ON SET-HARMONIOUS LABELING OF SOME GRAPHS

Seema Mehra¹, Rani Puneet²§
¹,²Department of Mathematics
Maharshi Dayanand University
Rohtak, Haryana, INDIA

Abstract: The aim of this paper is to introduce a new concept of set-harmonious labeling which is an extension of harmonious labeling and also characterize some graph classes with this new labeling.

AMS Subject Classification: 47H10, 54H25
Key Words: set-labeling, harmonious labeling, set-harmonious labeling

1. Introduction

Graph labeling of graphs is the process of assigning the labels (numbers) to the elements of the graph under some conditions. Due to its many practical applications in X-ray crystallography, computer science, coding theory and many other branches, it becomes the active area of research. All terms and definitions are not mentioned in this paper, we refer to [10] and [5] for more details. The concept of Graph Labeling as β-valuation was introduced by Rosa [6] in 1967. Subsequently, in 1972, Golomb [3] called it graceful-labeling. Due to phasing the problem in error correcting codes and channel assignments, Graham and Sloane [4] introduced the concept of harmonious labeling by producing the modular version over the graceful labeling. In 1983, Acharya [1] introduced the concept of set-labeling. After words, many authors Acharya et al. [2], Sudev and Germina...
[8], Thomas and Mathew [9] obtained results on set-labeling.

In this paper we introduce a new concept of set-harmonious labeling which is an extension of harmonious labeling and also characterize some graph classes with this new labeling.

2. Preliminaries

**Definition 2.1** (Graceful Labeling [6]). Let $G(V, E)$ be a graph with $p$, $q$ number of vertices and edges respectively. If $f$ is any injective function in $G$ defined as $f : V(G) \to \{0, 1, 2, \ldots, q\}$ produced a bijective function $f^* : E(G) \to \{0, 1, 2, \ldots, q\}$ defined as $f^*(uv) = |f(u) - f(v)|$ then $f$ is called graceful labeling.

**Definition 2.2** (Harmonious Labeling [4]). A graph $G(V, E)$ with $p$ vertices and $q$ edges is said to be harmonious if an injective function $f : V(G) \to \mathbb{Z}_q$ produced a bijective function $f^* : E(G) \to \mathbb{Z}_q$ defined as $f^*(uv) = (f(u) + f(v))(\text{mod } q)$ and $f$ is called harmonious labeling.

**Definition 2.3** (Set-Labeling [1]). A set-labeling of a graph $G(V, E)$ is the assignment of unique elements of the power set $2^X$ of any non-empty set $X$ to the vertices or edges or both in the graph.

**Definition 2.4** (Sumset [7]). The sumset of two non-empty sets $A$ and $B$ denoted by $A + B$ and it have the elements as $A + B = \{c; c = a + b, a \in A, b \in B\}$. If any of the sets $A$ and $B$ is countable infinite then their sum sets $A+B$ is also countable infinite. Here, we assume $A$ and $B$ as non-empty finite sets.

**Definition 2.5** (Cardinality [7]). The number of elements in any set $A$ is called its cardinality. It is denoted as $|A|$.

3. Main Results

Motivated by the concept of set-labeling and harmonious labeling of graphs, we introduce set-harmonious labeling. We also study the properties of set-harmonious labeling on certain classes of graphs like path, star, double star etc.

**Definition 3.1.** Let $G(V, E)$ be any graph with non-empty finite set $X$ of non-negative integers. Let $f : V(G) \to 2^X$ be an injective function and it produced a bijective mapping $f^* : E(G) \to 2^X - \phi$ defined as $f^*(uv) = |f(u) - f(v)|$.
ON SET-HARMONIOUS LABELING OF SOME GRAPHS

395

\( f(u) + f(v) \) where \(|X|\) denotes the cardinality of set \( X \) and \( + \) is defined as 
\( f(u) + f(v) = \{ r; a + b = r(\text{mod}|X|), a \in f(u), b \in f(v) \} \), then \( f \) is called set-harmonious labeling and the graph \( G \) is called set-harmonious graph.

