**D-DECOMPOSITION METHOD FOR STABILITY CHECKING
FOR TRINOMIAL LINEAR DIFFERENCE EQUATION
WITH TWO DELAYS**

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Abstract: We give asymptotic stability boundaries in the parameter space of trinomial linear difference equation, and also we give explicit inequalities for stability checking of the equation. We study a generalization of the notion of stability which we call r-stability.

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1. Introduction

We study stability of zero solution of the difference equation

$$x(n) = ax(n-m) + bx(n-k), \quad n = 1, 2, \dots \quad (1)$$

where $x(n): \mathbb{N} \rightarrow \mathbb{R}$, and natural numbers k, m are delays, $a, b \in \mathbb{R}$.

The characteristic polynomial for (1) is

$$f(\lambda) = \lambda^k - a\lambda^{k-m} - b. \quad (2)$$

We call (1) stable if all the roots of polynomial (2) satisfy $|\lambda_j| \leq 1$ ($1 \leq j \leq k$),

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and if for any j the equality $|\lambda_j| = 1$ implies that λ is simple. We call equation (1) asymptotically stable if all the roots of polynomial (2) satisfy $|\lambda_j| < 1$.

The point (a, b) in the ab -plane is called the point of asymptotic stability of (1) if all the roots of (2) with parameters a, b satisfy $|\lambda_j| < 1$ and is called unstable point if $|\lambda_j| > 1$ for some j . The stability domain of (1) is a set of all points of asymptotic stability in the ab -plane. Clearly, on the boundary of stability domain there exists $\omega \in [0, \pi]$, such that $f(e^{-i\omega}) = 0$. The method of D-decomposition is based on this fact.

Special cases of the stability problem for (1) were investigated in [1], [2]. The problem of stability of (1) was considered in papers [3], [4], [5], [6], [7], [8]. In [5] there were described the boundaries of stability domain in the ab -plane, and the delays k, m appeared in the rather complicated manner in the description of the boundaries. The results of [5] are obtained by hodograph method.

In [8] there were given stability criterion clearly showing the dependence of stability on delays. The method of [8] is based on the Schur-Cohn type stability criterion and consecutive reduction of equation degree and consecutive transformation of its coefficients. The purpose of this article to demonstrate elementary proofs of the above results about the boundary domains and explicit formulas for dependence of stability on delay. We also study a generalization of the notion of stability.

Our main method is D-decomposition method [9], [10], created by Yu. Neimark [11].

The paper is organized as follows. In the second section we describe the D-decomposition method for (1). In the next sections we give the main results about the stability of (1) by Lyapunov. In the fifth section we spread the results to the generalization of the stability notion which we call r -stability. In Section 6 we give proofs.

2. D-Decomposition Method for (1)

In characteristic equation $f(\lambda) = 0$ (see (2)) let $\lambda = e^{-i\omega}$, where $0 \leq \omega \leq \pi$. We get

$$a e^{im\omega} + b e^{ik\omega} = 1. \quad (3)$$

If $\omega = 0$ and $\omega = \pi$ in (3) we get

$$a + b = 1, \quad (-1)^m a + (-1)^k b = 1. \quad (4)$$

If $\omega = \frac{\pi u}{k-m}$, $u \in \mathbb{N}$ then equation (3) does not have solutions in a, b . If $\omega \in (0, \pi)$, and if $\omega(k-m)$ is not proportional π , then (3) has a unique solution

$$a = \frac{\sin k\omega}{\sin(k-m)\omega}, \quad b = -\frac{\sin m\omega}{\sin(k-m)\omega}. \quad (5)$$

Equations (4), (5) define a self-intersecting curve in the ab -plane. The curve divides the plane into several disjoint simply-connected domains. These domains are called the domains of D-decomposition. The curve (4), (5) is called the curve of D-decomposition for (1), and the points (a, b) of the curve are called points of D-decomposition. Equation (3) is also called the curve of D-decomposition.

