SIMPLE GRAPHOIDAL COVERING NUMBER OF PRODUCT OF GRAPHS

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Abstract: A graphoidal cover of G is a set ψ of (not necessarily open) paths in G , such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most one path in ψ and every edge of G is in exactly one path in ψ . The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by η . If every two paths in ψ have at most one common vertex, then it is called simple graphoidal cover of G . The minimum cardinality of a simple graphoidal cover of G is called simple graphoidal covering number of G and is denoted by η_s . Here we determine the simple graphoidal covering number of Product of Graphs.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected graph without loop or multiple edges. The order and size of the G are denoted by p and q respectively. For theoretical terminology of graph we refer Harary [10]. All the graphs considered in this paper are assumed to be connected and non-trivial. If $P = (v_1, v_2, v_3, \dots, v_n)$ be a path or cycle in a graph G , the vertices $v_2, v_3, v_4, \dots, v_{n-1}$ are called internal vertices of P and v_1, v_n are called external vertices of P . Two paths P and Q are said to be internally disjoint if no vertex of G is an internal vertex of both P and Q . The concept of graphoidal cover was introduced by Acharya and Sampath Kumar [1][2]. Arumugam and Pakkiam [5],[6],[7],[8],[9] determined the graphoidal covering number of several families of curve. Further Arumugam and Suresh Suseela [3] introduced acyclic graphoidal cover. The concept of simple graphoidal cover was introduced by Arumugam and Shahul Hamid [4].

Definition 1. (see [1]) A graphoidal cover of G is a set ψ of (not necessarily open) paths in G satisfying the following conditions.

- (i) Every path in ψ has at least two vertices.
- (ii) Every vertex of G is an internal vertex of at most one path in ψ .
- (iii) Every edge of G is in exactly one path in ψ .

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by η .

Definition 2. (see [4]) A simple graphoidal cover of a graph G is a graphoidal cover ψ of G such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple graphoidal cover of G is called simple graphoidal covering number of G and is denoted by η_s .

Definition 3. (see [4]) Let ψ be a collection of internally disjoint paths in G . A vertex of G is said to be an interior vertex of if it is an internal vertex of some path in ψ . Any vertex which is not an interior vertex of is said to be an exterior vertex of ψ .

Theorem 4. For any simple graphoidal cover ψ of a (p, q) of graph G , let t_ψ denote the number of exterior vertices of ψ and let $t = \min t_\psi$, where the minimum is taken overall simple graphoidal covers ψ of G , Then $\eta_s(G) = q - p + t$.

Corollary 5. For any graph G , $\eta_s(G) \geq q - p$. Moreover, the following are equivalent

- (i) $\eta_s(G) = q - p$.
- (ii) There exists a simple graphoidal cover of G without exterior vertices
- (iii) There exists a set P of internally disjoint and edge disjoint paths without exterior vertices such that any two paths in P have at most one vertex in common.

Theorem 6. For the complete graph $K_n (n \geq 3)$, we have

$$\eta_s(K_n) = \begin{cases} 4 & \text{if } n = 4 \\ \frac{(n-1)(n-2)}{2} & \text{if } n \text{ is odd} \\ \frac{n(n-3)}{2} & \text{if } n > 4 \text{ and } n \text{ is even} \end{cases}$$

Given two graphs G and H , their product is a graph whose vertex set is $V(G) \times V(H) = \{(g, h) : g \in V(G) \text{ and } h \in V(H)\}$, while the edge set varies according to the nature of the product. Various kinds of graph products are defined in [12]. We consider only Cartesian product, Strong product and Lexicographic product of two paths or cycles.

The Cartesian Product of two graphs G and H denoted by $G \circ H$ has the edge set $E(G \circ H) = \{((g, h), (g', h')) : g = g', (h, h') \in E(H) \text{ or } h = h', (g, g') \in E(G)\}$.

The Strong Product of graphs G and H is denoted by $G \otimes H$ where $E(G \otimes H) = \{((g, h), (g', h')) : g = g', (h, h') \in E(H); (g, g') \in E(G), h = h' \text{ or } (g, g') \in E(G) \text{ and } (h, h') \in E(H)\}$.

The Lexicographic Product or Composition of two graphs G and H is denoted by $G \odot H$ where $E(G \odot H) = \{((g, h), (g', h')) : (g, g') \in E(G) \text{ or } g = g', (h, h') \in E(H)\}$. For the positive integer m and n , we use the following notations

$$P_m = (u_1, u_2, \dots, u_m)$$

$$P_n = (u_1, u_2, \dots, u_n)$$

$$C_m = (u_1, u_2, \dots, u_m, u_1)$$

$$C_n = (u_1, u_2, \dots, u_n, u_1)$$

For all the product graphs G , we have considered the vertex set $V(G) = \{w_{ij} : 1 \leq i \leq m; 1 \leq j \leq n\}$ where $w_{i,j} = (u_i, v_j)$.