Major characteristics of set-harmonious graphs are described as under:

**Proposition 3.2.** If \( f : V(G) \rightarrow 2^X \) is a set-harmonious labeling on a graph \( G \) with \( p = q + 1 \) where \( p, q \) are number of vertices and edges respectively. Then, the ground set \( X \) must contain 0 as an element.

**Proof.** Suppose \( X \) does not contain 0 as an element. Let \( A \) and \( B \) are the set-labels of vertices \( u \) and \( v \) respectively. After taking the sumset of \( A \) and \( B \) over the modulo (\(|X|\)), the label for the edge \( uv \) contain 0. But 0 does not belong to \( X \).

So, when a graph admits a set-harmonious labeling then \( X \) must have 0 as an element. \( \square \)

**Proposition 3.3.** Let \( G(V, E) \) be a graph with \( p = q + 1 \). If \( f : V(G) \rightarrow 2^X \) is a set-harmonious labeling on the graph \( G \) then, \( X \) must be assigned to the pendant vertex in \( G \).

**Proof.** Suppose \( X \) is not assigned to the pendant vertex in a graph \( G \) with \( p = q + 1 \). So, it is assigned to a label of degree at least 2. Then, \( X \) must be the set-label of at least two edges under modulo (\(|X|\)) sumset operation. So, the function along the edges does not become bijective. Hence, it is contradiction to the assumed condition. \( \square \)

**Proposition 3.4.** If \( f : V(G) \rightarrow 2^X \) is a set-harmonious labeling on a graph \( G(V, E) \) with finite set of non-negative integers \( X \) then, \( \{0\} \) is either assigned to the vertex which is adjacent to the vertex of set-label \( \phi \) or if an element \( X_r \) in \( X \) is the set-label of the vertex \( V \) of \( G \) then its adjacent vertex have set-label \( X_S \) if \( X_r + X_S = |X| \), where \(|X|\) is the cardinality of the set \( X \).

**Proof.** Let \( G \) be a graph with set-harmonious labeling \( f \) then, all the subsets of \( X \) except \( \phi \) should be the set-label of all edges in \( G \). So, \( \{0\} \) must be the set-label of one of the edge among them. After applying the sumset modulo (\(|X|\)) among the vertices, the edge have set-label \( \{0\} \) if and only if either \( \{0\} \) is adjacent to \( \phi \) or when non-zero any element \( X_r \) is added into another element \( X_S \) of the ground set \( X \) such that \( X_r + X_S = |X| \). After taking the modulo (\(|X|\)), we get \( \{0\} \). \( \square \)
Proposition 3.5. Let $X_r$ be any non-empty, non-zero subset of $X$ and $f : V(G) \to 2^X$ is a set-harmonious labeling on a graph $G$ then, $\{0\}$ and $\phi$ cannot be the assigned to adjacent vertices of $X_r$.

Proof. Suppose $v_1$ and $v_2$ are two non-adjacent vertices in a graph $G$ and $v_r$ is a vertex which is adjacent to both $v_1$ and $v_2$. Let $\{0\}$ and $\phi$ be the set-labels of $v_1$ and $v_2$ respectively. Let $X_r$ is the set-label of the vertex $v_r$ then the set-label of edge $v_1v_r$ and edge $v_2v_r$ will be same. It creates the contradiction in bijectiveness of the mapping for edges. $\square$

Proposition 3.6. If $f : V(G) \to 2^X$ is a set-harmonious labeling on a graph $G(V, E)$ with non-empty finite set of non-negative integers. Then, every singleton element of $2^X$ should be adjacent to the vertex having the set-label either $\{0\}$ or $\phi$.

Proof. Let $G$ be a graph and $f$ be a set-harmonious labeling then the labeling assigned to the edges should be bijective and labeled as the modulo ($|X|$) of sumset of adjacent vertices. So, all the non-empty sub sets of $X$ should be exactly once assigned to the edges. This is possible only when all singletons are assigned either adjacent with $\{0\}$ or with $\phi$. $\square$

Theorem 3.7. A graph $G(V, E)$ have a set-harmonious labeling with the ground set $X$ then, it have $2^{|X|} - 1$ number of edges.

