EFFECTIVE PURSUIT STRATEGIES IN SIMPLE MOTION DIFFERENTIAL GAME OF THREE PURSUERS AND ONE EVADER

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Abstract: We consider a simple motion pursuit differential game of three pursuers and one evader with equal dynamic possibilities in the plane. Geometric constraints are imposed on the controls of players and the control set of each player is unit circle centered at the origin. Pursuit is said to be completed if the state of at least one of the pursuers coincides with that of evader. In this paper, we construct a new effective strategy for the pursuers. We prove that new strategy has advantage over the P-strategy.

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Key Words: differential game, many pursuers, evader, control, strategy

1. Introduction

Two person zero-sum differential games was originated by Isaac [12]. Since then, a number of books that relates to this area have been published such as Friedman [5], Pontryagin [19], Krasovskii and Subbotin [14], Petrosyan [18], Lewin [17], Chikrii [3], Grigorenko [6], Satimov [22].

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There are many works devoted to differential games of many players, e.g. [1], [2], [3], [4], [7], [8], [9], [10], [11], [13], [15], [16], [20], [21]. For the simple motion differential game where players' maximal speeds are equal to 1, Pshenichnii [20] showed that if the initial state of evader belongs to the interior of convex hull of initial states of pursuers, then evader can be captured, else evader can avoid from capturing. In that paper, pursuers used P-strategy.

In the present paper, we consider a simple motion pursuit differential game of three pursuers and one evader in the plane. Control parameters of players are subjected to geometric constraints. More precisely, the control sets of players are unit circle centered at the origin. We construct new effective strategies for the pursuers and show its advantage over the P-strategy.

2. Statement of the Problem

Consider a differential game of three pursuers and one evader described by the following equations

$$\dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \quad |u_i| \leq 1, \quad i = 1, 2, 3. \quad (2.1)$$

$$\dot{y} = v, \quad y(0) = y_0, \quad |v| \leq 1, \quad (2.2)$$

where $x_i, u_i, y, v \in \mathbb{R}^2$, u_i is the control parameter of the pursuer x_i , v is that of the evader y , $x_{i0} \neq y_0$, $i = 1, 2, 3$.

Definition 2.1. A measurable function $u_i(t)$, $|u_i(t)| \leq 1$, $t \geq 0$, is called control of the pursuer x_i , $i \in \{1, 2, 3\}$.

Definition 2.2. A measurable function $v(t)$, $|v(t)| \leq 1$, $t \geq 0$, is called control of the evader.

Let $H(0, 1) = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$.

Definition 2.3. A function $U_i(x_i, y, v)$, $U_i : \mathbb{R}^2 \times \mathbb{R}^2 \times H(0, 1) \rightarrow H(0, 1)$, is called strategy of the pursuer x_i , if for any control of the evader $v = v(t)$, $t \geq 0$, the initial value problem

$$\begin{cases} \dot{x}_i = U_i(x_i, y, v), & x_i(0) = x_{i0}, \\ \dot{y} = v, & y(0) = y_0, \end{cases} \quad (2.3)$$

has a unique solution $(x_i(t), y(t))$ with absolutely continuous components $x_i(t)$ and $y(t)$.

Diag.1. Current positions of players.

If $x_i(\tau) = y(\tau)$ for some $i \in \{1, 2, 3\}$ at some $\tau > 0$, then pursuit is said to be completed. Purpose of the pursuers is to complete the pursuit, that of the evader is to avoid from capturing.

Problem 1. Construct more effective strategy for the pursuers than P-strategy to complete the pursuit.

3. New Strategy for Pursuers

3.1. Construction Strategies for the Pursuers

It is known that if $y_0 \notin \text{int conv}\{x_{10}, x_{20}, x_{30}\}$, then evasion is possible (see, [20]). Therefore, we assume that $y_0 \in \text{int conv}\{x_{10}, x_{20}, x_{30}\}$. In this case, pursuit can be completed [20]. Note that P-strategy was used by pursuers to complete the game in that paper. In the present paper, we suggest another new effective strategy for pursuers that is more effective than P-strategy to complete the game.

Let $x_i = x_i(t)$, $y = y(t)$ be current positions of the players, and let

$$m_i = \frac{1}{2}(x_i + y), \quad e_i = \frac{y - x_i}{|y - x_i|}, \quad y \neq x_i, \quad i = 1, 2, 3.$$

Clearly, m_i is midpoint of segment with endpoints x_i and y . Let $x_i = (x_{i1}, x_{i2})$, $m_i = (m_{i1}, m_{i2})$, and $y_i = (y_1, y_2)$. Coordinates of the point m_i are

$$m_{i1} = \frac{1}{2}(x_{i1} + y_1), \quad m_{i2} = \frac{1}{2}(x_{i2} + y_2), \quad i = 1, 2, 3.$$

Diag.2. Strategy for x_1 .

