CYCLIC SUBGROUP GRAPH OF A FINITE GROUP

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Abstract: The cyclic subgroup graph $\Gamma_z(\mathbf{G})$ of a finite group $\mathbf{G}$ is a graph in which the cyclic subgroups are vertices and two distinct subgroups are adjacent if one of them is a subset of the other. In this paper few parameters in cyclic subgroup graph of a finite group such as independence number, chromatic number and matching number are studied. Also the independence number of power graph and cyclic subgroup graph of a finite group are found to be same.

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1. Introduction

Algebraic graph theory has developed in various dimensions in the past two decades. In this paper, we relate a group with a graph by its subgroup structure.

By a graph $\Gamma = (V, E)$, we mean a connected, finite, undirected graph with neither loops nor multiple edges. For notations and terminology, refer [1, 2]. The power graph $\mathcal{G}(G)$ of a finite group G is a graph in which the group elements are the vertices and two distinct elements x and y are adjacent if one of them

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is a power of the other. Power graph is studied in detail in [4]. The cyclic subgroup graph $\Gamma_z(G)$ is a graph in which the cyclic subgroups are vertices and two distinct subgroups are adjacent if one of them is a subset of the other. In this paper few parameters in cyclic subgroup graph of a finite group such as independence number, chromatic number and matching number are studied. Also the independence number of a power graph and the independence number of cyclic subgroup graph are found to be same. Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set and let $\mathcal{E} = (E_i/1 \leq i \leq m)$ be a family of nonempty subsets of X satisfying $\bigcup_{i=1}^m E_i = X$. Then $H = (X, \mathcal{E})$ is said to be a hypergraph. We need the following results.

Definition 1. [3] Let $H = (X, \mathcal{E})$ be a hypergraph. A set $D \subset X$ is called a hyperdominating set if for each $y \in X - D$ there exist some edge E containing y with $|E| \geq 2$ such that $E - y \subset D$. The hyperdomination number $\gamma_h(H)$ is the minimum cardinality of all hyperdominating sets.

Definition 2. [3] Let G be an abelian group. Then G can be represented as $G \cong Z_{p_1^{n_1}} \otimes \dots \otimes Z_{p_k^{n_k}}$ where $\otimes$ represents direct product of groups and p_i 's are primes. G_D denotes a graph with vertex set

$$V(G_D) = Z_{(n_1+1)} \times \dots \times Z_{(n_k+1)},$$

where $\times$ represents cartesian product. Two tuples $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ in $V(G_D)$ are adjacent if and only if either $x_i \geq y_i$ for all i or $x_i \leq y_i$ for all i .

Lemma 3. [3] For any abelian group

$$G \cong Z_{p_1^{n_1}} \otimes \dots \otimes Z_{p_k^{n_k}},$$

p_i 's are primes, the sets $S_l = \{(x_1, x_2, \dots, x_k) \in Z_{(n_1+1)} \times \dots \times Z_{(n_k+1)} : \sum x_i = l\}$ for $0 \leq l \leq \sum n_i$ are independent in G_D .

Theorem 4. [3] For any group G of order n with

$$G \cong Z_{p_1^{n_1}} \otimes \dots \otimes Z_{p_k^{n_k}}, (k \geq 2),$$

$$\gamma_h(G) \leq n - \sum_{i_1=0}^{\min\{n_1, n_2\}} \sum_{i_2=0}^{\min\{n_1+n_2-2i_1, n_3\}} \dots \sum_{i_{k-2}=0}^{\min\{\sum_{j=1}^{k-2} n_j - 2 \sum_{j=1}^{k-3} i_j, n_{k-1}\}} \min\{\sum_{j=1}^{k-1} n_j - 2 \sum_{j=1}^{k-2} i_j + 1, n_k + 1\}.$$

Moreover, equality holds whenever G is cyclic.

Theorem 5. [3] For any group G of order n , $\gamma_h(G) = n - \beta_o(\mathcal{G}(G))$ where $\mathcal{G}(G)$ is the undirected power graph of G .

2. Main Results

Definition 6. The cyclic subgroup graph $\Gamma_z(G)$ of a finite group G is a simple undirected graph in which the cyclic subgroups of G are the vertices and two distinct subgroups are adjacent if one of them is a subset of the other.

Theorem 7. For any finite group G , the independence number of the cyclic subgroup graph of G is same as that of the independence number of the power graph of G . That is $\beta_o(\Gamma_z(G)) = \beta_o(\mathcal{G}(G))$

Proof. Let $\beta_o(\mathcal{G}(G)) = s$ and $A = \{x_1, x_2, x_3, \dots, x_s\}$ be a maximum independent set of $\mathcal{G}(G)$ having s vertices. Take $x_i, x_j \in A$ and $i \neq j$, then x_i and x_j are not adjacent. Therefore $x_i \neq x_j^k$ and $x_j \neq x_i^k$ for all positive integer k . From this $x_i \notin \langle x_j \rangle$ and $x_j \notin \langle x_i \rangle$. So that $\langle x_i \rangle \not\subseteq \langle x_j \rangle$ and $\langle x_j \rangle \not\subseteq \langle x_i \rangle$. Therefore $\{\langle x_1 \rangle, \langle x_2 \rangle, \dots, \langle x_s \rangle\}$ is an independent set of $\Gamma_z(G)$. So that any maximum independent set of $\Gamma_z(G)$ has atleast s vertices. Therefore $\beta_o(\Gamma_z(G)) \geq \beta_o(\mathcal{G}(G))$.

