A SYSTEM OF DIFFERENCE EQUATIONS OF 8-TH ORDER FOR SOME MEJER'S FUNTIONS

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Abstract: In [4] we've considered a difference system connected with some Mejer's funtions, which satisfy to a differential equation of fourth order. Arithmetical applications f these considerations can be found in [5], [6]. Here we analogously deduce a difference system for some Mejer's functions, which satisfy to a differential equation of eighth order.

In memory of Professor A.A. Buchstab.

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1. ?? Introduction. Formulation of Theorem ??1

We fix $\alpha \in \mathbb{N}_0$. As usually, $m_\alpha := \Gamma(m + \alpha)/\Gamma(m)$ denotes Pochhammer's symbol. Let $|z| > 1$, $-3\pi/2 < \arg(z) \leq \pi/2$, $\log(z) = \ln(|z|) + i \arg(z)$,

$$l = 0, 1, 2, \nu \in \mathbb{N}_0, g_{l,1}(s) = g_{l,1}(s, \alpha, z, \nu) = \\ (-(-1)^l z)^s \Gamma(-s) (\Gamma(1+s))^{-3-2l} (\Gamma(1+\nu+s))^{2+2l} (\Gamma(1+\nu+\alpha-s))^{2+2l},$$

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$$f_{\alpha,l,1}^{\vee}(z, \nu) = f_{\alpha,l,1}(z, \nu) = -(-1)^{\nu l} \times \text{Mejer's function} \quad (1)$$

$$G_{4+2l,4+2l}^{(1,2+l)} \left(-(-1)^l z \left| \begin{array}{cccccc} \overbrace{-\nu, \dots, -\nu}^{l+2 \text{ times}} & \overbrace{\nu+1+\alpha, \dots, \nu+1+\alpha}^{l+2 \text{ times}} \\ 0, \dots, 0, & 0, \dots, 0 \end{array} \right. \right) =$$

$$-(-1)^{l\nu} \int_{L_1} \frac{g_{l,1}(s)}{2i\pi} ds, \text{ where the curve } L_1 \text{ passes from } +\infty \text{ to } +\infty$$

in the negative direction such that the set $\mathbb{N}_0$ lies to the right from it, but the set $(-\infty, -1) \cap \mathbb{Z}$ lies to the left from it. All unremovable singular points of $g_{l,1}(s)$, encircled by the curve L_1 , are the points $s = 0, \dots, \nu + 2\alpha$, each of them is a pole of the first order, and $\text{Res}(g_{l,1}(s); k) = \lim_{s \rightarrow k} ((s+k)g_{l,1}(s))$, where $l = 0, 1, 2$ and $k = 0, \dots, \nu$. Let $s = k + u$,

$$H_{l,1}(u, k, \nu) = g_{l,1}(s)(k+u) = (-(-1)^l z)^{k+u} \Gamma(-k-u) (\Gamma(1+k+u))^{-3-2l} \times$$

$$\left(\frac{\Gamma(1+\nu+k+u)}{\Gamma(1+\nu+\alpha-k-u)} \right)^{2+2l} = \left(\prod_{\kappa=1}^k (-k+\kappa-u)^{-1} \right) (-(-1)^l z)^{k+u} \times$$

$$\Gamma(1-u) (\Gamma(1+k+u))^{-3-2l} ((\Gamma(1+\nu+k+u))/\Gamma(1+\nu+\alpha-k-u))^{2+l},$$

where $l = 0, 1, 2$ and $k \in [0, +\infty) \cap \mathbb{Z}$. Hence,

$$((\nu+1)_{\alpha})^{2+l} \text{Res}(g_{l,1}(s); k) = ((\nu+\alpha)!/\nu!)^{2+l} \lim_{u \rightarrow 0} (u g_{4+2l,4+2l}^{(1,2+l)}(k+u)) =$$

$$-(-1)^{lk} \left(\frac{(\nu+\alpha)!(\nu+k)!}{\nu!(k!)^2(\nu+\alpha-k)!} \right)^{2+l} = -(-1)^{lk} z^k \left(\binom{\nu+\alpha}{k} \binom{\nu+k}{k} \right)^{2+2l},$$

where $l = 0, 1, 2$ and $k = 0, \dots, \nu$. Consequently,

$$f_{\alpha,l,1}^{*\vee}(z, \nu) := f_{\alpha,l,1}^*(z, \nu) := ((\nu+\alpha)!/\nu!)^{2+l} f_{\alpha,l,1}(z, \nu) = \quad (2)$$

$$((\nu+1)_{\alpha})^{2+l} f_{\alpha,l,1}(z, \nu) = \sum_{k=0}^{\nu+\alpha} (-1)^{(\nu+k)l} (z)^k \binom{\nu+\alpha}{k}^{2+l} \binom{\nu+k}{\nu}^{2+l},$$

where $l = 0, 1, 2$, $\nu \in \mathbb{N}_0$. Let $\kappa \in [3, 6] \cap \mathbb{Z}$, $k \in \{2, 3\}$ for $l = 0$, $k \in 2, 3, 5$ for $l = 1$, $k \in \{2, 3, 5, 7\}$ for $l = 2$, $g_{l,\kappa}^{\sim} = g_{l,\kappa}^{\sim}(s) = g_{l,\kappa}^{\sim}(s)(s, \alpha, z, \nu) =$

$$((-1)^{\kappa} z)^s (\Gamma(-s))^{\kappa+l} (\Gamma(1+s))^{-l-4+\kappa} (\Gamma(1+\nu+s)/\Gamma(1+\nu+\alpha, -s))^{2+l}.$$

$$f_{\alpha, l, \max(2, 2\kappa-5)}^{\vee}(z, \nu) = (-1)^{l\nu+\kappa} \times \text{Mejer's function} \quad (3)$$

$$G_{4+2l, 4+2l}^{(\kappa, 2+l)} \left((-1)^{\kappa} z \left| \begin{array}{cccccc} \overbrace{-\nu, \dots, -\nu}^{l+2 \text{ times}}, & \overbrace{\nu+1+\alpha, \dots, \nu+1+\alpha}^{l+2 \text{ times}} \\ 0, & \dots, & 0, & 0, & \dots, & 0 \end{array} \right. \right) =$$

$$(-1)^{l\nu+\kappa} \int_{L_2} \frac{\widetilde{g_{l, \kappa}(s)}}{2i\pi} ds, \text{ where } \kappa \in [3, 4+l] \cap \mathbb{Z}, \text{ and the curve } L_2 \text{ passes}$$

from $-\infty$ to $-\infty$ in the positive direction such that $\mathbb{Z} \cap (-\infty, -1]$ lies to the left from it, but $\mathbb{N}_0$ lies to the right from it. We put $f_{\alpha, l, k}(z, \nu) := f_{\alpha, l, k}^{\vee}(z, \nu)$ for $k \in \{2, 3\}$. All unremovable singular points of $\widetilde{g_{l, \kappa}(s)}$, encircled by L_2 , compose the set $\{-1 - \nu - k : k \in \mathbb{N}_0\}$, and we put $s = -\nu - 1 - k + u$. Then

