1, \alpha_2, \dots, \alpha_n}(z)$ and obtained their properties on the classes $\mathcal{S}_\delta(b)$ and $\mathcal{C}_\delta(b)$ of starlike and convex functions of complex order b and type δ . Later on Noor et al. [11] considered the same integral operators and investigated the mapping properties for the classes $\mathcal{V}_k^\lambda(\rho, b)$ and $\mathcal{R}_k^\lambda(\rho, b)$.

Motivated by the aforementioned work, in this paper, the author investigates some mapping properties of the classes $\mathcal{V}_k^\lambda(\rho, b)$ and $\mathcal{R}_k^\lambda(\rho, b)$ under generalized integral operator defined in (8) when $\gamma = 1$. The results obtained in this paper are generalized results of Breaz and Güney [5] and Noor et al. [11].

We recall the following lemma which is useful for our investigation:

Lemma 3. ([12]). *Let $f(z) \in \mathcal{V}_k(\alpha)$, $0 \leq \alpha < 1$. Then $f(z) \in \mathcal{R}_k(\beta)$ where*

$$\beta = \frac{1}{4} \left[(2\alpha - 1) + \sqrt{4\alpha^2 - 4\alpha + 9} \right]. \quad (11)$$

2. Main Results

Theorem 4. *Let $f_i, \phi_i \in \mathcal{S}_\alpha^\lambda(b)$ for all $i = 1, 2, 3, \dots, n$ with $0 \leq \rho < 1$, $b \in \mathbb{C} \setminus \{0\}$ and $\alpha_i, \beta_i \in \mathbb{R}^+$ for $1 \leq i \leq n$. If*

$$0 \leq (\rho - 1)n + (\rho - 1) \sum_{i=1}^n \beta_i + 1 < 1, \quad (12)$$

then the integral operator

$$\mathcal{I}_1^{\alpha_i, \beta_i}(f_1, f_2, \dots, f_n)(z) = \int_0^z \prod_{i=1}^n (f_i(t))^{\alpha_i} \left(\frac{f_i(t)}{t} \right)^{\beta_i} dt, \quad (13)$$

belong to the class $\mathcal{C}_\psi^\lambda(b)$ with

$$\psi = 1 + (\rho - 1)n + (\rho - 1) \sum_{i=1}^n \beta_i. \quad (14)$$

Proof. For the sake of simplicity, in the proof, we shall write $\mathcal{H}(z)$ instead of $\mathcal{I}_1^{\alpha_i, \beta_i}(f_1, f_2, \dots, f_n)(z)$.

Differentiation (13) with respect to z , we obtain

$$\mathcal{H}(z) = \prod_{i=1}^n (f_i(z))^{\alpha_i} \left(\frac{f_i(z)}{z} \right)^{\beta_i}. \quad (15)$$

Let us define

$$\phi_i(z) = z (f_i(z))^{\alpha_i}. \quad (16)$$

Clearly, $\phi_i(z) \in \mathcal{A}$. Equation (2.4) becomes

$$\mathcal{H}(z) = \prod_{i=1}^n \frac{\phi_i(z)}{z^n} \left(\frac{f_i(z)}{z} \right)^{\beta_i}. \quad (17)$$

Logarithmic differentiation of (16) yields

$$\frac{\mathcal{H}'(z)}{\mathcal{H}(z)} = \sum_{i=1}^n \left[\beta_i \left(\frac{f_i'(z)}{f_i(z)} - \frac{1}{z} \right) + \left(\frac{\phi_i'(z)}{\phi_i(z)} - \frac{1}{z} \right) \right]. \quad (18)$$

Multiplying (17) by z and simplifying, we get

$$\begin{aligned} 1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z \mathcal{H}'(z)}{\mathcal{H}(z)} \right) &= 1 - n - \sum_{i=1}^n \beta_i \\ &+ \sum_{i=1}^n \left\{ \beta_i \left[1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z f_i'(z)}{f_i(z)} - 1 \right) \right] + 1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z \phi_i'(z)}{\phi_i(z)} - 1 \right) \right\}. \end{aligned} \quad (19)$$