For the other inequality take $\beta_o(\Gamma_z(G)) = s$.

Let $B = \{\langle y_1 \rangle, \langle y_2 \rangle, \langle y_3 \rangle, \dots, \langle y_s \rangle\}$ be a maximum independent set of $\Gamma_z(G)$ having s vertices. Let $\langle y_i \rangle, \langle y_j \rangle \in B$ and $i \neq j$, then $\langle y_i \rangle$ and $\langle y_j \rangle$ are not adjacent in $\Gamma_z(G)$. Therefore $\langle y_i \rangle \not\subseteq \langle y_j \rangle$ and $\langle y_j \rangle \not\subseteq \langle y_i \rangle$. From this $y_i \notin \langle y_j \rangle$ and $y_j \notin \langle y_i \rangle$. Hence $y_i \neq y_j^k$ and $y_j \neq y_i^k$ for all k . It follows that y_i and y_j are not adjacent to each other in $\mathcal{G}(G)$. Since y_i and y_j are taken arbitrary, $\{y_1, y_2, y_3, \dots, y_s\}$ is an independent set in $\mathcal{G}(G)$. Therefore the maximum independent set of $\mathcal{G}(G)$ has atleast s vertices. Therefore $\beta_o(\Gamma_z(G)) \leq \beta_o(\mathcal{G}(G))$. $\square$

Theorem 8. The independence number of the cyclic subgroup graph of any finite cyclic group Z_n ,

$$\beta_o(\Gamma_z(Z_n)) = \sum_{i_1=0}^{\min\{n_1, n_2\}} \sum_{i_2=0}^{\min\{n_1+n_2-2i_1, n_3\}} \dots \sum_{i_{k-2}=0}^{\min\{\sum_{j=1}^{k-2} n_j - 2 \sum_{j=1}^{k-3} i_j, n_{k-1}\}} \min\{\sum_{j=1}^{k-1} n_j - 2 \sum_{j=1}^{k-2} i_j + 1, n_k + 1\}.$$

Proof. From theorem 4 and theorem 5 it follows that the independence number of the power graph of Z_n ,

$$\beta_o(\mathcal{G}(Z_n)) = \sum_{i_1=0}^{\min\{n_1, n_2\}} \sum_{i_2=0}^{\min\{n_1+n_2-2i_1, n_3\}} \dots \sum_{i_{k-2}=0}^{\min\{\sum_{j=1}^{k-2} n_j - 2 \sum_{j=1}^{k-3} i_j, n_{k-1}\}} \min\{\sum_{j=1}^{k-1} n_j - 2 \sum_{j=1}^{k-2} i_j + 1, n_k + 1\}.$$

Now applying theorem 7 result follows. $\square$

Theorem 9. For any cyclic group $Z_n \cong Z_{p_1^{n_1}} \otimes \dots \otimes Z_{p_k^{n_k}}$, p_i 's are distinct primes, $\Gamma_z(Z_n)$ is isomorphic to Z_{nD} .

Proof. In Z_n every subgroup is cyclic and each subgroup is of the form

$$\langle p_1^{r_1} \rangle \otimes \langle p_2^{r_2} \rangle \otimes \dots \otimes \langle p_k^{r_k} \rangle,$$

where $0 \leq r_i \leq n_i$. Define $f : V(\Gamma_z(Z_n)) \rightarrow V(Z_{nD})$ by

$$f(\langle p_1^{r_1} \rangle \otimes \langle p_2^{r_2} \rangle \otimes \dots \otimes \langle p_k^{r_k} \rangle) = (r_1, r_2, \dots, r_k).$$

Then f is a bijection. Also

$$\langle p_1^{r_1} \rangle \otimes \langle p_2^{r_2} \rangle \otimes \dots \otimes \langle p_k^{r_k} \rangle \subseteq \langle p_1^{s_1} \rangle \otimes \langle p_2^{s_2} \rangle \otimes \dots \otimes \langle p_k^{s_k} \rangle$$

holds if and only if $r_i \leq s_i \forall i$ holds. So that $\Gamma_z(Z_n)$ is isomorphic to Z_{nD} . $\square$

Theorem 10. For any cyclic group $Z_n \cong Z_{p_1^{n_1}} \otimes \dots \otimes Z_{p_k^{n_k}}$, p_i 's are distinct primes, the chromatic number of the cyclic subgroup graph of Z_n , $\chi(\Gamma_z(Z_n)) = n_1 + n_2 + \dots + n_k + 1$.

Proof. From theorem 10 its enough to find the chromatic number of Z_{nD} .