$$\widetilde{g_{l, \kappa}}(-\nu - 1 - k + u) = ((-1)^{\kappa} z)^{-\nu-1-k+u} \times \quad (4)$$

$$(\Gamma(\nu+1+k-u))^{\kappa+l} (\Gamma(-\nu-k+u))^{-l-4+\kappa} (\Gamma(-k+u)/\Gamma(2+2\nu+\alpha+k-u))^{2+l} =$$

$$((-1)^{\kappa} z)^{-\nu-1-k+u} (-1)^{(\nu+k)(l+\kappa)+kl} (\pi/\sin(u\pi))^{\kappa-2} \times$$

$$((\Gamma(\nu+1+k-u))^2/(\Gamma(1+k-u)\Gamma(2+2\nu+\alpha+k-u)))^{2+l}.$$

Let

$$H_l^*(\alpha, u, k, \nu) = \left(\frac{(\Gamma(\nu+1+k-u))^2}{\Gamma(1+k-u)\Gamma(2+2\nu+\alpha+k-u)} \right)^{2+l}, \quad (5)$$

$$H_l^{**}(z, \kappa, \alpha, u, k, \nu) = ((-1)^{\kappa} z)^{-\nu-1-k+u} (-1)^{(\nu+k)(l+\kappa)+kl} H_l^*(\alpha, u, k, \nu), \quad (6)$$

$$R(\alpha, t, \nu) = \left(\prod_{j=1}^{\nu} (t-j) \right) / \prod_{j=0*}^{\nu+\alpha} (t+j), \quad R(t, \nu) = R(1, t, \nu). \quad (7)$$

where $\nu \in \mathbb{N}_0$. Clearly,

$$H_l^*(\alpha, u, k, \nu) = R(\alpha, \nu+1+k-u, \nu)^{2+l}, \quad (8)$$

$$\frac{\partial}{\partial u} H_l^*(\alpha, u, k, \nu) = -\frac{\partial}{\partial t} (R^{2+l})(\alpha, \nu+1+k-u, \nu), \quad (9)$$

where $\nu \in \mathbb{N}_0$. In view of (4) - (8),

$$\widetilde{g_{l, \kappa}}(-\nu - 1 - k + u) = \left(\frac{\pi}{\sin(u\pi)} \right)^{\kappa-2} H_l^{**}(z, \kappa, \alpha, u, k, \nu). \quad (10)$$

In view of (10),(6) the function $g_{l,\kappa}^{\sim}(s)$ has at the point $-\nu-1-k$ the pole of the order $\kappa-2$. Therefore, in view of (10), (6),

$$Res(g_{l,\kappa}^{\sim}(s); -1-\nu-k) = 1/(\kappa-3)! \times \quad (11)$$

$$\lim_{u \rightarrow 0} \left(\left(\frac{\partial}{\partial u} \right)^{\kappa-3} (u^{\kappa-2} g_{4+2l,4+2l}^{(\kappa+l,2+l)}(-1-\nu-k+u)) \right) = \frac{1}{(\kappa-3)!} \times$$

$$\lim_{u \rightarrow 0} \left(\left(\frac{\partial}{\partial u} \right)^{\kappa-3} \left(\frac{u\pi}{\sin(u\pi)} \right)^{\kappa-2} H_l^{**}(z, \kappa, \alpha, u, k, \nu) \right).$$

In view of (9), (6),

$$\lim_{u \rightarrow 0} \left(\left(\frac{\partial}{\partial u} \right)^{\eta} H_l^{**}(z, \kappa, \alpha, u, k, \nu) \right) = z^{-1-\nu-k} \times \quad (12)$$

$$(-1)^{l\nu+\kappa} \sum_{\sigma=0}^{\eta} \binom{\eta}{\sigma} (\log((-1)^{\kappa} z))^{\eta-\sigma} \left(\left(\frac{\partial}{\partial u} \right)^{\sigma} H_l^{*} \right) (\alpha, 0, k, \nu) = z^{-1-\nu-k} \times$$

$$(-1)^{l\nu+\kappa} \sum_{\sigma=0}^{\eta} \binom{\eta}{\sigma} (\log((-1)^{\kappa} z))^{\eta-\sigma} \left(\left(-\frac{\partial}{\partial t} \right)^{\sigma} (R^{2+l}) \right) (\alpha, \nu+1+k, \nu).$$

Since $u\pi/\sin(u\pi)$ is an even function, it follows from (11),(12), that

$$Res(g_{l,\kappa}^{\sim}; -1-\nu-k) = \frac{1}{(\kappa-3)!} (-1)^{l\nu+\kappa} z^{-1-\nu-k} \mathfrak{S}, \text{ where } \mathfrak{S} = \quad (13)$$

$$\sum_{\theta \in [0, [(\kappa-3)/2]] \cap \mathbb{Z}} \binom{\kappa-3}{2\theta} \left(\lim_{u \rightarrow 0} \left(\left(\frac{\partial}{\partial u} \right)^{2\theta} \left(\frac{u\pi}{\sin(u\pi)} \right)^{\kappa-2} \right) \right) \mathfrak{S}^{*} \text{ with } \mathfrak{S}^{*} =$$

$$\sum_{\sigma=0}^{\kappa-3-2\theta} \binom{\kappa-3-2\theta}{\sigma} (\log((-1)^{\kappa} z))^{\kappa-3-2\theta-\sigma} \left(\left(-\frac{\partial}{\partial t} \right)^{\sigma} R^{2+l} \right) (\alpha, \nu+1+k, \nu).$$

If $\kappa=3$ then $k=2$, and in view of (13),

$$f_{\alpha,l,2}^{*}(z, \nu) := f_{\alpha,l,2}^{\vee*}(z, \nu) := ((\nu+1)_{\alpha})^{2+l} f_{\alpha,l,2}^{\vee}(z, \nu) = \quad (14)$$

$$\sum_{k=0}^{+\infty} z^{-(1+\nu+k)} ((\nu+1)_{\alpha} R(\alpha, 1+\nu+k, \nu))^{2+l},$$

where $l = 0, 1, 2$ and $\nu \in \mathbb{N}_0$. Let $t = 1 + \nu + k$ with $k \in [0, +\infty) \cap \mathbb{Z}$; then

$$f_{\alpha,l,2}^*(z, \nu) = \sum_{t=1+\nu}^{+\infty} z^{-t} ((\nu+1)_\alpha R(\alpha, t, \nu))^{2+l}, \quad (15)$$

where $l = 0, 1, 2$ and $\nu \in \mathbb{N}_0$. If $\kappa = 4$ then $k = 3$ and $l \in \{0, 1, 2\}$. Let

$$f_{\alpha,l,4}(z, \nu)^\vee := f_{\alpha,l,4}(z, \nu) := - \sum_{t=1+\nu}^{+\infty} z^{-t} \left(\frac{\partial}{\partial t} (R^{2+l}) \right) (\alpha, t, \nu), \quad (16)$$

$$f_{\alpha,l,k}^{\vee*}(z, \nu) := f_{\alpha,l,k}^*(z, \nu) := ((\nu+1)_\alpha)^{2+l} f_{\alpha,l,k}(z, \nu), \quad (17)$$

where $k = 3, 4, l = 0, 1, 2$ and $\nu \in \mathbb{N}_0$. Then, in view of (13),

$$f_{\alpha,l,3}^*(z, \nu) = f_{\alpha,l,2}^*(z, \nu) \log(z) + f_{\alpha,l,4}^*(z, \nu). \quad (18)$$