2. Main Results

Theorem 7. If $G = P_m \circ P_n$, where $m, n \geq 2$ then

$$\eta_s(G) = \begin{cases} q - p + 1 & \text{if } m = n = 2 \text{ or } m, n \geq 3 \\ q - p + 2 & \text{if } m = 2 \text{ and } n \geq 3 \end{cases}$$

Proof. The Edge set $E(G) = \{w_{ij}w_{i(j+1)} : 1 \leq i \leq m, 1 \leq j \leq (n-1)\} \cup \{w_{ij}w_{(i+1)j} : 1 \leq i \leq (m-1), 1 \leq j \leq n\}$.

When $m = 2$ and $n \geq 3$, The collection of paths of G are

$$P = (w_{11}, w_{12}, w_{22}, w_{21}, w_{11})$$

$$Q = (w_{12}, w_{13}, \dots, w_{1n}, w_{2n})$$

$$R = (w_{22}, w_{23}, \dots, w_{2n})$$

$$S = E(G) - P \cup Q \cup R$$

is a simple graphoidal cover of G in which w_{11}, w_{2n} are not internal. Therefore $\eta_s(G) \leq q - p + 2$. Now, let ψ be any simple graphoidal cover of G . If ψ contains only paths, then atleast three vertices can not be made internal. If ψ contains atleast one cycle, then atleast two vertices can not be made internal. Therefore $\eta_\psi \geq 2$. Hence $\eta_s(G) \geq q - p + 2$. Thus $\eta_s(G) = q - p + 2$.

When $m, n \geq 3$, then we have the following four cases.

Case (i) when $m = n = \text{odd}$, The collection of paths of G are

$$P = (w_{11}, w_{12}, w_{22}, w_{21}, w_{11})$$

$$Q_i = (w_{2i}, w_{2(i+1)}, w_{3(i+1)}, w_{3i}, w_{2i}) : i = 2, 4, 6, \dots, (n-1)$$

$$R_i = (w_{2i}, w_{1i}, w_{1(i+1)}, w_{2(i+1)}, w_{2i}) : i = 3, 5, 7, \dots, (n-2)$$

$$S_i = (w_{i2}, w_{i1}, w_{(i+1)1}, w_{(i+1)2}, w_{i2}) : i = 3, 5, 7, \dots, (m-2)$$

$$T_i = (w_{ij}, w_{i(j+1)}, w_{(i+1)(j+1)}, w_{(i+1)j}, w_{ij}) : i = 4, 6, \dots, (m-1); j = 2, 4, \dots, (n-1)$$

$$U_i = (w_{ij}, w_{i(j+1)}, w_{(i-1)(j+1)}) : i = 4, 6, 8, \dots, (m-1); j = 3, 5, \dots, (n-2)$$

$$V = (w_{1(n-1)}, w_{1n}, w_{2n})$$

$$W = (w_{(m-1)1}, w_{m1}, w_{m2})$$

$$X = E(G) - P \cup Q_i \cup R_i \cup S_i \cup T_i \cup U_i \cup V \cup W$$

is a simple graphoidal cover of G in which w_{11} is not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now, let ψ be any simple graphoidal cover of G . If ψ contains only paths, then atleast three vertices can not be made internal. If ψ contains atleast

one cycle, then atleast one vertex can not be made internal. *Therefore* $t_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

Case(ii) When $m = \text{odd}$ and $n = \text{even}$, The collection of paths of G are

$$\begin{aligned}
 P &= (w_{11}, w_{12}, w_{22}, w_{21}, w_{11}) \\
 Q_i &= (w_{2i}, w_{2(i+1)}, w_{3(i+1)}, w_{3i}, w_{2i}) : i = 2, 4, 6, \dots, (n-2) \\
 R_i &= (w_{2i}, w_{1i}, w_{1(i+1)}, w_{2(i+1)}, w_{2i}) : i = 3, 5, 7, \dots, (n-1) \\
 S_i &= (w_{i2}, w_{i1}, w_{(i+1)1}, w_{(i+1)2}, w_{i2}) : i = 3, 5, 7, \dots, (m-2) \\
 T_i &= (w_{i(n-1)}, w_{in}, w_{(i+1)n}, w_{(i+1)(n-1)}, w_{i(n-1)}) : i = 3, 5, \dots, (m-2) \\
 U_i &= (w_{i2}, w_{i3}, w_{(i+1)3}, w_{(i+1)2}, w_{i2}) : i = 4, 6, \dots, (m-1) \\
 V_i &= (w_{i(n-1)}, w_{i(n-2)}, w_{(i+1)(n-2)}, w_{(i+1)(n-1)}, w_{i(n-1)}) : i = 4, 6, \dots, (m-1) \\
 W_i &= (w_{ij}, w_{i(j+1)}, w_{(i+1)(j+1)}, w_{(i+1)j}, w_{ij}) : i = 4, 6, \dots, (m-1), j = 4, 6, \dots, (n-4) \\
 X_i &= (w_{ij}, w_{i(j+1)}, w_{(i-1)j}) : i = 4, 6, \dots, (m-1), j = 3, 5, \dots, (n-5) \\
 Y &= (w_{(m-1)1}, w_{m1}, w_{m2}) \\
 Z &= (w_{m(n-1)}, w_{mn}, w_{(m-1)n}) \\
 Z' &= E(G) - P \cup Q_i \cup R_i \cup S_i \cup T_i \cup U_i \cup V_i \cup W_i \cup X_i \cup Y \cup Z
 \end{aligned}$$

is a simple graphoidal cover of G in which w_{11} is not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now, let ψ be any simple graphoidal cover of G . If ψ contains only paths, then atleast three vertices can not be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. *Therefore* $t_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

