ON SET-HARMONIOUS LABELING OF SOME GRAPHS

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Abstract: The aim of this paper is to introduce a new concept of set-harmonious labeling which is an extension of harmonious labeling and also characterize some graph classes with this new labeling.

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1. Introduction

Graph labeling of graphs is the process of assigning the labels (numbers) to the elements of the graph under some conditions. Due to its many practical applications in X-ray crystallography, computer science, coding theory and many other branches, it becomes the active area of research. All terms and definitions are not mentioned in this paper, we refer to [10] and [5] for more details. The concept of Graph Labeling as β -valuation was introduced by Rosa [6] in 1967. Subsequently, in 1972, Golomb [3] called it graceful-labeling. Due to phasing the problem in error correcting codes and channel assignments, Graham and Sloane [4] introduced the concept of harmonious labeling by producing the modular version over the graceful labeling. In 1983, Acharya [1] introduced the concept of set-labeling. After words, many authors Acharya et al. [2], Sudev and Germina

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[8], Thomas and Mathew [9] obtained results on set-labeling.

In this paper we introduce a new concept of set-harmonious labeling which is an extension of harmonious labeling and also characterize some graph classes with this new labeling.

2. Preliminaries

Definition 2.1 (Graceful Labeling [6]). Let $G(V, E)$ be a graph with p , q number of vertices and edges respectively. If f is any injective function in G defined as $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$ produced a bijective function $f : E(G) \rightarrow \{0, 1, 2, \dots, q\}$ defined as $f(uv) = |f(u) - f(v)|$ then f is called graceful labeling.

Definition 2.2 (Harmonious Labeling [4]). A graph $G(V, E)$ with p vertices and q edges is said to be harmonious if an injective function $f : V(G) \rightarrow Z_q$ produced a bijective function $f : E(G) \rightarrow Z_q$ defined as $f(uv) = (f(u) + f(v)) \pmod{q}$ and f is called harmonious labeling.

Definition 2.3 (Set-Labeling [1]). A set-labeling of a graph $G(V, E)$ is the assignment of unique elements of the power set 2^X of any non-empty set X to the vertices or edges or both in the graph.

Definition 2.4 (Sumset [7]). The sumset of two non-empty sets A and B denoted by $A + B$ and it have the elements as $A + B = \{c; c = a + b, a \in A, b \in B\}$. If any of the sets A and B is countable infinite then their sum sets $A+B$ is also countable infinite. Here, we assume A and B as non-empty finite sets.

Definition 2.5 (Cardinality [7]). The number of elements in any set A is called its cardinality. It is denoted as $|A|$.

3. Main Results

Motivated by the concept of set-labeling and harmonious labeling of graphs, we introduce set-harmonious labeling. We also study the properties of set-harmonious labeling on certain classes of graphs like path, star, double star etc.

Definition 3.1. Let $G(V, E)$ be any graph with non-empty finite set X of non-negative integers. Let $f : V(G) \rightarrow 2^X$ be an injective function and it produced a bijective mapping $f : E(G) \rightarrow 2^X - \phi$ defined as $f(uv) =$

$f(u) \hat{+} f(v)$ where $|X|$ denotes the cardinality of set X and $\hat{+}$ is defined as $f(u) \hat{+} f(v) = \{r; a + b = r \pmod{|X|}, a \in f(u), b \in f(v)\}$, then f is called set-harmonious labeling and the graph G is called set-harmonious graph.

Major characteristics of set-harmonious graphs are described as under:

Proposition 3.2. *If $f : V(G) \rightarrow 2^X$ is a set-harmonious labeling on a graph G with $p = q + 1$ where p, q are number of vertices and edges respectively. Then, the ground set X must contain 0 as an element.*

Proof. Suppose X does not contain 0 as an element. Let A and B are the set-labels of vertices u and v respectively. After taking the sumset of A and B over the modulo $(|X|)$, the label for the edge uv contain 0. But 0 does not belongs to X .

So, when a graph admits a set-harmonious labeling then X must have 0 as an element. $\square$

Proposition 3.3. *Let $G(V, E)$ be a graph with $p = q + 1$. If $f : V(G) \rightarrow 2^X$ is a set-harmonious labeling on the graph G then, X must be assigned to the pendant vertex in G .*

Proof. Suppose X is not assigned to the pendant vertex in a graph G with $p = q + 1$. So, it is assigned to a label of degree at least 2. Then, X must be the set-label of at least two edges under modulo $(|X|)$ sumset operation. So, the function along the edges does not become bijective. Hence, it is contradiction to the assumed condition. $\square$

Proposition 3.4. *If $f : V(G) \rightarrow 2^X$ is a set-harmonious labeling on a graph $G(V, E)$ with finite set of non-negative integers X then, $\{0\}$ is either assigned to the vertex which is adjacent to the vertex of set-label ϕ or if an element X_r in X is the set-label of the vertex V of G then its adjacent vertex have set-label X_S if $X_r + X_S = |X|$, where $|X|$ is the cardinality of the set X .*