If $\kappa = 5$ then $k = 5$ and $l \in \{1, 2\}$. Let

$$f_{\alpha,l,5}(z, \nu) = \frac{1}{2} \sum_{t=\nu+1}^{+\infty} z^{-t} \sum_{k=0}^2 \left(\binom{2}{k} (-\log(z))^{2-k} \left(\frac{\partial}{\partial t} \right)^k R^{2+l} \right) (\alpha, t, \nu),$$

$$f_{\alpha,l,6}(z, \nu) = 2^{-1} \sum_{t=\nu+1}^{\infty} z^{-t} \left(\left(\frac{\partial}{\partial t} \right)^2 (R^{2+l}) \right) (\alpha, t, \nu), \quad (19)$$

$$f_{\alpha,l,5}^{\vee*}(z, \nu) := ((\nu+1)_\alpha)^{2+l} f_{\alpha,l,5}^\vee(z, \nu), f_{\alpha,l,k}^*(z, \nu) := ((\nu+1)_\alpha)^{2+l} f_{\alpha,l,k}(z, \nu)$$

for $k = 5, 6$. In view of (13),

$$f_{\alpha,l,5}^\vee(z, \nu) = \quad (20)$$

$$\begin{aligned} & i\pi \sum_{t=\nu+1}^{\infty} \left(\left(\sum_{k=0}^1 (-\log(z))^{1-k} \left(\frac{\partial}{\partial t} \right)^k \right) (R)^{2+l} \right) (\alpha, t, \nu) + \\ & \frac{1}{2} \sum_{t=\nu+1}^{+\infty} z^{-t} \sum_{k=0}^2 \left(\binom{2}{k} (-\log(z))^{2-k} \left(\frac{\partial}{\partial t} \right)^k (R)^{2+l} \right) (\alpha, t, \nu) = \\ & -i\pi f_{\alpha,l,3}^\vee(z, \nu) + f_{\alpha,l,5}(z, \nu), \end{aligned}$$

and

$$f_{\alpha,l,5}(z, \nu) = \sum_{\sigma=0}^2 \frac{1}{(2-\sigma)!} (\log(z))^{2-\sigma} f_{\alpha,l,2+2\sigma}^\vee(z, \nu). \quad (21)$$

If $\kappa = 6$ then $k = 7$ and $l = 2$. Let

$$f_{\alpha,l,8}(z, \nu) = -6^{-1} \sum_{t=\nu+1}^{\infty} z^{-t} \left(\left(\frac{\partial}{\partial t} \right)^3 (R^{2+l}) \right) (\alpha, t, \nu), \quad (22)$$

$$f_{\alpha,l,7}(z, \nu) = \sum_{t=\nu+1}^{\infty} \frac{1}{6z^t} \sum_{\sigma=0}^3 \binom{3}{\sigma} (\log(z))^{3-\sigma} \left(-\frac{\partial}{\partial t} \right)^{\sigma} R^{2+l}(\alpha, t, \nu) = \quad (23)$$

$$\sum_{\sigma=0}^3 \frac{1}{(3-\sigma)!} (\log(z))^{3-\sigma} f_{\alpha,2,2\sigma+2}(z, \nu).$$

$$f_{\alpha,l,7}^{\vee*}(z, \nu) := ((\nu+1)_{\alpha})^{2+l} f_{\alpha,l,7}^{\vee}(z, \nu), f_{\alpha,l,k}^*(z, \nu) := ((\nu+1)_{\alpha})^{2+l} f_{\alpha,l,k}(z, \nu)$$

for $k = 7, 8$. In view of (13),

$$f_{\alpha,l,7}^{\vee*}(z, \nu) = f_{\alpha,l,7}^*(z, \nu) + (2\pi^2/3) f_{\alpha,l,3}^*(z, \nu). \quad (24)$$

Since $(R(\alpha, t, \nu))^{2+l}$ for $\nu \in \mathbb{N}$ has in the points $t = 1, \dots, \nu$, the zeros of the order $2+l$, and $k-1 < 2+l$ for k in $f_{\alpha,l,2k}(z, \nu)$, it follows that

$$f_{\alpha,l,2k}(z, \nu) = \frac{(-1)^{k-1}}{(k-1)!} \sum_{t=1}^{\infty} z^{-t} \left(\frac{\partial}{\partial t} \right)^{k-1} ((\nu+1)_{\alpha} R)^{2+l}(\alpha, t, \nu) \quad (25)$$

for $k \in [1.4] \cap \mathbb{Z}$ and $\nu \in \mathbb{N}_0$. For $m \in \mathbb{N}$ we denote by E_m the unit $m \times m$ -matrix. We denote by $\vec{e}_{m,k}$ and $\bar{e}_{m,k}$ respectively the k -th column and k -th row of the matrix E_m , if $k \in [1, m] \cap \mathbb{Z}$. We put $\vec{e}_{m,k} = 0\vec{e}_{m,1}$, $\bar{e}_{m,k} = 0\bar{e}_{m,1}$, if $k \in \mathbb{Z} \setminus [1, m]$. We denote by N_m the $m \times m$ -matrix, k -th column of which coincides with $\vec{e}_{m,k-1}$ for $k \in [1, m] \cap \mathbb{N}$. If A is a matrix, then we denote by A^t the matrix transposed to the matrix A . Let $\delta^0 f_{\alpha,2,k}(z, \nu) = f_{\alpha,2,k}(z, \nu)$,

$$X_{\alpha,2,k}(z; \nu) = \sum_{\kappa=1}^8 \delta^{\kappa-1} f_{\alpha,2,k}(z, \nu) \vec{e}_{8,k} \text{ for } k \in \mathfrak{K}. \quad (26)$$