Case (iii) When $m = \text{even}$ and $n = \text{odd}$, The collection of paths of G are

$$\begin{aligned}
 P &= (w_{11}, w_{12}, w_{22}, w_{21}, w_{11}) \\
 Q_i &= (w_{2i}, w_{2(i+1)}, w_{3(i+1)}, w_{3i}, w_{2i}) : i = 2, 4, 6, \dots, (n-1) \\
 R_i &= (w_{2i}, w_{1i}, w_{1(i+1)}, w_{2(i+1)}, w_{2i}) : i = 3, 5, 7, \dots, (n-2) \\
 S_i &= (w_{i2}, w_{i1}, w_{(i+1)1}, w_{(i+1)2}, w_{i2}) : i = 3, 5, 7, \dots, (m-1) \\
 T_i &= (w_{ij}, w_{i(j+1)}, w_{(i+1)(j+1)}, w_{(i+1)j}, w_{ij}) : i = 4, 6, \dots, (m-2), j = 2, 4, \dots, (n-1) \\
 U_i &= (w_{mi}, w_{m(i+1)}, w_{(m-1)(i+1)}) : i = 2, 3, \dots, (n-1) \\
 V_i &= (w_{ij}, w_{i(j+1)}, w_{(i-1)j}) : i = 4, 6, \dots, (m-2), j = 3, 5, \dots, (n-2) \\
 W &= (w_{1(n-1)}, w_{1n}, w_{2n}) \\
 X &= E(G) - P \cup Q_i \cup R_i \cup S_i \cup T_i \cup U_i \cup V_i \cup W
 \end{aligned}$$

is a simple graphoidal cover of G in which w_{11} is not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now, let ψ be any simple graphoidal cover of G . If ψ contains only paths, then atleast three vertices can not be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $\eta_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

Case(iv) When $m = n = \text{even}$ or $m = n = 2$. The collection of paths of G are

$$\begin{aligned}
 P &= (w_{11}, w_{12}, w_{22}, w_{21}, w_{11}) \\
 Q_i &= (w_{2i}, w_{2(i+1)}, w_{3(i+1)}, w_{3i}, w_{2i}) : i = 2, 4, \dots, (n-2) \\
 R_i &= (w_{2i}, w_{1i}, w_{1(i+1)}, w_{2(i+1)}, w_{2i}) : i = 3, 5, \dots, (n-1) \\
 S_i &= (w_{i2}, w_{i1}, w_{(i+1)1}, w_{(i+1)2}, w_{i2}) : i = 3, 5, \dots, (m-1) \\
 T_i &= (w_{i(n-1)}, w_{in}, w_{(i+1)n}, w_{(i+1)(n-1)}, w_{i(n-1)}) : i = 3, 5, \dots, (m-1) \\
 U_i &= (w_{i2}, w_{i3}, w_{(i+1)3}, w_{(i+1)2}, w_{i2}) : i = 4, 6, 8, \dots, (m-2) \\
 V_i &= (w_{i(n-1)}, w_{i(n-2)}, w_{(i+1)(n-2)}, w_{(i+1)(n-1)}, w_{i(n-1)}) : i = 4, 6, \dots, (m-2) \\
 W_i &= (w_{ij}, w_{i(j+1)}, w_{(i+1)(j+1)}, w_{(i+1)j}, w_{ij}) : i = 4, 6, \dots, (m-2), j = 4, 6, \dots, (n-4) \\
 Y_i &= (w_{ij}, w_{i(j+1)}, w_{(i-1)(j+1)}) : i = 4, 6, \dots, (m-2), j = 3, 5, \dots, (n-5) \\
 Z_i &= (w_{mi}, w_{m(i+1)}, w_{(m-1)(i+1)}) : i = 2, 3, \dots, (n-3)
 \end{aligned}$$

$$Z = E(G) - P \cup Q_i \cup R_i \cup S_i \cup T_i \cup U_i \cup V_i \cup W_i \cup Y_i \cup Z_i$$

is a simple graphoidal cover of G in which w_{11} is the only vertex which is not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now, let ψ be any simple graphoidal cover of G . If ψ contains only paths, then atleast three vertices can not be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $\eta_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$. $\square$

Theorem 8. If $G = C_m \circ C_n$, then $\eta_s(G) = q - p + 1$.

Proof. The edge set $E(G) = \{w_{ij}w_{i(j+1)} : 1 \leq i \leq m, 1 \leq j \leq (n-1)\} \cup \{w_{in}w_{i1} : 1 \leq i \leq m\} \cup \{w_{ij}w_{(i+1)j} : 1 \leq i \leq (m-1), 1 \leq j \leq n\} \cup \{w_{mj}w_{1j} : 1 \leq j \leq n\}$. The collection of paths of G are

$$\begin{aligned}
 P_i &= (w_{i1}, w_{i2}, w_{i3}, \dots, w_{in}, w_{i1}), 1 \leq i \leq m \\
 Q &= (w_{11}, w_{21}, \dots, w_{m1}, w_{11}) \\
 R &= E(G) - P_i \cup Q
 \end{aligned}$$

is a simple graphoidal cover of G in which w_{11} is not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now, let ψ be any simple graphoidal cover of G . Here we have two cases.