Adding and subtracting α on the right hand side of (18) gives

$$\begin{aligned} &\left[\left(1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z \mathcal{H}'(z)}{\mathcal{H}(z)} \right) \right) - \psi \right] \\ &= \sum_{i=1}^n \beta_i \left[\left(1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z f_i'(z)}{f_i(z)} - 1 \right) \right) - \alpha \right] \\ &+ \sum_{i=1}^n \left[\left(1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z \phi_i'(z)}{\phi_i(z)} - 1 \right) - \alpha \right) \right], \end{aligned} \quad (20)$$

where ψ is given by (14). Taking real part of (20) and after simplification, we get

$$\int_0^{2\pi} \left| \operatorname{Re} \left[\left(1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z \mathcal{H}'(z)}{\mathcal{H}(z)} \right) \right) - \psi \right] \right| d\theta$$

$$\begin{aligned}
&\leq \sum_{i=1}^n \beta_i \int_0^{2\pi} \left| \operatorname{Re} \left[\left(1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z f_i(z)}{f_i(z)} - 1 \right) \right) - \alpha \right] \right| d\theta \\
&\quad + \sum_{i=1}^n \int_0^{2\pi} \left| \operatorname{Re} \left[\left(1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z \phi_i(z)}{\phi_i(z)} - 1 \right) \right) - \alpha \right] \right| d\theta. \quad (21)
\end{aligned}$$

Since, by hypothesis $f_i, \phi_i \in \mathcal{S}_\alpha^\lambda(b)$ for $1 \leq i \leq n$, we have

$$\int_0^{2\pi} \left| \operatorname{Re} \left[\left(1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z f_i(z)}{f_i(z)} - 1 \right) \right) - \alpha \right] \right| d\theta \leq (1 - \alpha) k\pi, \quad (22)$$

$$\int_0^{2\pi} \left| \operatorname{Re} \left[\left(1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z \phi_i(z)}{\phi_i(z)} - 1 \right) \right) - \alpha \right] \right| d\theta \leq (1 - \alpha) k\pi. \quad (23)$$

Using (22) and (23) in (21), we have

$$\int_0^{2\pi} \left| \mathcal{R} \left[\left(1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z \mathcal{H}(z)}{\mathcal{H}(z)} \right) \right) - \psi \right] \right| d\theta \leq (1 - \psi) k\pi, \quad (24)$$

where ψ is given by (14). Hence $\mathcal{H} \in \mathcal{S}_\alpha^\lambda(b)$. Thus, the proof of Theorem 4 is completed.

Put $\alpha_i = 0$ for all $i = 1, 2, 3, \dots, n$ in Theorem 2.1. Notice that, in such case $\mathcal{H}(z) = \mathcal{F}_n(z)$ and $\phi_i(z) = z$ which shows $\frac{z \phi_i(z)}{\phi_i(z)} - 1 = 0$.

Therefore, from (20), it follows that

$$\begin{aligned}
&\left[\left(1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z \mathcal{H}(z)}{\mathcal{H}(z)} \right) \right) - \psi \right] \\
&= \sum_{i=1}^n \beta_i \left[\left(1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z f_i(z)}{f_i(z)} - 1 \right) \right) - \alpha \right],
\end{aligned}$$

where $\psi = 1 + (1 - \alpha) \sum_{i=1}^n \beta_i$. □

Corollary 5. (cf. [11], Theorem 2.1). Let $f_i(z) \in \mathcal{S}_\alpha(b)$ for all $1 \leq i < n$ with $0 \leq \rho < 1, b \in \mathbb{C} \setminus \{0\}$ and $\beta_i > 0, 1 \leq i < n$. If

$$0 \leq 1 + (\rho - 1) \sum_{i=1}^n \beta_i < 1,$$

then $\mathcal{F}_n(z) \in \mathcal{C}_\delta^\lambda(b)$ with $\delta = 1 + (1 - \rho) \sum_{i=1}^n \beta_i$.

Next, we take $\beta_i = 0, 1 \leq i \leq n$ in Theorem 4. In this case

$$\mathcal{H}(z) = \mathcal{F}_{\alpha_1, \alpha_2, \dots, \alpha_n}(z) = \int_0^z \prod_{i=1}^n (f_i(z))^{\alpha_i} dz,$$

which implies $\mathcal{H}(z) = \prod_{i=1}^n \frac{\phi_i(z)}{z^n}$.

$$\frac{z\mathcal{H}(z)}{\mathcal{H}(z)} = \sum_{i=1}^n \left(\frac{z\phi_i(z)}{\phi_i(z)} - 1 \right) = \sum_{i=1}^n \alpha_i \frac{zf_i(z)}{f_i(z)},$$

we have the following results due to Noor et al. [11].