The connection of $X_{\alpha,2,k}(z; \nu)$ with $X_{\alpha,2,k}(z; \nu-1)$ was studied in [7]. Related proofs are long and connected with heavy calculations. Therefore we decided to present here relatively easy case $l = 2, \alpha = 1$. We apply the same method, as in [4], where was considered the case $l = 0, \alpha = 1$; arithmetical applications of corresponding results can be found in [5] - [6]. Results for the case $\alpha = 0$ can be found in [3]. Let w be a independent variable,

$$P(\alpha, w, \nu) = \nu^7(\nu + \alpha)^4 + (2\nu + \alpha)P^{(h)}(\alpha, w, \nu) \text{ with } P^{(h)}(\alpha, w, \nu) = \quad (27)$$

$$\begin{aligned}
& \nu^5(\nu + \alpha)^2(-4\nu(\nu + \alpha)w + 2(8\nu + 5\alpha)w^2) - 4(11\nu^2 + 15\nu\alpha + 5\alpha^2) \times \\
& \nu^4(\nu + \alpha)w^3 + \nu^3(96\nu^3 + 210\nu^2\alpha + 150\nu\alpha^2 + 35\alpha^3)w^4 - (3\nu + 2\alpha) \times \\
& 4\nu^2(15\nu^2 + 21\nu\alpha + 7\alpha^2)w^5 + 2\nu(152\nu^3 + 301\nu^2\alpha + 196\nu\alpha^2 + 42\alpha^3)w^6 - 4w^7 \times \\
& (119\nu^3 + 228\nu^2\alpha + 144\nu\alpha^2 + 30\alpha^3), P^*(w, \nu) = \sum_{k=0}^7 p_k^*(\nu)w^k = P(1, w, \nu), \\
& Q^{(h)}(\alpha, w, \nu) = \nu^3(704\nu^3 + 1314\nu^2\alpha + 810\nu\alpha^2 + 165\alpha^3) + 4w\nu^2 \times \\
& (455\nu^3 + 859\nu^2\alpha + 535\nu\alpha^2 + 110\alpha^3) + 2w^2\nu \times \\
& (800\nu^3 + 1523\nu^2\alpha + 956\nu\alpha^2 + 198\alpha^3) + 4w^3 \times \\
& (119\nu^3 + 228\nu^2\alpha + 144\nu\alpha^2 + 30\alpha^3), Q(\alpha, w, \nu) = (2\nu + \alpha)Q^{(h)}(\alpha, w, \nu), \\
& Q^*(\alpha, w, \nu) = Q(\alpha, w + 1, \nu)(w - \nu - \alpha)^4, Q(w, \nu) = Q(1, w, \nu), Q^*(w, \nu) = \\
& \sum_{k=0}^7 q_k^*(\nu)w^k = Q^*(1, w, \nu), T^*(z, w, \nu) = P^*(w, \nu) + zQ^*(w, \nu). \text{ Let}
\end{aligned} \tag{28}$$

$$\tau = \tau_1(\nu) = \nu + 1, \mu = \mu_1(\nu) = (\nu + 1)^2 = \tau^2, \kappa^2 = \kappa, \mathfrak{k} \in [0, 3] \cap \mathbb{Z}, \tag{29}$$

$$\begin{aligned}
r_{2\mathfrak{k}+2-\kappa}^*(\nu) &= \kappa \binom{4}{\mathfrak{k}} (-\tau^2)^{4-\mathfrak{k}}, \bar{R}(\nu) = \sum_{k=1}^8 r_k^*(\nu) \bar{e}_{8,k}, h_{i,k}^*(\nu) = p_{i-1}^*(\nu) r_k^*(\nu), \\
S^*(\nu) &= p_0^*(\nu) E_8 + \sum_{k=1}^7 p_k^*(\nu) N_8^k,
\end{aligned} \tag{30}$$

$$v_{i,k}^*(\nu) = \left(\sum_{\sigma=1}^k \binom{i-1}{k-\sigma} q_{\sigma-1}^*(\nu) \right) + \sum_{\sigma=1}^k \sum_{\kappa=1}^{i-1} h_{9-\kappa,\sigma}^*(\nu) \binom{i-1-\kappa}{k-\sigma}. \tag{31}$$

where $\{i, k\} \subset [1, 8] \cap \mathbb{Z}$. Let $V^*(\nu)$ be the 8×8 -matrix which has $v_{i,k}^*(\nu)$ on intersection of its i -th row and k -th column. Let further

$$A^*(z, \nu) = S^*(\nu) + zV^*(\nu), X_k(z; \nu) = X_{1,2,k}(z; \nu). \tag{32}$$

We prove here the following

Theorem ??1. *The domain of definition of column $X_k(z; \nu)$ can be extended to the set $\Omega_0 \times (\mathbb{Z} \setminus \{-1\})$, and this column satisfies to the equation*

$$\nu^{11} X_k(z; \nu - 1) = A^*(z; \nu) X_k(z; \nu), \tag{33}$$

for $\nu \in M_1^* = (-\infty, -2] \cup [1, +\infty) \cap \mathbb{Z}$, $k = 1, 2, 3, 5, 7$, $z \in \Omega_0$.

2. 1 Some Relations. Begin of the Proof of Theorem ?? .1

Let α be an independent variable.

Lemma 1.1. Let $T^\wedge(\alpha, w, \nu) := (w + \nu)^4 P(\alpha, w, \nu) + w^8 Q(\alpha, w, \nu) - \nu^{11}(w - \nu - \alpha)^4$. Then

$$T^\wedge(\alpha, w, \nu) = 0 \quad (34)$$

Proof. If $T^\wedge(\alpha, w, \nu) \neq 0$, then $\deg_\alpha(T^\wedge(\alpha, w, \nu)) \leq 4$. Moreover, coefficient at α^4 in $T^\wedge(\alpha, w, \nu)$ is equal to

$$\begin{aligned} & (\nu^7 - 4\nu^6 w + 10\nu^5 w^2 - 20\nu^4 w^3 + 35\nu^3 w^4 - 56\nu^2 w^5 + 84\nu w^6 - 120w^7) \times \\ & (w + \nu)^4 + w^8(165\nu^3 + 440\nu^2 w + 396\nu w^2 + 120w^3) - \nu^{11} = \\ & (\nu^{11}(1 - 1) + (4 - 4)\nu^{10})w + 2(3 - 8 + 5)\nu^9 w^2 + 4(1 - 6 + 10 - 5)\nu^8 w^3 + \\ & (1 - 16 + 60 - 80 + 35)\nu^7 w^4 - 4(1 - 10 + 30 - 35 + 14)\nu^6 w^5 + 12(7 - 40 + 33)\nu^5 w^6 + \\ & (10 - 80 + 210 - 224 + 84)\nu^4 w^7 + \\ & (35 - 224 + 504 - 480 + 165)\nu^3 w^8 - 8(7 - 42 + 90 - 55)\nu^2 w^9 - 120(1 - 1)w^{11} = 0. \end{aligned}$$