Case(i) when $m=n$, If ψ contains only paths, then atleast $mn-(m+n)>1$ vertices can not be made internal, since every path is of length 2. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $\eta_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

Case(ii) when $m \neq n$, If ψ contains only paths, then atleast $mn-(m+n)-1$ vertices can not be made internal, since every path is of length 2. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $\eta_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$. $\square$

Theorem 9. If $G = P_m \circ C_n$ where $m \geq 2, n \geq 3$, then $\eta_s(G) = q - p + 2$

Proof. The edge set $E(G) = \{w_{ij}w_{i(j+1)} : 1 \leq i \leq m, 1 \leq j \leq (n-1)\} \cup \{w_{in}w_{i1} : 1 \leq i \leq m\} \cup \{w_{ij}w_{(i+1)j} : 1 \leq i \leq (m-1), 1 \leq j \leq n\}$. The collection of paths of G are

$$P_i = (w_{i1}, w_{i2}, w_{i3}, \dots, w_{in}, w_{i1}), 1 \leq i \leq m.$$

$$Q = (w_{11}, w_{21}, \dots, w_{m1}, w_{11})$$

$$R = E(G) - P_i \cup Q$$

is a simple graphoidal cover of G in which w_{11} and w_{n1} are not internal. Therefore $\eta_s(G) \leq q - p + 2$. Now, let ψ be any simple graphoidal cover of G . If ψ contains only paths, then atleast four vertices can not be made internal. If ψ contains atleast one cycle, then atleast two vertices can not be made internal. Therefore $\eta_\psi \geq 2$. Hence $\eta_s(G) \geq q - p + 2$. Thus $\eta_s(G) = q - p + 2$. $\square$

Theorem 10. For the graph $G = P_m \odot P_n$, where $m, n \geq 2$ then

$$\eta_s(G) = \begin{cases} 4 & \text{if } m = n = 2 \\ q - p + 1 & \text{if } m = 2 \text{ and } n = 3 \text{ or } m = n = 3 \\ q - p & \text{if } m \geq 3 \text{ and } n \geq 4 \text{ or } m = 2 \text{ and } n \geq 4 \end{cases}$$

Proof. The edge set $E(G) = \{w_{ij}w_{(i+1)k} : 1 \leq i \leq (m-1), i \leq j \leq n, 1 \leq k \leq n\} \cup \{w_{ij}w_{i(j+1)} : 1 \leq i \leq m, 1 \leq j \leq (n-1)\}$.

When $m = n = 2$, Then $G = K_4$, by Theorem 6, $\eta_s(G) = 4$

When $m=2$ and $n=3$, the collection of paths of G are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}), 1 \leq i \leq n-1$$

$$Q = (w_{11}, w_{2n}, w_{1n})$$

$$R = E(G) - P_i \cup Q$$

is a simple graphoidal graph in which w_{11} is not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now let ψ be any graphoidal cover of G . If ψ contains only paths, then atleast four vertices can not be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $\eta_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

when $m=2$ and $n \geq 4$, the collection of path of G are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}), 1 \leq i \leq (n - 1).$$

$$Q = (w_{22}, w_{11}, w_{2n})$$

$$R = (w_{12}, w_{2n}, w_{1n})$$

$$S = E(G) - P_i \cup Q \cup R$$

is a simple graphoidal graph in which all vertices made internal. By corollary (5), $\eta_s(G) = q - p$.

When $m=3$ and $n=3$, the collection of paths of G are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}), 1 \leq i \leq (n - 1)$$

$$Q = (w_{21}, w_{31}, w_{32}, w_{21})$$

$$R = (w_{22}, w_{23}, w_{33}, w_{22})$$

$$S = E(G) - P_i \cup Q \cup R$$

is a simple graphoidal graph in which w_{11} is the only vertex not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now let ψ be any graphoidal cover of G . If ψ contains only paths, then atleast four vertices can not be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $\eta_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

When $m \geq 3$ and $n \geq 4$, then we have following cases.

Case (i) when $m=n$ =odd, the collection of paths of G are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}); 1 \leq i \leq (n - 1)$$

$$Q_i = (w_{ij}, w_{(i+1)j}, w_{(i+1)(j+1)}, w_{ij}) : i = 2, 3, 4 \dots (m - 1), j = 1, 3, 5 \dots (n - 2)$$

$$R_i = (w_{i(n-1)}, w_{in}, w_{(i+1)n}, w_{i(n-1)}); i = 2, 4, 6 \dots (m - 1)$$

$$S = (w_{22}, w_{11}, w_{2n})$$

$$T = E(G) - P_i \cup Q_i \cup R_i \cup S$$

is a simple graphoidal cover of G in which all the vertices made internal. By corollary (5), $\eta_s(G) = q - p$.