Proof. Let $G$ be any set-harmonious graph and $f : V(G) \to 2^X$ is a set-harmonious labeling. Then, the associated mapping $f^* : E(G) \to 2^X - \phi$ is bijective. So, $|E(G)| = 2^{|X|} - 1$. $\square$

Theorem 3.8. Let $G$ be a set-harmonious labeling with non-empty finite set $X$ of non-negative integers. Then, the cardinality of the ground set $X$ is $\log_2[|E(G)| + 1]$.

Proof. $G$ be a graph with set-harmonious labeling then by above Theorem 3.7,

\[
|E(G)| = 2^{|X|} - 1.
\]

\[
\Rightarrow \log(|E(G)| + 1) = \log 2^X.
\]

\[
\Rightarrow \log(|E(G)| + 1) = X \log 2.
\]

\[
\Rightarrow \frac{\log(|E(G)| + 1)}{\log 2} = X.
\]

\[
\Rightarrow X = \log_2[|E(G)| + 1].
\]
Theorem 3.9. A star graph $S_{1,n}$ have set-harmonious labeling with non-empty finite ground set $X$ if and only if $n = 2^{|X|} - 1$.

Proof. Let $G = S_{1,n}$ be a star graph with vertices set $\{v_1, v_2, \ldots, v_n\}$ those are adjacent with the central vertex $v$. Suppose, $G$ have set-harmonious labeling. Then, by above Theorem 3.7, the number of edges in $G = n = 2^{|X|} - 1$.

Conversely, assume that number of adjacent vertices of $v$ in $S_{1,n}$ are $2^{|X|} - 1$ and $X$ is a finite non-empty ground set. We want to prove $S_{1,n}$ have set-harmonious labeling. Let $\phi$ be assigned to the central vertex and remaining subsets of $2^X$ are assigned to $\{v_1, v_2, \ldots, v_n\}$ vertices. In this way all the edges of $S_{1,n}$ contains bijective modulo ($|X|$) sumsets labeling among the vertices.

Hence, $S_{1,n}$ have set-harmonious labeling.

Figure 1 proves the admissibility of set-harmonious labeling for the star graph $S_{1,7}$ for $X = \{0, 1, 2\}$ where $n = 7 = 2^{|X|} - 1$.

Theorem 3.10. A double star graph $S_{m,n}$ have a set-harmonious labeling.

Proof. Let $G = S_{m,n}$ be a double star graph with $|E(G)| = m + n - 1$. Let $\{v_1, v_2, \ldots, v_m\}$ and $\{u_1, u_2, \ldots, u_n\}$ are two sets of vertices of $S_{m,n}$. Without loss of generality, we can assume that $v_1$ and $u_1$ are only adjacent vertices in the graph $G$. Let $X$ is the non-empty finite set of non-negative integers. Here, we can label $v_1$ as $\phi$ and $u_1$ as $\{0\}$ and assign all non-empty sub-sets of $X$ to the remaining vertices. Now, the labeling assign to the edges are modulo ($|X|$) to the sumset of value assign to the adjacent vertices. Hence, in this way
all sub sets of $|X|$ are bijectively assign among the edges. So, $G = S_{m,n}$ is a set-harmonious graph.

Figure 2 proves the admissibility of set-harmonious labeling for the double star graph $S_{2,2}$ for $X = \{0, 1\}$ and $|E(G)| = m + n - 1 = 2 + 2 - 1 = 2^2 - 1$.

**Theorem 3.11.** A complete graph $K_m$ does not have a set-harmonious labeling.

**Proof.** $K_2$ is a complete graph with 1 edge. By Theorem 3.7, it does not have set-harmonious labeling. $K_3$ is a complete graph with $2^2 - 1 = 3$ edges but it does not form a bijective mapping along the edges of the graph.