The key point of the D-decomposition method is the following statement. Let two points (a_1, b_1) and (a_2, b_2) belong to one connected domain of D-decomposition. Let two polynomials be obtained from (2) by substitution of (a_1, b_1) and (a_2, b_2) instead of (a, b) . Then these two polynomials have the same number of roots inside the unit disc of complex plane. In the other case on the curve connecting (a_1, b_1) and (a_2, b_2) there would be a point (a_3, b_3) such that $f(e^{-i\omega}) = 0$, which is a point of D-decomposition, which contradicts the definition of D-decomposition domain.

Therefore if one point of connected domain of D-decomposition is asymptotically stable, then all the points of this domain are asymptotically stable. The last stage of the method consists of checking of D-decomposition domains on stability. It is enough to check one point in each domain of D-decomposition.

In the Figure 1 is D-decomposition for $k = 5$, $m = 3$. All the points of D-decomposition for (1) lie in the domain

$$|a| + |b| \geq 1, \quad ||a| - |b|| \leq 1. \quad (6)$$

Inequalities (6) are triangle inequalities for possibly degenerate triangle defined by (3), composed by vectors $ae^{im\omega}$, $be^{ik\omega}$ and 1 in the complex plane (Figure 2).

3. The Boundaries of the Stability Domains for (1)

If the delays k, m have common divisor d , and $k = k_1 d$, $m = m_1 d$, then the characteristic polynomial (2) can be written as $f_1(\lambda) = (\lambda^d)^{k_1} - a(\lambda^d)^{k_1-m_1} - b$, therefore the asymptotic stability checking is reduced to the case of k, m coprime.

For coprime k, m ($k > m > 0$) there exists a unique quadruple of integers (j, s, p, q) such that

$$ks - mj = 1, \quad kp - mq = -1, \quad 0 \leq j < k, \quad 0 \leq p < m. \quad (7)$$

Figure 1: D -decomposition of the ab -plane into 11 domains, $k = 5$, $m = 3$.

Following theorem gives the boundaries of stability domain for (1). This theorem is different in some detail from the results of [5].

Theorem 1. *Let k, m be coprime, let $k > m > 0$ and let the quadruple of integers (j, s, p, q) satisfy (7). Then (1) asymptotically stable if and only if the point (a, b) lies inside the domain having the following boundaries:*

I-II) lines (4), III) curve (5) with $\omega \in [\frac{\pi j}{k}, \frac{\pi s}{m})$ and IV) curve (5) with $\omega \in (\frac{\pi p}{m}, \frac{\pi q}{k}]$.

If the point lies outside the domain, then (1) is unstable.

Stability boundaries of (1) shown in Figure 4.

Remark 1. In (7) $s + p = m$, $j + q = k$.

Remark 2. If $m > 1$, then the curve III connects $(0, (-1)^j)$ to $((-1)^s, 0)$, and the curve IV connects $((-1)^p, 0)$ to $(0, (-1)^q)$. If $m = 1$, then the curve III connects $(0, (-1)^{k-1})$ to $(-\frac{k}{k-1}, \frac{(-1)^{k-1}}{k-1})$, the curve IV connects $(\frac{k}{k-1}, \frac{-1}{k-1})$ to $(0, -1)$.

4. Inequalities for Stability Checking for (1)

The method of D -decomposition allows to find the explicit inequalities for stability checking of (1).