Case (ii) when $m = \text{odd}$ and $n = \text{even}$, then the collection of path of G are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}) : 1 \leq i \leq (n - 1)$$

$$Q_i = (w_{ij}, w_{(i+1)j}, w_{(i+1)(j+1)}, w_{ij}) : i = 2, 3, 4 \dots (m - 1), j = 1, 3, 5 \dots (n - 1)$$

$$R = (w_{1n}, w_{2n}, w_{3n})$$

$$S = (w_{22}, w_{11}, w_{2n})$$

$$T = E(G) - P_i \cup Q_i \cup R \cup S$$

is a simple graphoidal cover of G in which all the vertices made internal. By corollary (5), $\eta_s(G) = q - p$.

Case (iii) when $m = \text{even}$ and $n = \text{odd}$, then the collection of path of G are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}) : 1 \leq i \leq (n - 1)$$

$$Q_i = (w_{ij}, w_{(i+1)j}, w_{(i+1)(j+1)}, w_{ij}) : i = 2, 3, 4 \dots (m - 1), j = 1, 3, 5 \dots (n - 2)$$

$$R_i = (w_{i(n-1)}, w_{in}, w_{(i+1)n}, w_{i(n-1)}) : i = 3, 5, 7, \dots (m - 1)$$

$$S = (w_{1n}, w_{2n}, w_{3n})$$

$$T = (w_{22}, w_{11}, w_{2n})$$

$$U = E(G) - P_i \cup Q_i \cup R_i \cup S \cup T$$

is a simple graphoidal cover of G in which all the vertices made internal. By Corollary (5), $\eta_s(G) = q - p$.

Case (iv) when $m=n = \text{even}$, then the collection of path of G are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}), 1 \leq i \leq (n - 1)$$

$$Q_i = (w_{ij}, w_{(i+1)j}, w_{(i+1)(j+1)}, w_{ij}) : i = 2, 3, 4 \dots (m - 1), j = 1, 3, 5 \dots (n - 1)$$

$$R = (w_{1n}, w_{2n}, w_{3n})$$

$$S = (w_{22}, w_{11}, w_{2n})$$

$$T = E(G) - P_i \cup Q_i \cup R \cup S$$

is a simple graphoidal cover of G in which all the vertices made internal. By corollary (5), $\eta_s(G) = q - p$. $\square$

Theorem 11. For the graph $G = C_m \odot C_n$, where $m, n \geq 3$, then

$$\eta_s(G) = \begin{cases} q - p + 1 & \text{if } m = 3 \text{ and } n \geq 3 \\ q - p & \text{if } m = n \geq 4 \end{cases}$$

Proof. Let $E(G) = \{w_{ij}w_{(i+1)k} : 1 \leq i \leq (n-1); 1 \leq j \leq n; 1 \leq k \leq n\} \cup \{w_{ij}w_{i(j+1)} : 1 \leq i \leq m; 1 \leq j \leq (n-1)\} \cup \{w_{i1}w_{in} : 1 \leq i \leq m\} \cup \{w_{1j}w_{mk} : 1 \leq j \leq n; 1 \leq k \leq n\}$

When $m = 3$ and $n \geq 3$, the collection of paths of G are

$$P_i = (w_{i1}, w_{i2}, w_{i3} \dots w_{in}, w_{i1}) : 1 \leq i \leq 3$$

$$Q = (w_{11}, w_{21}, w_{31}, w_{11})$$

$$R = E(G) - P_i \cup Q$$

is a simple graphoidal cover of G in which w_{11} is not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now let ψ be any graphoidal cover of G . If ψ contains only paths, then no vertices can be made internal, since every path is an edge. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $\eta_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

When $m \geq 4$ and $n \geq 4$, the collection of paths of G are

$$P_i = (w_{i1}, w_{i2}, w_{i3} \dots w_{in}, w_{i1}), 1 \leq i \leq m$$

$$Q = (w_{11}, w_{21}, w_{31}, \dots w_{m1}, w_{11})$$

$$R = (w_{22}, w_{11}, w_{2n})$$

$$S = E(G) - P_i \cup Q \cup R$$

is a simple graphoidal cover of G in which all vertices becomes internal. By corollary (5), $\eta_s(G) = q - p$. $\square$

Theorem 12. For the graph $G = P_m \odot C_n$, where $m \geq 2, n \geq 3$, then

$$\eta_s(G) = \begin{cases} q - p + 2 & \text{if } m = 2 \text{ and } n = 3 \\ q - p + 1 & \text{if } m = n = 3 \\ q - p & \text{if } m = 2 \text{ and } n \geq 4 \text{ or } m \geq 3 \text{ and } n \geq 4 \end{cases}$$

Proof. Let $E(G) = \{w_{ij}w_{(i+1)k} : 1 \leq i \leq (m-1), 1 \leq j \leq n, 1 \leq k \leq n\} \cup \{w_{ij}w_{i(j+1)} : 1 \leq i \leq n, 1 \leq j \leq (n-1)\} \cup \{w_{i1}w_{in} : 1 \leq i \leq m\}$.