Hence, $\deg_\alpha(T^\wedge(\alpha, w, \nu)) \leq 3$, if $T^\wedge(\alpha, w, \nu) \neq 0$, and it is enough to check the equality (34) for $\alpha = -2\nu$, $\alpha = -\nu$, $\alpha = 0$, and $\alpha = \nu$. If $\alpha = -2\nu$, then

$$P(-2\nu, w, \nu) = \nu^{11}, Q(-2\nu, w, \nu) = 0, (w - \nu - (-2\nu))^4 = (w + \nu)^4,$$

and (34) holds. If $\alpha = -\nu$, then $P(-\nu, w, \nu) = \nu^7 w^4 - 4\nu^6 w^5 + 10\nu^5 w^6 -$

$$\begin{aligned} & 20\nu^4 w^7, Q(-\nu, w, \nu) = 35\nu^7 + 84\nu^6 w + 70\nu^5 w^2 + 20\nu^4, T^\wedge(-\nu, w, \nu) = \\ & (\nu^7 w^4 - 4\nu^6 w^5 + 10\nu^5 w^6 - 20\nu^4 w^7)(w + \nu)^4 - \nu^{11} w^4 + \\ & w^8(35\nu^7 + 84\nu^6 w + 70\nu^5 w^2 + 20\nu^4) = 20(-1 + 1)\nu^4 w^{11} + \\ & (1 - 1)\nu^{11} w^4 + (-4 + 4)\nu^{10} w^5 + (10 - 16 + 6)\nu^9 w^6 + (-20 + 40 - 24 + 4)\nu^8 w^7 + \\ & (-80 + 60 - 16 + 1 + 35)\nu^7 w^8 + (-120 + 40 - 4 + 84)\nu^6 w^9 + 10(-8 + 1 + 7)\nu^5 w^{10} = 0. \end{aligned}$$

$$\begin{aligned} & \text{If } \alpha = 0, \text{ then } P(0, w, \nu) = \nu^{11} - 8\nu^{10} w + 32\nu^9 w^2 - 88\nu^8 w^3 + \\ & 8(24\nu^7 w^4 - 45\nu^6 w^5 + 76\nu^5 w^6 - 119\nu^4 w^7), Q(0, w, \nu) = 8 \times \\ & (176\nu^7 + 455\nu^6 w + 400\nu^5 w^2 + 119\nu^4 w^3, T^\wedge(0, w, \nu) = (w + \nu)^4 \times \\ & (\nu^{11} - 8\nu^{10} w + 32\nu^9 w^2 - 88\nu^8 w^3 + 192\nu^7 w^4 - 360\nu^6 w^5 + 608\nu^5 w^6 - 952\nu^4 w^7) + \\ & w^8(1408\nu^7 + 3640\nu^6 w + 3200\nu^5 w^2 + 952\nu^4 w^3) - \nu^{11}(w - \nu)^4 = \end{aligned}$$

$$\begin{aligned}
& \nu^4 w^{11} 952(-1+1) + \nu^{15}(1-1) + \nu^{14} w(4-8+4) + \nu^{13} w^2(32-32+6-6) + \\
& \nu^{12} w^3(-88+128-48+4+4) + \nu^{11} w^4(192-352+192-32+1-1) + \\
& \nu^{10} w^5(-360+768-528+128-8) + \nu^9 w^6 32(19-45+36-11+1) + \\
& \nu^8 w^7(-952+2432-2160+768-88) + \nu^7 w^8 32(-119+114-45+6+44) + \\
& \nu^6 w^9 8(-714+304-45+455) + \nu^5 w^{10} 32(-119+19+100) = 0.
\end{aligned}$$

If $\alpha = \nu$, then $P(\nu, w, \nu) = 16\nu^{11} - 96\nu^{10}w + 312\nu^9w^2 - 744\nu^8w^3 +$
 $1473\nu^7w^4 - 2580\nu^6w^5 + 4146\nu^5w^6 - 6252\nu^4w^7$, $Q(\nu, w, \nu) =$
 $3\nu^4(2993\nu^3 + 5877\nu^2w + 6954\nu w^2 + 2084w^3)$, $T_2^\wedge(\nu, w, \nu) =$
 $(w + \nu)^4(16\nu^{11} - 96\nu^{10}w + 312\nu^9w^2 - 744\nu^8w^3) +$
 $(w + \nu)^4(1473\nu^7w^4 - 2580\nu^6w^5 + 4146\nu^5w^6 - 6252\nu^4w^7) +$
 $w^8 3\nu^4(2993\nu^3 + 5877\nu^2w + 6954\nu w^2 + 2084w^3) - \nu^{11}(w - 2\nu)^4 =$
 $16(1-1)\nu^{15} + 32(-3+2+1)\nu^{14}w + 24(13-16+4-1)\nu^{13}w^2 +$
 $8(-93+156-72+8+1)\nu^{12}w^3 + (16-384+1872-2976+1473-1)\nu^{11}w^4 +$
 $12(-8+104-372+491-215)\nu^{10}w^5 + 6(52-496+1473-1720+691)\nu^9w^6 +$
 $12(-62+491-1290+1382-521)\nu^8w^7 +$
 $3(491-3440+8292-8336+2993)\nu^7w^8 + 12(-215+1382-3126+1959)\nu^6w^9 +$
 $6(691-4168+3477)\nu^5w^{10} + 6252(-1+1)\nu^4w^{11} = 0. \square.$

Let $\nu \in \mathfrak{M}_0 = \mathbb{Z} \setminus \{-1\}$, $\tau = \nu + 1$,

$$D(z, \nu, w) = z(w^2 - \tau^2)^4 - w^8. \quad (35)$$

and w is independent variable. Clearly,

$$D(z, \nu, w) = D(z, -\nu - 2, w) \text{ for } \nu \in \mathbb{Z}. \quad (36)$$

Let $\delta := z \frac{\partial}{\partial z}$. According to the general properties of the Mejer's functions,

$$D(z, \nu, \delta)f = 0, (\delta + \nu + 1)^4 f_k^\vee(z, \nu) = (\delta - \nu - 2)^4 f_k^\vee(z, \nu + 1), \quad (37)$$

where $f = f_k^\vee(z, \nu)$, $\nu \in \mathbb{N}_0$, $k \in \mathfrak{K} = \{1, 2, 3, 5, 7\}$. Therefore

$$(\delta + \nu)^4 f_k^\vee(z, \nu - 1) = (\delta - \nu - 1)^4 f_k^\vee(z, \nu), \quad (38)$$

where $\nu \in \cap \mathbb{N}$, $k \in \mathfrak{K}$. Let

$$f_k^\vee(z, -\nu - 2) = f_k^\vee(z, \nu), \text{ for } \nu \in \mathbb{N}_0, k \in \mathfrak{K}. \quad (39)$$

So, $f_k^\vee(z, \nu)$ is defined for $\nu \in \mathfrak{M}_0$. If $\nu \in (-\infty, -2] \cap \mathbb{Z}$, then $\nu_1 = -\nu - 2 \in \mathbb{N}_0$, and, in view of (39), $f_k^\vee(z, \nu) = f_k^\vee(z, -\nu_1 - 2) = f_k^\vee(z, \nu_1) = f_k^\vee(z, -\nu - 2)$. Therefore (39) holds for all the $\nu \in \mathfrak{M}_0$. Furthermore,

$$\delta^s f_k^\vee(z, -\nu - 2) = \delta^s f_k^\vee(z, \nu) \text{ for } s \in \mathbb{N}_0, \nu \in \mathfrak{M}_0, k \in \mathfrak{K}. \quad (40)$$

In view of (36) and (40), the first equality in (37) holds for $\nu \in \mathfrak{M}_0$. Furthermore, if $\nu \in (-\infty, -2] \cap \mathbb{Z}$, then $\nu_1 = -\nu - 2 \in \mathbb{N}_0$, $-\nu - 1 \in \mathbb{N}$, and in view of (37), (40),

$$(\delta - \nu - 1)^4 f_k^\vee(z, \nu) = (\delta + \nu_1 + 1)^4 f_k^\vee(z, \nu_1) = \quad (41)$$

$$(\delta - \nu_1 - 2)^4 f_k^\vee(z, \nu_1 + 1) = (\delta + \nu)^4 f_k^\vee(z, -\nu - 1) = (\delta + \nu)^4 f_k^\vee(z, \nu - 1).$$