Corollary 6. (cf. [11], Theorem 2.5). Let $f_i(z) \in \mathcal{C}_\alpha(b)$ for all $1 \leq i < n$ with $0 \leq \rho < 1$, $b \in \mathbb{C} \setminus \{0\}$ and $\alpha_i > 0, 1 \leq i < n$. If

$$0 \leq 1 + (\rho - 1) \sum_{i=1}^n \alpha_i < 1,$$

then $\mathcal{F}_{\alpha_1, \alpha_2, \dots, \alpha_n}(z) \in \mathcal{C}_{\delta_1}(b)$ with $\delta_1 = 1 + (1 - \rho) \sum_{i=1}^n \alpha_i$.

Remark 7. If $k = 2$, in Corollary 2.2, we obtain the results of ([5], Theorem 1) and ([11], Corollary 2.2).

Remark 8. If $k = 2$, in Corollary 2.3, we obtain the results of ([5], Theorem 3) and ([11], Corollary 2.6).

Theorem 9. Let $f_i, \phi_i \in \mathcal{C}_\alpha(1)$ for all $i = 1, 2, 3, \dots, n$ with $0 \leq \alpha < 1$. Let $\alpha_i, \beta_i \in \mathbb{R}^+$ for $1 \leq i \leq n$. If

$$0 \leq (\beta - 1) \sum_{i=1}^n (1 + \beta_i) + 1 < 1, \quad (25)$$

then, $\mathcal{H}(z) \in \mathcal{C}_l(1)$ with $l = 1 + (\beta - 1) + \sum_{i=1}^n (1 + \beta_i)$ and β is given by (11).

Proof. proceeding as Theorem 4 with $b = 1$ and $\lambda = 0$. we have

$$\begin{aligned} & \int_0^{2\pi} \left| \operatorname{Re} \left[\left(1 + \left(\frac{z\mathcal{H}(z)}{\mathcal{H}(z)} \right) \right) - l \right] \right| d\theta \\ & \leq \sum_{i=1}^n \beta_i \int_0^{2\pi} \left| \operatorname{Re} \left[\left(\frac{zf_i(z)}{f_i(z)} \right) - \beta \right] \right| d\theta \\ & \quad + \sum_{i=1}^n \int_0^{2\pi} \left| \operatorname{Re} \left[\left(\frac{z\phi_i(z)}{\phi_i(z)} \right) - \beta \right] \right| d\theta. \end{aligned} \quad (26)$$

Since, by hypothesis $f_i, \phi_i \in \mathcal{C}_\alpha(l)$ for $1 \leq i \leq n$ and by using Lemma 3, we have

$$\int_0^{2\pi} \left| \operatorname{Re} \left[\left(\frac{zf_i(z)}{f_i(z)} \right) - \beta \right] \right| d\theta \leq (1 - \beta) k\pi, \quad (27)$$

and

$$\int_0^{2\pi} \left| \operatorname{Re} \left[\left(\frac{z\phi_i(z)}{\phi_i(z)} \right) - \beta \right] \right| d\theta \leq (1 - \beta) k\pi, \quad (28)$$

and β is given by (11). Using (27) and (28) in (26), we obtain

$$\int_0^{2\pi} \left| \operatorname{Re} \left[\left(1 + \left(\frac{z\mathcal{H}(z)}{\mathcal{H}(z)} \right) \right) - l \right] \right| d\theta \leq (1 - l) k\pi. \quad (29)$$

Thus, $\mathcal{H}(z) \in \mathcal{C}^\lambda(1, l)$ with $l = 1 + (\beta - 1) + \sum_{i=1}^n (1 + \beta_i)$. □

Remark 10. For $\alpha_i = 0$, we obtain the results of ([11], Theorem 2.3).

For $n = 1, \lambda = 0, \alpha_1 = 0, \beta_1 = 0, f_1 = f$ in Theorem 9. We get the following results due to [11].

Corollary 11. [11]. Let $f(z) \in \mathcal{C}_\alpha(1)$. Then the Alexander operator $\mathcal{I}(z) = \int_0^z \frac{f(t)}{t} dt$ (see [1]) belongs to the class $\mathcal{C}_\alpha(1)$, where β is given by (11).

Remark 12. For $\alpha = 0$ and $k = 2$, in the above Corollary 11, we have the well known result $f(z) \in \mathcal{C}(0) \Rightarrow \mathcal{I}(z) \in \mathcal{C}(\frac{1}{2})$.

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