From now on, denote the straight line passing through the given two points $a, b \in \mathbb{R}^2$, $a \neq b$, by (ab) , and the ray going out of the point a and passing through the point b by $[ab)$.

Any point $z = (\xi, \eta)$ of the straight line that passes through the point m_i and orthogonal to e_i satisfies the equation

$$(z - m_i)e_i = 0$$

or in coordinate form

$$\begin{aligned} \xi(y_1 - x_{i1}) + \eta(y_2 - x_{i2}) &= \frac{1}{2}(y_1^2 + y_2^2 - x_{i1}^2 - x_{i2}^2) \\ &= \frac{1}{2}(|y|^2 - |x_i|^2), \quad i = 1, 2, 3. \end{aligned} \quad (2.3i)$$

Let a_{12} , a_{23} , a_{31} be intersection points of straight lines (2.3₁) and (2.3₂), (2.3₂) and (2.3₃), (2.3₃) and (2.3₁), respectively (Diag.1). Next, find coordinates of the points a_{12} , a_{23} , a_{31} . Since a_{ij} , $(i, j) = (1, 2), (2, 3), (3, 1)$, is the intersection point of the lines (2.3_i) and (2.3_j), then for the coordinates (ξ_{ij}, η_{ij}) of the point a_{ij} , we have

$$\begin{aligned} \xi_{ij} &= \frac{1}{2} \frac{\begin{vmatrix} |y|^2 - |x_i|^2 & y_2 - x_{i2} \\ |y|^2 - |x_j|^2 & y_2 - x_{j2} \end{vmatrix}}{\begin{vmatrix} y_1 - x_{i1} & y_2 - x_{i2} \\ y_1 - x_{j1} & y_2 - x_{j2} \end{vmatrix}}, \\ \eta_{ij} &= \frac{1}{2} \frac{\begin{vmatrix} y_1 - x_{i1} & |y|^2 - |x_i|^2 \\ y_1 - x_{j1} & |y|^2 - |x_j|^2 \end{vmatrix}}{\begin{vmatrix} y_1 - x_{i1} & y_2 - x_{i2} \\ y_1 - x_{j1} & y_2 - x_{j2} \end{vmatrix}}. \end{aligned}$$

Diag.3. Strategy for x_1

Next, construct a strategy for each pursuer x_i , $i \in \{1, 2, 3\}$. Make a convention that

$$a_{01} = a_{31}, \quad a_{34} = a_{31}, \quad e_{34} = e_{31}, \quad e_{10} = e_{13}, \quad p_{34} = p_{31}, \quad p_{10} = p_{13}$$

Let (see Diag. 2).

$$e_{i,i-1} = \frac{a_{i-1,i} - x_i}{|a_{i-1,i} - x_i|}, \quad e_{i,i+1} = \frac{a_{i,i+1} - x_i}{|a_{i,i+1} - x_i|},$$

$$p_{i,i-1} = \frac{a_{i-1,i} - m_i}{|a_{i-1,i} - m_i|}, \quad p_{i,i+1} = -p_{i,i-1}.$$

Note that $e_i \perp p_{i,i-1}$, $e_i \perp p_{i,i+1}$.

Next, construct strategies of pursuers. Set,

$$u_i = \begin{cases} e_{i,i-1}, & (v, p_{i,i-1}) > (e_{i,i-1}, p_{i,i-1}), \\ e_{i,i+1}, & (v, p_{i,i+1}) > (e_{i,i+1}, p_{i,i+1}), \\ v - (v, e_i)e_i + e_i \sqrt{1 - |v|^2 + (v, e_i)^2}, & \text{otherwise} \end{cases} \quad (3.1)$$

Constructed strategy needs some comments. To imagine when the inequality $(v, p_{i,i-1}) > (e_{i,i-1}, p_{i,i-1})$ in (3.1) may occur, consider the case $i = 1$. Then this inequality takes the form $(v, p_{13}) > (e_{13}, p_{13})$.