The next theorem is slightly different from Theorem 1.3 of [8]. We need the following notations.

$$\psi_a(a, b) = \begin{cases} 0 & \text{if } |a| + |b| < 1; \\ \arccos \frac{1+b^2-a^2}{2|a|}, & \text{if } |a| + |b| \geq 1, ||a| - |b|| < 1, a^k b^m < 0; \\ \pi & \text{otherwise.} \end{cases} \quad (8)$$

$$\psi_b(a, b) = \begin{cases} 0 & \text{if } |a| + |b| < 1; \\ \arccos \frac{1+a^2-b^2}{2|b|}, & \text{if } |a| + |b| \geq 1, ||a| - |b|| < 1, a^k b^m < 0; \\ \pi & \text{otherwise.} \end{cases} \quad (9)$$

Theorem 2. For any k, m coprime, such that $k > m > 0$, (1) is asymptotically stable if and only if

$$m\psi_a(a, b) + k\psi_b(a, b) < \pi. \quad (10)$$

According the Theorem 2 for stability checking of (1) with given real a, b and k, m coprime we need to make several steps with clear geometric motivation. Let us show these steps in case when there exists nondegenerate triangle ABC with sides $|a|, |b|, 1$, *i.e.* when

$$|a| + |b| > 1, \quad ||a| - |b|| < 1. \quad (11)$$

Here and subsequently φ_a, φ_b stand for angles opposite the sides $|a|, |b|$ in ABC (see Fig. 2),

$$\arccos \varphi_a = \frac{1 + b^2 - a^2}{2|a|}, \quad \arccos \varphi_b = \frac{1 + a^2 - b^2}{2|b|}. \quad (12)$$

In the Fig. 3 there are the domains for inequalities $m\varphi_a + k\varphi_b < \pi$ in the ab -plane.

To examine stability of (1) one has to compare $m\varphi_a + k\varphi_b$ and π . If $m\varphi_a + k\varphi_b > \pi$, then (1) is unstable. If $m\varphi_a + k\varphi_b = \pi$, then (1) is not asymptotically stable. If $m\varphi_a + k\varphi_b < \pi$, then (1) is asymptotically stable for $a^k b^m < 0$ and unstable for $a^k b^m > 0$.

Figure 2: The equality (3) and the triangle with the sides 1, $|a|$, $|b|$. Parameters: $(k, m, s, j) = (3, 1, 1, 2)$, $\omega = 0.8\pi$, $\varphi_a = 0.4\pi$, $\varphi_b = 0.2\pi$, $a = -1$, $b \simeq 0.618$.

5. r -Stability of (1)

Definition 3. Let $r > 0$. Equation (1) is called r -stable if all the roots λ_j ($1 \leq j \leq k$) of polynomial (2) satisfy $|\lambda_j| \leq r$, and for any j the equality $|\lambda_j| = r$ implies that λ is simple. Equation (1) is called asymptotically r -stable, if all the roots λ_j ($1 \leq j \leq k$) of polynomial (2) satisfy $|\lambda_j| < r$.

If (x_n) is a solution of r -stable (asymptotically r -stable) equation, then the sequence $(x(n)/r^n)$ is bounded (goes to 0 as $n \rightarrow \infty$). (Asymptotic) stability by Lyapunov is as (asymptotic) r -stability with $r = 1$. r -stability checking is not much harder then checking of classical stability. In the characteristic polynomial (2) let $\lambda = r\mu$ and we get characteristic equation $\mu^k - (a/r^m)\mu^{k-m} - (b/r^k) = 0$. Therefore Theorems 1, 2 let us check r -stability, if we make a substitution $a \rightarrow a/r^m$, $b \rightarrow b/r^k$. For example, Theorem 1 has the following consequence.

Theorem 4. Let $r > 0$, let k, m be coprime, let $k > m > 0$ and let the quadruple of integers (j, s, p, q) satisfy (7). Then (1) is asymptotically r -stable if and only if the point (a, b) lies inside the domain having the following boundaries:

$$\begin{aligned} I r) \quad & \frac{a}{r^m} + \frac{b}{r^k} = 1, & II r) \quad & (-1)^m \frac{a}{r^m} + (-1)^k \frac{b}{r^k} = 1, \\ III r) \quad & a = r^m \frac{\sin k\omega}{\sin(k-m)\omega}, & b = -r^k \frac{\sin m\omega}{\sin(k-m)\omega}, & \omega \in \left[\frac{\pi j}{k}, \frac{\pi s}{m} \right), \end{aligned}$$