If $\nu \in (-\infty, -3] \cap \mathbb{Z}$, then $\nu + 1 \in (-\infty, -2] \cap \mathbb{Z}$, and, in view (41), we have the equality $(\delta - \nu - 2)^4 f_k^\vee(z, \nu + 1) = (\delta + \nu + 1)^4 f_k^\vee(z, \nu)$. So, (38) holds for $\nu \in \mathfrak{M}_1 = ((-\infty, -2] \cap \mathbb{Z}) \cup \mathbb{N}$, and second equality in (37) holds for $\nu \in \mathfrak{M}_2 = ((-\infty, -3] \cap \mathbb{Z}) \cup \mathbb{N}_0$. In view of (34), (38), (41), (35), (37) and equalities $Q(\delta, \nu)(\delta + \nu)^4 = (\delta + \nu)^4 Q(\delta, \nu)$, $Q(\delta, \nu)z = zQ(\delta + 1, \nu)$,

$$\nu^{11}(\delta + \nu)^4 f_k^\vee(z, \nu - 1) = (\delta + \nu)^4 P^*(\delta, \nu) f_k^\vee(z, \nu) + \quad (42)$$

$$\delta^8 Q(\delta, \nu) f_k^\vee(z, \nu) = (\delta + \nu)^4 (P(\delta, \nu) + zQ^*(\delta, \nu)) f_k^\vee(z, \nu).$$

Clearly, $\mathfrak{M}_i = ((-\infty, -1 - i] \cup [2 - i, +\infty)) \cap \mathbb{Z} = \mathfrak{M}_0 \setminus \{2 - 2i\}$, for $i \in \{1, 2\}$, and the map $\nu \rightarrow \nu_1 = -\nu - 2$ is bijection of $\mathfrak{M}_i$ onto $\mathfrak{M}_{3-i}$ for $i \in \{1, 2\}$. We denote by $\mathfrak{V}_i$ the set of all functions defined on $\Omega_0 \times \mathfrak{M}_i$ for $i = 0, 1, 2$ and regular relatively $z \in \Omega_0$ for fixed ν . For $f \in \mathfrak{V}_0$, $z \in \Omega_0$, we put

$$\mathfrak{W}_1(f, z, \nu) = \nu^{11} f(z, \nu - 1) - T^*(z, \delta, \nu) f(z, \nu) \text{ if } \nu \in \mathfrak{M}_1, \quad (43)$$

$$\mathfrak{W}_2(f, z, \nu) = \mathfrak{W}_1(f, z, -\nu - 2) \text{ if } \nu \in \mathfrak{M}_2. \quad (44)$$

The map $f \rightarrow \mathfrak{W}_i(f, z, \nu)$ turns $\mathfrak{V}_0$ into $\mathfrak{V}_i$ for $i = 1, 2$. In view of (42),

$$(\delta + \nu)^4 \mathfrak{W}_1(f_k^\vee, z, \nu) = 0 \text{ for } \nu \in \mathfrak{M}_1, k \in \mathfrak{K}. \quad (45)$$

If $\nu \in \mathfrak{M}_2$, $k \in \mathfrak{K}$, then $-\nu - 3 \in \mathfrak{M}_2$, $\nu_1 = -\nu - 2 \in \mathfrak{M}_1$, and, in view of (45), (40),

$$(\delta - \nu - 2)^4 \mathfrak{W}_2(f_k^\vee, z, \nu) = 0. \quad (46)$$

Theorem 1.1. *Instead of (45) and (46) the following equality holds*

$$\mathfrak{M}_i(f_k^\vee, z, \nu) = 0. \quad (47)$$

where $i \in \{1, 2\}$, $\nu \in \mathfrak{M}_i$, $k \in \mathfrak{K}$. In other words, this means that $(\delta + \nu)^4$ and $(\delta - \nu - 2)^4$ can be omit from the left side of respectively (45) and (46).

Proof. Let $\theta = 2\nu + 1$, $i \in \mathbb{N}$.

$$S_i(t, \nu) = \left(\frac{\partial}{\partial t}\right)^{i-1} \left(\left(\sum_{k=1}^{\nu} \frac{1}{t-k} \right) - \sum_{k=0}^{\nu+1} \frac{1}{t+k} \right) = (-1)^{i-1} (i-1)! \times \quad (48)$$

$$\begin{aligned} & \left(\left(\sum_{k=1}^{\nu} \frac{1}{(t-k)^i} \right) - \sum_{k=0}^{\nu+1} \frac{1}{(t+k)^i} \right), H_i^\wedge(\nu) = S_i(\nu, \nu-1) - S_i(\nu+1, \nu) = \\ & (-1)^i (i-1)! (2/\nu^i - 1/(2\nu+1)^i - 1/(2\nu+2)^i), H_i^\vee(\nu) = H_i^\wedge(\nu) \times \\ & (2\nu(2\nu+1)(2\nu+2))^i = (-1)^i (i-1)! (2^{i+1}(\theta+1)^i \theta^i - (\theta-1)^i (\theta^i + (\theta+1)^i)) = \\ & (-1)^i (i-1)! ((2^{i+1}(\theta+1)^i - (\theta-1)^i) \theta^i - (\theta^2 - 1)^i). \text{ Then} \\ & H_1^\vee(\nu) = -2\theta^2 - 5\theta - 1, H_2^\vee(\nu) = 6\theta^4 + 18\theta^3 + 9\theta^2 - 1, (H_1^\vee(\nu))^2 = 4\theta^4 + \\ & 20\theta^3 + 29\theta^2 + 10\theta + 1, 4(H_1^\vee(\nu))^2 + 3H_2^\vee(\nu) = 34\theta^4 + 134\theta^3 + 143\theta^2 + 40\theta + 1, \\ & 4(H_1^\vee(\nu))^3 + 3H_1^\vee(\nu)H_2^\vee(\nu) = -68\theta^6 - 438\theta^5 - 990\theta^4 - 929\theta^3 - \\ & 345\theta^2 - 45\theta - 1, H_3^\vee(\nu) = -28\theta^6 - 102\theta^5 - 96\theta^4 - 34\theta^3 + 6\theta^2 - 2, \\ & 16(H_1^\vee(\nu))^3 + 12H_1^\vee(\nu)H_2^\vee(\nu) + H_3^\vee(\nu) = \quad (49) \\ & -300\theta^6 - 1854\theta^5 - 4056\theta^4 - 3750\theta^3 - 1374\theta^2 - 180\theta - 6, \\ & (H_1^\vee(\nu))^3 = -8\theta^6 - 60\theta^5 - 162\theta^4 - 185\theta^3 - 81\theta^2 - 15\theta - 1, H_i^{\wedge\wedge}(\nu) := \\ & S_i^2(\nu, \nu-1) - S_i^2(\nu+1, \nu) = 2H_i^\wedge(\nu)S_i(\nu+1, \nu) + (H_i^\wedge(\nu))^2. \text{ In view of (7)} \end{aligned}$$