It may occur only if v goes out from the angle between the rays $[ya_{31})$ and $[yb_{31})$ shown in the Diag. 3, where the ray $[yb_{31})$ parallel to $[x_1a_{31})$. It should be noted, however, the fact that v belongs to this angle doesn't guarantee that the inequality $(v, p_{13}) > (e_{13}, p_{13})$ is satisfied. In case this inequality fails to hold, according to (3.1), the pursuer x_1 applies P-strategy. If v doesn't belong to these angles shown in the Diag.3, the pursuer x_1 applies P-strategy as well.

3.2. Advantage of Constructed Strategy

We compare the constructed strategy with the P-strategy and show advantage of the new strategy. Consider the rate of decrease of the expression

$$\alpha(t) = \sum_{i=1}^3 |y(t) - x_i(t)|^2$$

at any time t , that is

$$\frac{d}{dt}\alpha_i(t) = 2 \sum_{i=1}^3 |y(t) - x_i(t)| ((v, e_i) - (u_i, e_i)).$$

Compare the value of the quantity $(v, e_i) - (u_i, e_i)$ corresponding to the strategy (3.1) and P-strategy. Let $u_i(t)$ be defined by (3.1). If $u_i(t)$ is defined by formula (see the third line of (3.1))

$$u_i(t) = v - (v, e_i)e_i + e_i\sqrt{1 - |v|^2 + (v, e_i)^2},$$

then, clearly, $(v, e_i) - (u_i, e_i)$ is the same as the value corresponding to P-strategy.

Let $u_i(t) = e_{i,i-1}$, $(v, p_{i,i-1}) > (e_{i,i-1}, p_{i,i-1})$ (see the first line of (3.1)). Then

$$(u_i, e_i) - (v, e_i) = (e_{i,i-1}, e_i) - (v, e_i). \quad (3.2)$$

When the pursuer x_i uses P-strategy

$$(u_i, e_i) - (v, e_i) = \sqrt{1 - |v|^2 + (v, e_i)^2} - (v, e_i), \quad (3.3)$$

We compare (3.2) and (3.3), and show that right hand side of (3.2) is greater than that of (3.3). To this end it suffices to show that

$$(e_{i,i-1}, e_i) > \sqrt{1 - |v|^2 + (v, e_i)^2}. \quad (3.4)$$

Indeed, square both sides of (3.4) to obtain

$$(e_{i,i-1}, e_i)^2 > 1 - |v|^2 + (v, e_i)^2 \quad (3.5)$$

Since

$$(e_{i,i-1}, e_i)^2 = 1 - (e_{i,i-1}, p_{i,i-1})^2, \quad (v, e_i)^2 = |v|^2 - (v, p_{i,i-1})^2$$

Diag.4. Time $T = |s - w| + |w - a_{31}|$

then (3.5) can be transformed to the inequality

$$1 - (e_{i,i-1}, p_{i-1})^2 > 1 - (v, p_{i,i-1})^2,$$

or equivalently

$$(e_{i,i-1}, p_{i,i-1}) < (v, p_{i,i-1}),$$

which is true according to the inequality in the first line of (3.1).

Thus, when the pursuer x_i uses (3.1), the expression $\alpha(t)$ decreases faster than x_i uses P-strategy.

In a similar fashion, it can be shown that when x_i uses

$$u_i(t) = e_{i,i+1},$$

$$(v, p_{i,i+1}) > (e_{i,i+1}, p_{i,i+1}),$$

the expression $\alpha(t)$ decreases faster than x_i uses P-strategy.

Therefore, the strategy (3.1) has an advantage over the P-strategy.

It is important to point out that when pursuers use P-strategy, the best control for the evader is as follows. Pass a straight line through the point y parallel the longest side of triangle with vertices a_{12}, a_{23}, a_{31} . Let $[sw]$ be a segment of this line between the sides of the triangle. In Diag.4

$$|a_{12} - a_{23}| \leq |a_{12} - a_{31}| \leq |a_{23} - a_{31}|.$$

First, the evader moves along the ray $[ys)$ to the distance $|s - y| - \epsilon/2$, where ϵ is a given sufficiently small positive number. Then, turns back and moves in the direction $[sw)$ to the distance $|w - y| - \epsilon/2$. Until this time the pursuer

x_3 moves parallel to the side $[a_{23}a_{31}]$ of the triangle. Finally, the evader moves parallel to the side of middle length, that is, $[a_{12}a_{31}]$ until he meets the pursuer x_3 . In this way, the evader can move not being captured until the time

$$|s - w| + |w - a_{31}| - \epsilon.$$

Therefore, $T = |s - w| + |w - a_{31}|$ is a guaranteed evasion time for the evader provided that pursuers use P-strategy. For any $\epsilon > 0$, the evader can move not being captured on the time interval $[0, T - \epsilon]$.