Figure 3: The set of the points (a, b) (shaded) in the ab -plane, such that $0 < m\varphi_a + k\varphi_b < \pi$. Left: $k > m > 1$ ($k = 3, m = 2$). Right: $k > m = 1$ ($k = 5$).

$$IVr) \quad a = r^m \frac{\sin k\omega}{\sin(k-m)\omega}, \quad b = -r^k \frac{\sin m\omega}{\sin(k-m)\omega}, \quad \omega \in \left(\frac{\pi p}{m}, \frac{\pi q}{k}\right].$$

If the point (a, b) lies outside the domain, then (1) is r -unstable.

Theorem 4 is illustrated by Figure 4.

6. Proofs

6.1. Proof of Theorem 1

Let the conditions of Theorem 1 be met. Each point (a, b) satisfying (3) defines a triangle with sides $|a|$, $|b|$, 1 (Figure 2). It may degenerate if either $m\omega$ or $k\omega$ is a multiple of π . The following three lemmas form part of the proof of theorem.

Lemma 1. *If none of the numbers $(k-m)\omega$, $m\omega$ or $k\omega$ is a multiple of π , then (3) implies the existence of an integer z such that*

$$0 < m\varphi_a + k\varphi_b = z\pi, \quad z \in \mathbb{Z}_+, \quad z \geq 1$$

in the triangle with the sides $|a|$, $|b|$, 1.

Proof. We adopt the convention that $\arg x \in (-\pi, \pi]$ for every $x \in \mathbb{C}$. From (3) it follows that $(\arg ae^{im\omega})(\arg be^{ik\omega}) < 0$. For example let $\arg ae^{im\omega} =$

k is even, m is odd,
 $m > 1$
 $(k = 4, m = 3)$

k is odd, m is even
 $(k = 3, m = 2)$

k is odd, m is odd,
 $m > 1$
 $(k = 5, m = 3)$

k is even ($k = 4$),
 $m = 1$

k is odd ($k = 3$), $m = 1$

Figure 4: Domains of r -stability of (1) in the ab -plane for $r = 0.8$, $r = 1$, $r = 1.2$.

$-\varphi_b < 0$, $\arg be^{ik\omega} = \varphi_a > 0$ (Figure 2). Then there exist $u \in \mathbb{Z}_+$ and $v \in \mathbb{Z}_+$ such that

$$\begin{aligned} a > 0 &\rightarrow \varphi_b = 2\pi u - m\omega, & a < 0 &\rightarrow \varphi_b = 2\pi u - m\omega - \pi, \\ b > 0 &\rightarrow \varphi_a = k\omega - 2\pi v, & b < 0 &\rightarrow \varphi_a = k\omega - \pi - 2\pi v, \end{aligned}$$

and the lemma follows. $\square$

Lemma 2. For all $\omega \in [\frac{\pi j}{k}, \frac{\pi s}{m})$ and $\omega \in (\frac{\pi p}{m}, \frac{\pi q}{k}]$ from (3) it follows that

$$m\varphi_a + k\varphi_b = \pi \quad (13)$$

in the triangle with the sides $|a|$, $|b|$, 1.

Proof. We restrict ourselves to the case $\omega \in [\frac{\pi j}{k}, \frac{\pi s}{m})$. From (7) we obtain $\pi j \leq k\omega < \pi j + \frac{\pi}{m}$, $\pi s - \frac{\pi}{k} \leq m\omega < \pi s$. Then there exist $\theta_a \in [0, \frac{\pi}{m})$, $\theta_b \in (0, \frac{\pi}{k}]$ such that

$$k\omega = \pi j + \theta_a, \quad 0 \leq \theta_a < \frac{\pi}{m}, \quad m\omega = \pi s - \theta_b, \quad 0 < \theta_b \leq \frac{\pi}{k}. \quad (14)$$

By (14) and (7) we get $m\theta_a + k\theta_b = \pi$. It remains to prove that $\theta_a = \varphi_a$, and the same for θ_b .