$$\left(\frac{\partial}{\partial t}\right) (R(t, \nu))^4 = (R(t, \nu))^4 4S_1(t, \nu), \left(\frac{\partial}{\partial t}\right)^2 (R(t, \nu))^4 = \quad (50)$$

$$(R(t, \nu))^4 (16(S_1^2(t, \nu)) + 4S_2(t, \nu)), \left(\frac{\partial}{\partial t}\right)^3 (R(t, \nu))^4 =$$

$$(R(t, \nu))^4 (64S_1^3(t, \nu) + 48S_1(t, \nu))S_2(t, \nu) + 4S_3(t, \nu)).$$

If the function $f(z)$ is regular in Ω_0 , then, according to the Leibnitz formula, we have the equality

$$\delta^n \left(\frac{(\log(z))^m}{m!} f(z) \right) = \sum_{k=0}^m \frac{((\log(z))^{m-k})}{(m-k)!} \binom{n}{k} w^{n-k} \Big|_{w=\delta} f(z) =$$

$$\left(\sum_{k=0}^m \frac{(\log(z))^{m-k}}{(m-k)!k!} \left(\left(\frac{\partial}{\partial w} \right)^k w^n \right) \Big|_{w=\delta} \right) f(z);$$

therefore, if $H(w) \in \mathbb{C}[w]$, then

$$H(\delta) \left(\frac{(\log(z))^m}{m!} f(z) \right) = \sum_{k=0}^m \frac{(\log(z))^{m-k}}{(m-k)!k!} \left(\left(\frac{\partial}{\partial w} \right)^k H(w) \right) \Big|_{w=\delta} f(z). \quad (51)$$

If $f \in \mathfrak{V}_0$, then $((\log(z))^m/m!)f(z) \in \mathfrak{V}_0$ for any $m \in \mathbb{N}$, and

$$\mathfrak{W}_i(((\log^m(z))/m!)f, z, \nu) = ((\log^m(z))/m!)\mathfrak{W}_i(f, z, \nu) - \quad (52)$$

$$\sum_{k=1}^m \frac{(\log(z))^{m-k}}{(m-k)!k!} \left(\left(\frac{\partial}{\partial w} \right)^k T^*(z, w, \eta_i(\nu)) \right) \Big|_{w=\delta} f(z, \eta_i(\nu))$$

with $\eta_i(\nu) = (3 - 2i)\nu + 2 - 2i$ for $i = 1, 2$. In view of (2), (43), $\mathfrak{W}_1^\vee(f_1, z, \nu)$ belongs to $\mathbb{C}[z]$ for arbitrary $\nu \in \mathfrak{M}_1$. In view of (45), $\mathfrak{W}_1(f_1, z, \nu)$ belongs to the null-space of the operator $(\delta + \nu)^4$ (as linear operator on $\mathbb{C}[z]$), and in the case, when $\nu \in \mathbb{N}$, this null-space consists of the single element – the zero element in $\mathbb{C}[z]$. Hence, $\mathfrak{W}_1(f_1^\vee, z, \nu) = 0$ for $\nu \in \mathbb{N}$. In view of (28),

$$Q(-\nu, \nu) = \nu^3(2\nu + 1)^4, \quad Q(\nu + 2, -\nu - 2) = -(\nu + 2)^3(2\nu + 3)^4. \quad (53)$$

In view of (2), (44), $\mathfrak{W}_2(f_1, z, \nu) \in \mathbb{C}[z]$ if $\nu \in \mathfrak{M}_2$. In view of (46), the null-space of the operator $(\delta - \nu - 2)^4$ (as linear operator on $\mathbb{C}[z]$) contains $\mathfrak{W}_2(f_1, z, \nu)$, and, if $\nu \in \mathbb{N}_0$, then $\mathfrak{W}_2(f_1, z, \nu) = \gamma_1 z^{\nu+2}$, with $\gamma_1 \in \mathbb{C}$.

Remark 1. Of course, the null-space of the operator $(\delta - \nu - 2)^4$ as linear operator on $\mathbb{C}[z, \log(z)]$, consists of all the functions having the form

$$z^{\nu+2}(\beta_0 + \beta_1 \log(z) + \beta_2 \log^2(z) + \beta_3 \log^3(z)) \text{ with } \beta_k \in \mathbb{C} \text{ for } k = 0, 1, 2, 3.$$

But in our case the space $\mathbb{C}[z]$ is an invariant subspace of the space $\mathbb{C}[z, \log(z)]$ relatively this operator. Therefore $\beta_k = 0$ for $k = 1, 2, 3$.

Clearly,

$$p(\delta)z^k = p(k)z^k \text{ for any } p \in \mathbb{C}, k \in \mathbb{Z}. \quad (54)$$

In view of (2), $\deg_z(f_1^\vee(z, \nu + \kappa) = \nu + 1 + \kappa$, coefficient at $z^{\nu+1+\kappa}$ in $f_1^\vee(z, \nu + \kappa)$ is equal to $((2\nu + 2\kappa + 1)!)^4 / ((\nu + \kappa + 1)!)^8$, for $\kappa \in \{0, 1\}$.

In view of (54) and (53), $zQ(\delta + 1, -\nu - 2)(\delta + \nu + 1)^4 z^{\nu+1} =$

$$Q(\nu + 2, -\nu - 2)(2\nu + 2)^4 z^{\nu+2} = -(\nu + 2)^3(2\nu + 3)^4(2\nu + 2)^4 z^{\nu+2}.$$

In view of (44), $(\nu + 1)!)^8 \gamma_1 / (2\nu + 1)!)^4 =$

$$-(\nu + 2)^{11}(2\nu + 3)^4(2\nu + 2)^4 / (\nu + 2)^8 - (-(\nu + 2)^3(2\nu + 3)^4(2\nu + 2)^4) = 0.$$

Therefore (47) holds for $i = 2, k = 1, \nu \in \mathbb{N}_0$.

We denote by $\mathfrak{C}$ the set of all the functions $f(z)$, which are regular outside the closed unit circle and which tend to zero, when z tends to ∞ . Each such function is the sum of convergent for $|z| > 1$ Laurent series $\sum_{k=1}^{\infty} a_k z^{-k}$.