Proof. Let G be a graph with set-harmonious labeling f then, all the sub sets of X except ϕ should be the set-label of all edges in G . So, $\{0\}$ must be the set-label of one of the edge among them. After applying the sumset modulo $(|X|)$ among the vertices, the edge have set-label $\{0\}$ if and only if either $\{0\}$ is adjacent to ϕ or when non-zero any element X_r is added into another element X_S of the ground set X such that $X_r + X_S = |X|$. After taking the modulo $(|X|)$, we get $\{0\}$. $\square$

Proposition 3.5. *Let X_r be any non-empty, non zero subset of X and $f : V(G) \rightarrow 2^X$ is a set-harmonious labeling on a graph G then, $\{0\}$ and ϕ cannot be the assigned to adjacent vertices of X_r .*

Proof. Suppose v_1 and v_2 are two non-adjacent vertices in a graph G and v_r is a vertex which is adjacent to both v_1 and v_2 . Let $\{0\}$ and ϕ be the set-labels of v_1 and v_2 respectively. Let X_r is the set-label of the vertex v_r then set-label of edge v_1v_r and edge v_2v_r will be same. It creates the contradiction in bijectiveness of the mapping for edges. $\square$

Proposition 3.6. *If $f : V(G) \rightarrow 2^X$ is a set-harmonious labeling on a graph $G(V, E)$ with non-empty finite set of non-negative integers. Then, every singleton element of 2^x should be adjacent to the vertex having the set-label either $\{0\}$ or ϕ .*

Proof. Let G be a graph and f be a set-harmonious labeling then the labeling assigned to the edges should be bijective and labeled as the modulo ($|X|$) of sumset of adjacent vertices. So, all the non-empty sub sets of X should be exactly once assigned to the edges. This is possible only when all singletons are assigned either adjacent with $\{0\}$ or with ϕ . $\square$

Theorem 3.7. *A graph $G(V, E)$ have a set-harmonious labeling with the ground set X then, it have $2^{|X|} - 1$ number of edges.*

Proof. Let G be any set-harmonious graph and $f : V(G) \rightarrow 2^X$ is a set-harmonious labeling. Then, the associated mapping $f : E(G) \rightarrow 2^X - \phi$ is bijective. So, $|E(G)| = 2^{|X|} - 1$. $\square$

Theorem 3.8. *Let G be a set-harmonious labeling with non-empty finite set X of non-negative integers. Then, the cardinality of the ground set X is $\log_2[|E(G)| + 1]$.*

Proof. G be a graph with set-harmonious labeling then by above Theorem 3.7,

$$\begin{aligned}
 |E(G)| &= 2^{|X|} - 1. \\
 \Rightarrow \log(|E(G)| + 1) &= \log 2^X. \\
 \Rightarrow \log(|E(G)| + 1) &= X \log 2. \\
 \Rightarrow \frac{\log(|E(G)| + 1)}{\log 2} &= X. \\
 \Rightarrow X &= \log_2[|E(G)| + 1].
 \end{aligned}$$

Figure 1

□

Theorem 3.9. A star graph $S_{1,n}$ have set-harmonious labeling with non-empty finite ground set X if and only if $n = 2^{|X|} - 1$.

Proof. Let $G = S_{1,n}$ be a star graph with vertices set $\{v_1, v_2, \dots, v_n\}$ those are adjacent with the central vertex v . Suppose, G have set-harmonious labeling. Then, by above Theorem 3.7, the number of edges in $G = n = 2^{|X|} - 1$.

Conversely, assume that number of adjacent vertices of v in $S_{1,n}$ are $2^{|X|} - 1$ and X is a finite non-empty ground set. We want to prove $S_{1,n}$ have set-harmonious labeling. Let ϕ be assigned to the central vertex and remaining subsets of 2^X are assigned to $\{v_1, v_2, \dots, v_n\}$ vertices. In this way all the edges of $S_{1,n}$ contains bijective modulo $(|X|)$ sumsets labeling among the vertices.

Hence, $S_{1,n}$ have set-harmonious labeling. □

Figure 1 proves the admissibility of set-harmonious labeling for the star graph $S_{1,7}$ for $X = \{0, 1, 2\}$ where $n = 7 = 2^{|X|} - 1$.

Theorem 3.10. A double star graph $S_{m,n}$ have a set-harmonious labeling.

Proof. Let $G = S_{m,n}$ be a double star graph with $|E(G)| = m + n - 1$. Let $\{v_1, v_2, \dots, v_m\}$ and $\{u_1, u_2, \dots, u_n\}$ are two sets of vertices of $S_{m,n}$. Without loss of generality, we can assume that v_1 and u_1 are only adjacent vertices in the graph G . Let X is the non-empty finite set of non-negative integers. Here, we can label v_1 as ϕ and u_1 as $\{0\}$ and assign all non-empty sub-sets of X to the remaining vertices. Now, the labeling assign to the edges are modulo $(|X|)$ to the sumset of value assign to the adjacent vertices. Hence, in this way

Figure 2

all sub sets of $|X|$ are bijectively assign among the edges. So, $G = S_{m,n}$ is a set-harmonious graph. $\square$

Figure 2 proves the admissibility of set-harmonious labeling for the double star graph $S_{2,2}$ for $X = \{0, 1\}$ and $|E(G)| = m + n - 1 = 2 + 2 - 1 = 2^2 - 1$.