CASE 1: j is even. Then s is odd. By (3) and (14) we get $a < 0$, $b > 0$ (see Figure 2), therefore $\arg be^{ik\omega} = \theta_a$, $\arg ae^{im\omega} = -\theta_b$. Hence, θ_a is opposite $|a|$, and the same for θ_b .

Other cases are analogous (CASE 2: s is even, and CASE 3: j, s are odd). Lemma 2 is proved. □

Lemma 3. Let (13) holds in the triangle ABC with the sides $BC = |a|$, $AC = |b|$, $AB = 1$. Let $(a, b) \in \mathbb{R}^2$ and let

$$1 < |a| + |b| < |a| + |b|, \quad ||a| - |b|| = ||a| - |b||. \quad (15)$$

Then

$$m\varphi_{a'} + k\varphi_{b'} < \pi. \quad (16)$$

in the triangle with the sides $|a|$, $|b|$, 1 and opposite angles $\varphi_{a'}$, $\varphi_{b'}$, $\pi - \varphi_{a'} - \varphi_{b'}$.

Proof. There is no loss of generality in assuming that $a > 0$, $b > 0$, $a > 0$, $b > 0$ (see Figure 5). Let ABC denote the triangle with the sides $AC = b$, $BC = a$, $AB = 1$. By (15) the point C lies on the hyperbola having foci A , B (in the exceptional case $a = b$ hyperbola degenerates into a perpendicular bisector). The focal radii AC , BC make angles $\varphi_{a'} < \varphi_a$, $\varphi_{b'} < \varphi_b$ with AB , which gives (16). Lemma 3 is proved. □

Let us continue the proof of Theorem 1. Consider a domain D with boundary consisting of curves I-IV. Lemmas 1–3 show that there are no points of D-decomposition between III and the curve $|a| + |b| = 1$, and the same is true for IV. By (3) there are no points of D-decomposition inside the domain $|a| + |b| < 1$. It follows that there are no points of D-decomposition inside D .

Since the point $(a, b) = (0, 0)$ is the point of asymptotic stability for (1), we conclude that all the points inside D are the points of asymptotic stability for (1).

Figure 5: Left: bold curve consists of the points (a, b) in the ab -plane, such that $m\varphi_a + k\varphi_b = \pi$; $|a - b| = |a' - b'|$. Right: all the points C lies in a hyperbola, $|CA - CB| \equiv |a - b|$.

It remains to prove that all the points outside D are unstable points. Indeed, consider the curves Ir – IVr of Theorem 4. Given a point (a, b) on the curves, there exists a root λ of the polynomial (2) such that $|\lambda| = r$. While the parameter r increases from 1 to infinity, the curves Ir – IVr cover completely the ab -plane, except D . Therefore all the points outside D are unstable. Theorem 1 is proved.

6.2. Proof of Theorem 2

According to Theorem 1 the domain $|a| + |b| < 1$ consists of points of asymptotic stability for (1). This is reflected in the first lines of the formulas (8), (9) and in the inequality (10). In two quadrants of the ab -plane, where $a^k b^m < 0$, there are additional points of asymptotic stability in the bands $||a| - |b|| < 1$ between the curves $m\varphi_a + k\varphi_b = \pi$ and $|a| + |b| = 1$. This is reflected in the second lines of the formulas (8), (9) and in the inequality (10). Finally, there are no additional stability points in two quadrants, where $a^k b^m > 0$, since Theorem 1 indicates the lines (4) as the boundaries of the stability domain. This is reflected in the third lines of the formulas (8), (9) and in the inequality (10). Theorem 2 is proved.

7. Conclusion

In the sequence of papers [1]–[8] at complication of equations, proofs is sometimes simplified. In the present paper the problem is reduced to the elementary triangle geometry.

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