When $m = 2$ and $n = 3$, then the collection of paths of G are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}) : 1 \leq i \leq n - 1$$

$$Q = E(G) - P_i$$

is a simple graphoidal cover of G in which w_{11} and w_{2n} are not internal. Therefore $\eta_s(G) \leq q - p + 2$. Now let ψ be any graphoidal cover of G . If ψ contains only paths, then no vertices can be made internal, since every path is an edge. If ψ

contains atleast one cycle, then atleast two vertices can not be made internal. *Therefore* $\eta_\psi \geq 2$. Hence $\eta_s(G) \geq q - p + 2$. Thus $\eta_s(G) = q - p + 2$.

when $m = n = 3$, The collection of paths of G are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}) : 1 \leq i \leq n - 1$$

$$Q = (w_{21}, w_{31}, w_{32}, w_{21})$$

$$R = (w_{22}, w_{23}, w_{33}, w_{22})$$

$$S = E(G) - P_i \cup Q \cup R$$

is a graphoidal cover of G in which w_{11} is not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now let ψ be any graphoidal cover of G . If ψ contains only paths, then atleast three vertices can not be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. *Therefore* $\eta_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

when $m = 2$ and $n \geq 4$, The collection of paths of are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}) : 1 \leq i \leq (n - 1)$$

$$Q = (w_{12}, w_{2n}, w_{1n})$$

$$R = (w_{22}, w_{11}, w_{2n})$$

$$S = E(G) - P_i \cup Q \cup R$$

is a graphoidal cover of G in which all vertices made internal. By corollary (5), $\eta_s(G) = q - p$.

When $m \geq 3$ and $n \geq 4$, then we have following cases. Case (i) when $m=n=\text{odd}$, the collection of paths of G are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}); 1 \leq i \leq n - 1$$

$$Q_i = (w_{ij}, w_{(i+1)j}, w_{(i+1)(j+1)}, w_{ij}) : i = 2, 3, 4 \dots (m - 1), j = 1, 3, 5 \dots (n - 2)$$

$$R_i = (w_{i(n-1)}, w_{in}, w_{(i+1)n}, w_{i(n-1)}); i = 2, 4, 6 \dots (m - 1)$$

$$S = (w_{22}, w_{11}, w_{2n})$$

$$T = E(G) - P_i \cup Q_i \cup R_i \cup S$$

is a simple graphoidal cover of G in which all the vertices made internal. By corollary (5), $\eta_s(G) = q - p$.

Case (ii) when $m = \text{odd}$ and $n = \text{even}$, then the collection of path of G are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}) : 1 \leq i \leq (n - 1)$$

$$Q_i = (w_{ij}, w_{(i+1)j}, w_{(i+1)(j+1)}, w_{ij}) : i = 2, 3, 4 \dots (m - 1), j = 1, 3, 5 \dots (n - 1)$$

$$R = (w_{1n}, w_{2n}, w_{3n})$$

$$S = (w_{22}, w_{11}, w_{2n})$$

$$T = E(G) - P_i \cup Q_i \cup R \cup S$$

is a simple graphoidal cover of G in which all the vertices made internal. By corollary (5), $\eta_s(G) = q - p$.

Case (iii) when $m = \text{even}$ and $n = \text{odd}$, then the collection of path of G are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}) : 1 \leq i \leq (n - 1)$$

$$Q_i = (w_{ij}, w_{(i+1)j}, w_{(i+1)(j+1)}, w_{ij}) : i = 2, 3, 4 \dots (m - 1), j = 1, 3, 5 \dots (n - 2)$$

$$R_i = (w_{i(n-1)}, w_{in}, w_{(i+1)n}, w_{i(n-1)}) : i = 3, 5, 7, \dots (m - 1)$$

$$S = (w_{1n}, w_{2n}, w_{3n})$$

$$T = (w_{22}, w_{11}, w_{2n})$$

$$U = E(G) - P_i \cup Q_i \cup R_i \cup S \cup T$$

is a simple graphoidal cover of G in which all the vertices made internal. By Corollary (5), $\eta_s(G) = q - p$.

Case (iv) when $m=n= \text{even}$, then the collection of path of G are

$$P_i = (w_{1i}, w_{1(i+1)}, w_{2i}, w_{1i}), 1 \leq i \leq (n - 1)$$

$$Q_i = (w_{ij}, w_{(i+1)j}, w_{(i+1)(j+1)}, w_{ij}) : i = 2, 3, 4 \dots (m - 1), j = 1, 3, 5 \dots (n - 1)$$

$$R = (w_{1n}, w_{2n}, w_{3n})$$

$$S = (w_{22}, w_{11}, w_{2n})$$

$$T = E(G) - P_i \cup Q_i \cup R \cup S$$

is a simple graphoidal cover of G in which all the vertices made internal. By corollary (5), $\eta_s(G) = q - p$. $\square$

Theorem 13. For the graph $G = P_m \otimes P_n$, where $m, n \geq 2$ then

$$\eta_s(G) = \begin{cases} q - p + 2 & \text{if } m = 2 \text{ and } n \geq 2 \\ q - p + 1 & \text{if } m = 3 \text{ and } n \geq 3 \end{cases}$$