Theorem 3.11. *A complete graph K_m does not have a set-harmonious labeling.*

Proof. K_2 is a complete graph with 1 edge. By Theorem 3.7, it does not have set-harmonious labeling. K_3 is a complete graph with $2^2 - 1 = 3$ edges but it does not form a bijective mapping along the edges of the graph. $\square$

Now, we consider a complete graph with more than three vertices. Assume that the complete graph K_m , $m > 3$ have a set-harmonious labeling. Then, by Theorem 3.7,

$$|E(G)| = 2^{|X|} - 1 = \frac{m(m-1)}{2}$$

where X is a non-empty finite ground set of non-negative integers. Here, $2^{|X|} - 1$ is a positive odd integer.

$$\begin{aligned} \Rightarrow \quad & \frac{m(m-1)}{2} \text{ is also integer.} \\ \Rightarrow \quad & m(m-1) \text{ is a multiple of 2.} \\ \Rightarrow \quad & \text{Either } m = 2k \text{ or } m-1 = 2k. \end{aligned}$$

So,

$$\begin{aligned} 2^{|x|} - 1 &= \frac{2k(2k-1)}{2} = k(2k-1). \\ \Rightarrow \quad 2^{|x|} - 1 &= 2k^2 - k. \\ \Rightarrow \quad 2k^2 - k + 1 &= 2^n. \end{aligned}$$

Hence, K_m have a set-harmonious labeling if $2k^2 - k + 1 = 2^n$ have integral solutions. But, there does not exist any odd positive integer k which satisfies it. Hence, K_m does not have a set-harmonious labeling.

Theorem 3.12. *$A(m, n)$ -Tadpole graph have set-harmonious labeling when $m + n = 7$ for $m > 2$.*

Proof. A (m, n) -Tadpole contains a cycle graph C_m and a path graph P_n with $\{v_1, v_2, \dots, v_m\}$ and $\{u_1, u_2, \dots, u_n\}$ vertices set respectively. Let X is a non-empty set of non-negative integers. Without loss of generality, we can assume that C_m and P_n are joined together by a vertex u_1 and u_n is the pendant vertex of the path graph. Here, $m + n = 7$. So, the possible cases are as:

1. $m = 3, n = 4$.
2. $m = 4, n = 3$.
3. $m = 5, n = 2$.

We assign X to the pendant vertex of P_n and further assign the sub sets of X to the vertices in such a way that each of the edge contain modular sumset of all the vertices over $|X|$ exactly once and makes the bijective mapping corresponding to the edges. Hence, G has a set-harmonious labeling. $\square$

Proposition 3.13. *Let G be a set-harmonious graph with $p = q + 1$ and X is a non-empty finite set of non-negative integers with cardinality n . Then subsets of X with cardinality $n - 1$ cannot be assigned to the adjacent vertices.*

Proof. If G has a set-harmonious labeling and a set-label of cardinality $n - 1$ is assigned to u_1 . Then by assigning the sub set of X with cardinality $n - 1$ to the adjacent vertices of u_1 , we get the same label corresponding to both edges as X . It gives contraction. $\square$

Theorem 3.14. *No cycle graph C_n have set-harmonious labeling.*

Proof. Let C_n be a cycle graph with n vertices and edges. For being a set-harmonious labeling, we must have $2^{|X|} - 1$ edges. C_n have no pendant vertex. So, by Proposition 3.3, X can not be assigned to any of the vertex. Further, the set labels of cardinality $n - 1$ cannot be assigned more than two adjacent vertices because the set labels corresponding to the edges gives X . Similarly, nonsingleton sub sets of X with cardinality $n - 2$ cannot be assigned to adjacent vertices those have set label as sub set of X with cardinality $n - 1$.

In this way, we require more singletons for assigning to the vertices with the adjacent vertices of set labels of cardinality $n-1, n-2, \dots$. Hence, C_n does not have a set-harmonious labeling. $\square$

Theorem 3.15. *Every Tree except the path graph have set-harmonious labeling.*

Proof. Let G be a tree graph with $|E(G)| = 2^{|X|} - 1$, where X is a non-empty finite set of nonnegative integers. Then the labeling assign to a node of maximal degree as $\{0\}$ and the node with second maximal degree labeled as ϕ . The pendant node at highest level in tree labeled as X and labeled the remaining vertices in the tree as mention the results in propositions. Hence, we form a set-harmonious labeled tree. $\square$

4. Conclusion

In this paper, we have discussed the concept and characterization of set-harmonious graphs and find some graph classes those have set-harmonious labeling. Here, we find the essential requirement of number of edges in the graph for having the set-harmonious labeling. There is also a further scope of research as characterize the different graph classes, verify the existence for different graph operations and establish the necessary and sufficient condition for a graph to admit set harmonious labeling.

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