Now, we consider a complete graph with more than three vertices. Assume that the complete graph $K_m$, $m > 3$ have a set-harmonious labeling. Then, by Theorem 3.7,

$$|E(G)| = 2^{|X|} - 1 = \frac{m(m-1)}{2}$$

where $X$ is a non-empty finite ground set of non-negative integers. Here, $2^{|X|} - 1$ is a positive odd integer.

$$\Rightarrow \frac{m(m-1)}{2} \text{ is also integer.}$$

$$\Rightarrow m(m-1) \text{ is a multiple of } 2.$$ 

$$\Rightarrow \text{Either } m = 2k \text{ or } m - 1 = 2k.$$ 

So,

$$2^{|X|} - 1 = \frac{2k(2k-1)}{2} = k(2k-1).$$

$$\Rightarrow 2^{|X|} - 1 = 2k^2 - k.$$ 

$$\Rightarrow 2k^2 - k + 1 = 2^n.$$
Hence, $K_m$ have a set-harmonious labeling if $2k^2 - k + 1 = 2^n$ have integral solutions. But, there does not exist any odd positive integer $k$ which satisfies it. Hence, $K_m$ does not have a set-harmonious labeling.

**Theorem 3.12.** A $(m, n)$-Tadpole graph have set-harmonious labeling when $m + n = 7$ for $m > 2$.

**Proof.** A $(m, n)$-Tadpole contains a cycle graph $C_m$ and a path graph $P_n$ with $\{v_1, v_2, \ldots, v_m\}$ and $\{u_1, u_2, \ldots, u_n\}$ vertices set respectively. Let $X$ is a non-empty set of non-negative integers. Without loss of generality, we can assume that $C_m$ and $P_n$ are joined together by a vertex $u_1$ and $u_n$ is the pendant vertex of the path graph. Here, $m + n = 7$. So, the possible cases are as:

1. $m = 3$, $n = 4$.
2. $m = 4$, $n = 3$.
3. $m = 5$, $n = 2$.

We assign $X$ to the pendant vertex of $P_n$ and further assign the sub sets of $X$ to the vertices in such a way that each of the edge contain modular sumset of all the vertices over $|X|$ exactly once and makes the bijective mapping corresponding to the edges. Hence, $G$ has a set-harmonious labeling.

**Proposition 3.13.** Let $G$ be a set-harmonious graph with $p = q + 1$ and $X$ is a non-empty finite set of non-negative integers with cardinality $n$. Then subsets of $X$ with cardinality $n - 1$ cannot be assigned to the adjacent vertices.

**Proof.** If $G$ has a set-harmonious labeling and a set-label of cardinality $n - 1$ is assigned to $u_1$. Then by assigning the sub set of $X$ with cardinality $n - 1$ to the adjacent vertices of $u_1$, we get the same label corresponding to both edges as $X$. It gives contraction.

**Theorem 3.14.** No cycle graph $C_n$ have set-harmonious labeling.

**Proof.** Let $C_n$ be a cycle graph with $n$ vertices and edges. For being a set-harmonious labeling, we must have $2^{|X|} - 1$ edges. $C_n$ have no pendant vertex. So, by Proposition 3.3, $X$ can not be assigned to any of the vertex. Further, the set labels of cardinality $n - 1$ cannot be assigned more than two adjacent vertices because the set labels corresponding to the edges gives $X$. Similarly, nonsingleton sub sets of $X$ with cardinality $n - 2$ cannot be assigned to adjacent vertices those have set label as sub set of $X$ with cardinality $n - 1$. 


In this way, we require more singletons for assigning to the vertices with the adjacent vertices of set labels of cardinality \( n - 1, n - 2, \ldots \). Hence, \( C_n \) does not have a set-harmonious labeling.

**Theorem 3.15.** Every Tree except the path graph have set-harmonious labeling.

**Proof.** Let \( G \) be a tree graph with \( |E(G)| = 2^{|X|} - 1 \), where \( X \) is a non-empty finite set of nonnegative integers. Then the labeling assign to a node of maximal degree as \( \{0\} \) and the node with second maximal degree labeled as \( \phi \). The pendant node at highest level in tree labeled as \( X \) and labeled the remaining vertices in the tree as mention the results in propositions. Hence, we form a set-harmonious labeled tree.

**4. Conclusion**

In this paper, we have discussed the concept and characterization of set-harmonious graphs and find some graph classes those have set-harmonious labeling. Here, we find the essential requirement of number of edges in the graph for having the set-harmonious labeling. There is also a further scope of research as characterize the different graph classes, verify the existence for different graph operations and establish the necessary and sufficient condition for a graph to admit set harmonious labeling.

**References**