Proof. Let $E(G) = \{w_{ij}w_{i(j+1)} : 1 \leq i \leq m; 1 \leq j \leq (n - 1)\} \cup \{w_{ij}w_{(i+1)j} : 1 \leq i \leq (m - 1); 1 \leq j \leq n\} \cup \{w_{ij}w_{(i+1)(j+1)} : 1 \leq i \leq (m - 1); 1 \leq j \leq (n - 1)\} \cup \{w_{ij}w_{(i+1)(j-1)} : 1 \leq i \leq (m - 1); 2 \leq j \leq n\}$

When $m=2$ and $n \geq 2$, The collection of paths of G are

$$P_i = (w_{1j}, w_{1(j+1)}, w_{2j}, w_{1j}) : 1 \leq j \leq (n - 1)$$

$$Q = E(G) - P_i$$

is a simple graphoidal cover of G in which w_{11} and w_{2n} are not internal. Therefore $\eta_s(G) \leq q - p + 2$. Now let ψ be any graphoidal cover of G . If ψ contains only paths, then atleast four vertices can not be made internal. If ψ contains atleast one cycle, then atleast two vertices can not be made internal. Therefore $t_\psi \geq 2$. Hence $\eta_s(G) \geq q - p + 2$. Thus $\eta_s(G) = q - p + 2$.

When $m \geq 3$ and $n \geq 3$, Here we have following cases

Case (i) When $m=n$ = odd, The collection of paths of G are

$$P_i = (w_{1j}, w_{1(j+1)}, w_{2j}, w_{1j}) : 1 \leq j \leq (n - 1)$$

$$Q_i = (w_{ij}, w_{(i+1)j}, w_{(i+1)(j+1)}, w_{ij}) : i = 2, 3, \dots, (m - 1); j = 1, 3, 5, \dots, (n - 2)$$

$$R_i = (w_{i(n-1)}, w_{in}, w_{(i+1)n}, w_{i(n-1)}) : i = 2, 4, \dots, (m - 1)$$

$$S = E(G) - P_i \cup Q_i \cup R_i$$

is a simple graphoidal cover of G in which w_{11} is not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now let ψ be any graphoidal cover of G . If ψ contains only paths, then atleast four vertices can not be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $t_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

Case (ii) When m =odd and n = even, The collection of paths of G are

$$P_i = (w_{1j}, w_{1(j+1)}, w_{2j}, w_{1j}) : 1 \leq j \leq (n - 1)$$

$$Q_i = (w_{ij}, w_{(i+1)j}, w_{(i+1)(j+1)}, w_{ij}) : i = 2, 3, \dots, (m - 1); j = 1, 3, 5, \dots, (n - 1)$$

$$R = (w_{1n}, w_{2n}, w_{3n})$$

$$S = E(G) - P_i \cup Q_i \cup R$$

is a simple graphoidal cover of G in which w_{11} is not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now let ψ be any graphoidal cover of G . If ψ contains only paths, then atleast four vertices can not be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $t_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

Case (iii) When m =even and n =odd, The collection of paths of G are

$$P_i = (w_{1j}, w_{1(j+1)}, w_{2j}, w_{1j}); 1 \leq j \leq (n - 1)$$

$$Q_i = (w_{ij}, w_{(i+1)j}, w_{(i+1)(j+1)}, w_{ij}) : 2 \leq j \leq (m - 1); j = 1, 3, 5, \dots, (n - 2)$$

$$R_i = (w_{i(n-1)}, w_{in}, w_{(i+1)n}, w_{i(n-1)}) : i = 3, 5, \dots, (m - 1)$$

$$S = (w_{1n}, w_{2n}, w_{3n})$$

$$T = E(G) - P_i \cup Q_i \cup R_i \cup S$$

is a simple graphoidal cover of G in which w_{11} is not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now let ψ be any graphoidal cover of G . If ψ contains only paths, then atleast four vertices can not be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $\eta_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

Case (iv) When $m = n = \text{even}$, The collection of paths of G are

$$P_i = (w_{1j}, w_{1(j+1)}, w_{2j}, w_{1j}) : 1 \leq j \leq (n - 1)$$

$$Q_i = (w_{ij}, w_{(i+1)j}, w_{(i+1)(j+1)}, w_{ij}) : 2 \leq j \leq (m - 1); j = 1, 3, 5 \dots (n - 1)$$

$$R = (w_{1n}, w_{2n}, w_{3n})$$

$$S = E(G) - P_i \cup Q_i \cup R$$

is a simple graphoidal cover of G in which w_{11} is not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now let ψ be any graphoidal cover of G . If ψ contains only paths, then atleast four vertices can not be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $\eta_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$. $\square$

Theorem 14. For $C_m \otimes C_n, m, n \geq 3$ then

$$\eta_s(G) = \begin{cases} q - p + 1 & \text{if } m = n = 3 \\ q - p & \text{if } m \geq 3 \text{ and } n \geq 4 \end{cases}$$