Consider the sets $S_l = \{(x_1, x_2, \dots, x_k) \in Z_{(n_1+1)} \times \dots \times Z_{(n_k+1)} : \sum_{i=1}^k x_i = l\}$ for $0 \leq l \leq \sum n_i$. By Theorem 3 each of these sets S_l are independent. So that $\{S_0, S_1, S_2, \dots, S_{n_1+n_2+\dots+n_k}\}$ forms a chromatic partition of $\Gamma_z(Z_n)$. Hence $\chi(\Gamma_z(Z_n)) \leq n_1 + n_2 + \dots + n_k + 1$.

Now consider the set $K = \{(n_1, n_2, \dots, n_{i-1}, t_i, 0, 0, \dots, 0) : 0 \leq t_i \leq n_i, 1 \leq i \leq k\}$. Any two vertices in K are adjacent. So that K induces a complete subgraph with $|K| = n_1 + n_2 + \dots + n_k + 1$. Hence $\chi(\Gamma_z(Z_n)) \geq n_1 + n_2 + \dots + n_k + 1$ which completes the proof. $\square$

Theorem 11. For any cyclic group $Z_n \cong Z_{p_1^{n_1}} \otimes \dots \otimes Z_{p_k^{n_k}}$, p_i 's are distinct primes, the cyclic subgroup graph $\Gamma_z(Z_n)$ is Hamiltonian.

Proof. By Theorem 9 it's enough to prove Z_{nD} is Hamiltonian. The vertex $(0, 0, 0, \dots, 0) \in V(Z_{nD}) = Z_{(n_1+1)} \times \dots \times Z_{(n_k+1)}$ denoted by origin is adjacent to all other vertices. So it's enough to prove there is a Hamiltonian path in Z_{nD} starting from the origin. We prove this by induction on k . Suppose $k = 1$. Then $Z_n \cong Z_{p_1^{n_1}}$. Now $P = (0, 1, 2, \dots, n_1)$ is a Hamiltonian path starting from origin 0. So the result is true for $k = 1$. Assume the result is true for $k - 1$. To prove for k assume $Z_n \cong Z_{p_1^{n_1}} \otimes \dots \otimes Z_{p_k^{n_k}}$, p_i 's are distinct primes. Take $m = p_1^{n_1} \otimes \dots \otimes p_{k-1}^{n_{k-1}}$. Then $Z_m \cong Z_{p_1^{n_1}} \otimes \dots \otimes Z_{p_{k-1}^{n_{k-1}}}$. By our assumption Z_{mD} has a Hamiltonian path starting from origin say $P = (0, 0, \dots, 0) = (x_{11}, x_{12}, \dots, x_{1(k-1)}), (x_{21}, x_{22}, \dots, x_{2(k-1)}), \dots, (x_{t1}, x_{t2}, \dots, x_{t(k-1)})$ where $t = (n_1 + 1)(n_2 + 1) \dots (n_{k-1} + 1)$.

Consider the corresponding paths

$$P_i = ((0, 0, \dots, 0, i) = (x_{11}, x_{12}, \dots, x_{1(k-1)}, i), (x_{21}, x_{22}, \dots, x_{2(k-1)}, i), \dots, (x_{t1}, x_{t2}, \dots, x_{t(k-1)}, i))$$

for $0 \leq i \leq n_k$. Let the reverse path be denoted by

$$P_i^{-1} = ((x_{t1}, x_{t2}, \dots, x_{t(k-1)}, i), \dots, (x_{21}, x_{22}, \dots, x_{2(k-1)}, i), (0, 0, \dots, 0, i) = (x_{11}, x_{12}, \dots, x_{1(k-1)}, i)).$$

Now consider the path $P' = P_0 P_1^{-1} P_2 P_3^{-1} P_4 \dots P^*$ taking the P_i and their inverse paths alternatively in the sense the path starts from vertices of P_0 followed by vertices of P_1^{-1} followed by the vertices of P_2 and so on until the last path P^* which may be either P_{n_k} or $P_{n_k}^{-1}$ depending on whether n_k is even or odd. P' is the required Hamiltonian path in Z_{nD} starting from the origin. $\square$

Theorem 12. For any cyclic group $Z_n \cong Z_{p_1^{n_1}} \otimes \dots \otimes Z_{p_k^{n_k}}$, p_i 's are distinct primes, the matching number of $\Gamma_z(Z_n)$ is $\lfloor \frac{n}{2} \rfloor$.

Proof. By theorem 11 there is Hamiltonian cycle for $\Gamma_z(Z_n)$ say $C_n = (v_1, v_2, v_3, \dots, v_n, v_1)$. When n is even taking the alternative edges $\{v_1 v_2, v_3 v_4, v_5 v_6, \dots, v_{n-1} v_n\}$ a perfect matching is obtained. When n is odd $\{v_1 v_2, v_3 v_4, v_5 v_6, \dots, v_{n-2} v_{n-1}\}$ forms a maximum matching with $\frac{n-1}{2}$ vertices. Hence $\Gamma_z(Z_n)$ has matching number $\lfloor \frac{n}{2} \rfloor$. $\square$

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