If the pursuers use the strategy (3.1), the evader is captured earlier than the time T since if the evader moves in the direction $[ys)$ or $[yw)$ pursuer x_3 moves towards $[x_3a_{23})$ or $[x_3a_{31})$, respectively, and as shown above, the distance $|y - x_3|$ reduces faster. This shows advantage of the strategy (3.1).

4. Conclusion

A pursuit differential game of three pursuers and one evader in the plane has been studied when all the players have equal dynamic possibilities. New strategies for pursuers have been constructed. We have compared the new strategy with the familiar P-strategy and shown that new strategy has an advantage over the P-strategy. The present work can be extended by considering many pursuers and one evader.

References

- [1] A. I. Blagodatskikh, N. N. Petrov, *Conflict interaction between groups of controlled objects*, Izhevsk: Udmurt State University Press (2009).
- [2] F. L. Chernous'ko, A Problem of Evasion of Several Pursuers, *Prikl. Mat. Mekh.*, **40(1)**, (1976), 14–24.
- [3] A. A. Chikrii, *Conflict-controlled processes*, Kluwer, Dordrecht (1997).
- [4] H. T. Croft, Lion and man: a postscript, *J. London Math. Soc.*, **39**, (1964), 385–390.
- [5] A. Friedman, *Differential Games*, John Wiley and Sons, New York (1971).
- [6] N. L. Grigorenko, *Mathematical methods of control of several dynamic processes*, MSU Press, Moscow (1990).
- [7] G. I. Ibragimov, N. A. Rasid, A. Sh. Kuchkarov, F. Ismail, Multi Pursuer Differential Game of Optimal Approach with Integral Constraints on Controls of Players, *Taiwanese Journal of Mathematics*, **19(3)**, (2015), 963–976, DOI: 10.11650/tjm.19.2015.2288.
- [8] G. I. Ibragimov, M. Salimi, M. Amini, Evasion from many pursuers in simple motion differential game with integral constraints, *European Journal of Operational Research*, **218(2)**, (2012), 505–511.

- [9] G. I. Ibragimov, Optimal Pursuit with Countably Many Pursuers and One Evader, *Differential Equations*, **41(5)**, (2005), 627–635.
- [10] G. I. Ibragimov, On the optimal pursuit game of several pursuers and one evader, *Prikladnaya Matematika I Mekhanika*, **62(2)**, (1998), 199–205.
- [11] I. A. Alias, G. I. Ibragimov, A. Rakhmanov, Evasion Differential Game of Infinitely Many Evaders from Infinitely Many Pursuers in Hilbert Space, *Dynamic Games and Applications*, **6(2)**, (2016), 1–13, DOI: 10.1007/s13235-016-0196-0.
- [12] R. Isaacs, *Differential games*, John Wiley & Sons, New York, NY, USA (1965).
- [13] R. P. Ivanov, Simple pursuit-evasion on a compact convex set, *Doklady Akademii Nauk SSSR*, **254(6)**, (1980), 1318–1321.
- [14] N. N. Krasovskii, A. I. Subbotin, *Game-Theoretical Control Problems*, Springer, New York (1988).
- [15] F. H. Kuang, Sufficient conditions of capture in differential games of m pursuer and one evader, *Kibernetika*, **6**, (1986), 91–97.
- [16] A. S. Kuchkarov, G. I. Ibragimov, M. Khakestari, On a linear differential game of optimal approach of many pursuers with one evader. *Journal of Dynamical and Control Systems*, **19(1)**, (2013), 1–15, DOI: 10.1007/s10883-013-9161-z.
- [17] J. Lewin, *Differential Games*, Springer, Berlin, Germany (1994).
- [18] L. A. Petrosyan, *Differential Games of Pursuit*, World Scientific, Singapore, London (1993).
- [19] L. S. Pontryagin, *Selected works*, Nauka, Moscow (1988).
- [20] B. N. Pshenichnii, Simple pursuit by several objects, *Cybernetics and Systems Analysis*, **12(3)**, (1976), 145–146.
- [21] B. N. Pshenichnii, A. A. Chikrii, J. S. Rappoport, An efficient method of solving differential games with many pursuers, *Dokl. Akad. Nauk SSSR*, **256**, (1981), 530–535.
- [22] N. Yu. Satimov, B. B. Rikhsiev, *Methods of solving the problem of avoiding encounter in mathematical control theory*, Fan, Tashkent (2000).