Proof. Let Edge Set $E(G) = \{w_{ij}w_{i(j+1)} : 1 \leq i \leq m; 1 \leq j \leq (n - 1)\} \cup \{w_{in}w_{i1} : 1 \leq i \leq m\} \cup \{w_{in}w_{(i+1)1} : 1 \leq i \leq (m - 1)\} \cup \{w_{i1}w_{(i+1)n} : 1 \leq i \leq (m - 1)\} \cup \{w_{ij}w_{(i+1)j} : 1 \leq i \leq (m - 1); 1 \leq j \leq n\} \cup \{w_{mj}w_{1j} : 1 \leq j \leq n\} \cup \{w_{1j}w_{m(j+1)} : 1 \leq j \leq (n - 1)\} \cup \{w_{mj}w_{1(j+1)} : 1 \leq i \leq (m - 1)\} \cup \{w_{ij}w_{(i+1)(j+1)} : 1 \leq i \leq (m - 1); 1 \leq j \leq (n - 1)\} \cup \{w_{ij}w_{(i+1)(j-1)} : 1 \leq i \leq (m - 1); 2 \leq j \leq n\} \cup \{w_{11}w_{mn}\} \cup \{w_{1n}w_{m1}\}$

When $m = n = 3$, The collection of paths of G are

$$P_i = (w_{i1}, w_{i2}, w_{i3}, w_{i1}) : i = 1, 2, 3$$

$$Q = (w_{11}, w_{21}, w_{31}, w_{11})$$

$$R = E(G) - P_i \cup Q$$

is a simple graphoidal cover of G in which w_{11} is the only vertex which is not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now let ψ be any graphoidal cover of G . If

ψ contains only paths, then no vertex can be made internal, since every path is an edge. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $t_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

when $m \geq 3$ and $n \geq 4$, The collection of paths of G are

$$P_i = (w_{i1}, w_{i2}, w_{i3}, \dots, w_{in}, w_{i1}) : i = 1, 2, 3 \dots m$$

$$Q = (w_{11}, w_{21}, w_{31}, w_{41}, \dots, w_{m1}, w_{11})$$

$$R = (w_{22}, w_{11}, w_{mn})$$

$$S = E(G) - P_i \cup Q \cup R$$

is a simple graphoidal cover of G in which all the vertices made internal. By corollary (5), $\eta_s(G) = q - p$. $\square$

Theorem 15. For $P_m \otimes C_n, m \geq 2, n \geq 3$ then $\eta_s(G) = q - p + 2$.

Proof. Let Edge Set $E(G) = \{w_{ij}w_{i(j+1)} : 1 \leq i \leq m; 1 \leq j \leq (n-1)\} \cup \{w_{in}w_{i1} : 1 \leq i \leq m\} \cup \{w_{in}w_{(i+1)1} : 1 \leq i \leq (m-1)\} \cup \{w_{i1}w_{(i+1)n} : 1 \leq i \leq (m-1)\} \cup \{w_{ij}w_{(i+1)j} : 1 \leq i \leq (m-1), 1 \leq j \leq n\} \cup \{w_{ij}w_{(i+1)(j+1)} : 1 \leq i \leq (m-1), 1 \leq j \leq (n-1)\} \cup \{w_{ij}w_{(i+1)(j-1)} : 1 \leq i \leq m-1, 2 \leq j \leq n\}$. Here We consider two cases

Case(i) When $m=2$ and $n \geq 3$, The collection of path of G are

$$P_i = (w_{i1}, w_{i2}, w_{i3}, \dots, w_{in}, w_{i1}) : i = 1, 2, 3 \dots m$$

$$Q = E(G) - P_i$$

is a simple graphoidal cover of G in which w_{11} and w_{21} are not an internal. Therefore $\eta_s(G) \leq q - p + 2$. Now let ψ be any graphoidal cover of G . Here we have two sub cases.

Sub case(i) When $m=2$ and $n=3$, If ψ contains only paths, no vertices can not be made internal, since each path is an edge. If ψ contains atleast one cycle, two vertices can not be made internal, since each path is an edge. Therefore $t_\psi \geq 2$. Hence $\eta_s(G) \geq q - p + 2$. Thus $\eta_s(G) = q - p + 2$.

Sub case (ii) When $m = 2, n > 3$, If ψ contains only paths, four vertices can not be made internal. If ψ contains atleast one cycle, two vertices can not be made internal. Therefore $t_\psi \geq 2$. Hence $\eta_s(G) \geq q - p + 2$. Thus $\eta_s(G) = q - p + 2$.

Case (ii) When $m \geq 3$ and $n \geq 3$, The collection of path of G are

$$P_i = (w_{i1}, w_{i2}, w_{i3}, \dots, w_{in}, w_{i1}) : i = 1, 2, 3 \dots m$$

$$Q = (w_{11}, w_{21}, w_{31}, w_{11} \dots w_{m1})$$

$$R = E(G) - P_i \cup Q$$

is a simple graphoidal cover of G in which w_{11} and w_{m1} are not an internal. Therefore $\eta_s(G) \leq q - p + 2$. Now let ψ be any graphoidal cover of G . If ψ contains only paths, then atleast six vertices can not be made internal. If ψ contains atleast one cycle, then atleast two vertex can not be made internal. Therefore $\eta_\psi \geq 2$. Hence $\eta_s(G) \geq q - p + 2$. Thus $\eta_s(G) = q - p + 2$. $\square$

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