We denote by $\mathfrak{C}^{(n)}$ with $n \in \mathbb{N}_0$ the set of all the functions $f(z)$, can be represented in the form

$$f(z) = \sum_{k=0}^n f_k(z) \frac{(\log(z))^k}{k!} \quad (55)$$

with $f_k(z) \in \mathfrak{C}$ for $k = 0, \dots, n$. Clearly,

$$(\delta f)(z) = (\delta f_n)(z) \frac{(\log(z))^n}{n!} + \sum_{k=0}^{n-1} ((\delta f_k)(z) + f_{k+1}(z)) \frac{(\log(z))^k}{k!} \quad (56)$$

Clearly, if $\{m, n, \nu\} \subset \mathbb{N}_0$ and $m < n$, then $\mathfrak{C}^{(m)}$ and $z^{-\nu} \mathfrak{C}^{(n)}$ are δ -invariant $\mathbb{C}$ -linear subspaces of $\mathbb{C}$ -linear space $\mathfrak{C}^{(n)}$. Let $\mathbb{C}[[x]]$ denotes the ring of all formal power series over $\mathbb{C}$ with variable x . Let $n \in \mathbb{N}_0$; we consider

elements of $\oplus_{k=0}^n \mathbb{C}[[z^{-1}]] = \mathbb{C}[[z^{-1}]] \oplus, \dots, \oplus \mathbb{C}[[z^{-1}]]$ as columns

$$\Theta(z) = \sum_{k=0}^n T_k(z) \vec{e}_{n+1, k+1} \quad (57)$$

consisting of $n + 1$ elements $T_0(z), \dots, T_n(z)$, which lie in $\mathbb{C}[[z^{-1}]]$. We put

$$\delta T(z) = \sum_{k=0}^{\infty} -k a_k z^{-k} \text{ for } T(z) = \sum_{k=0}^{\infty} a_k z^{-k} \in \mathbb{C}[[z^{-1}]] \quad (58)$$

If $\Theta(z) \in \oplus_{k=0}^n \mathbb{C}[[z^{-1}]]$ represented in the form (57), we put

$$(\delta\Theta)(z) = \sum_{k=0}^n T_k^{(1)}(z) \vec{e}_{n+1, k+1} \quad (59)$$

where $T_n^{(1)}(z) = (\delta T_n)(z)$ and $T_k^{(1)}(z) = (\delta T_k)(z) + T_{k+1}(z)$, if $0 \leq k < n$.

Lemma 1.1. *If $n \in \mathbb{N}_0$ then $\mathfrak{C}^{(n)}$ can be considered as (or maybe, more precisely, embedded in) a δ -invariant $\mathbb{C}$ -linear subspace of $\oplus_{k=0}^n \mathbb{C}[[z^{-1}]]$.*

Proof. We need to prove only that if

$$f(z) = \sum_{k=0}^n f_k(z) \frac{(\log(z))^k}{k!} = 0, \text{ where } f_k(z) \in \mathfrak{C} \quad (60)$$

for $k = 0, \dots, n$, then $f_k(z) = 0$ for $k = 0, \dots, n$. The equality (60) can be prolonged by means of analytic continuation to the equality

$$H((r, \varphi)) = \sum_{k=0}^n f_k(\exp(\text{Log}(Z))) \frac{(\text{Log}(Z))^k}{k!} = \quad (61)$$

$$\sum_{k=0}^n f_k(\exp(\ln(r) + i\varphi)) \frac{(\ln(r) + i\varphi)^k}{k!} = \sum_{k=0}^n f_k(r \exp(i\varphi)) \frac{(\ln(r) + i\varphi)^k}{k!} = 0,$$

where $Z = (r, \phi)$ lies on Riemannian surface of $\text{Log}(Z)$, $r > 1$, $\varphi \in \mathbb{R}$ and H is uniquely defined by h .

$$\text{Then } H((r, \varphi + 2\kappa\pi)) = \sum_{k=0}^n f_k(r \exp(i\varphi)) \frac{(\ln(r) + i(\varphi + 2\kappa\pi))^k}{k!} = 0$$

for $\kappa = 0, \dots, n$. Making use of Vandermonde determinant we conclude that $f_k(r \exp(i\varphi)) = 0$ for $k = 0, \dots, n$. $\square$

In view of (15), (44), (13), (21), (20), (23), $\mathfrak{W}_2(f_k, z, \nu)$ belongs to $z^{-\nu} \mathfrak{C}^{(3)}$ for $\nu \in \mathbb{N}_0$. If $\{\nu \in \mathbb{N}_0$, then $(\delta - \nu - 2)^{2+l}$ as operator on linear over $\mathbb{C}$ space $z^{-\nu} \mathfrak{C}^{(0)}$ is injection. Therefore, in view of (46), $\mathfrak{W}_2(f_k^\vee, z, \nu) = 0$ for $k \in \{2, 3, 5, 7\}$, $\nu \in \mathbb{N}_0$. Let $\nu \in \mathbb{N}$. In view of (7), if $\kappa = \kappa^2$, then

$$(R((\nu + 1 - \kappa, \nu - \kappa)))^4 = ((\nu - \kappa)!)^8 / ((2\nu + 2 - 2\kappa)!)^4. \quad (62)$$

In view of (45), $\mathfrak{W}_1(f_2, z, \nu)$ belongs to the null-space of the operator $(\delta + \nu)^4$ as linear operator on $z^{-\nu} \mathfrak{C}^{(0)}$, and, if $\nu \in \mathbb{N}$, then $\mathfrak{W}_1(f_2, z, \nu) = \gamma_2 z^{-\nu}$.

Remark 2. The null-space of $(\delta + \nu)^4$ as linear operator on $\mathfrak{C}[\log(z)]$ coincides with $z^{-\nu}\mathfrak{C}^{(3)}$. But in our case the space $z^{-\nu}\mathfrak{C}^{(0)} = z^{-\nu}\mathfrak{C}$ is an invariant subspace of the space $z^{-\nu}\mathfrak{C}[\log(z)]$ relatively this operator. Therefore $\mathfrak{W}_2(f_k, z, \nu) \in z^{-\nu}\mathfrak{C}^{(0)}$ for $\nu \in \mathbb{N}_0$.

In view of (54), (14), (7), and (53),

$$\begin{aligned}\gamma_2 &= -\nu^{11}((\nu-1)!)^8/((2\nu)!)^4 + Q(-\nu, \nu)(-2\nu-2)^4(\nu!)^8/((2\nu+2)!)^4 = \\ &= -\nu^{11}((\nu-1)!)^8/((2\nu)!)^4 + \nu^3(2\nu+1)^4(2\nu+2)^4(\nu!)^8/((2\nu+2)!)^4 = 0.\end{aligned}$$

Hence, $\mathfrak{W}_1(f_2^\vee, z, \nu) = 0$ for $\nu \in \mathbb{N}$. Therefore, in view of (45), (13) and (52),

$$\mathfrak{W}_1(f_3, z, \nu) = \mathfrak{W}_1((\log(z))f_2, z, \nu) + \mathfrak{W}_1(f_4, z, \nu) = \mathfrak{W}_1(f_4, z, \nu) - \quad (63)$$

$$\begin{aligned}\left(\frac{\partial}{\partial w}T^*(z, w, \nu)\right)\Big|_{w=\delta} f_2(z, \nu) &= \nu^{11}f_4^\vee(z, \nu-1) - T^*(z, \delta, \nu)f_4^\vee(z, \nu) - \\ &\quad \left(\frac{\partial}{\partial w}T^*(z, w, \nu)\right)\Big|_{w=\delta} f_2(z, \nu) \in \mathbb{C}[[z^{-1}]].\end{aligned}$$

We deduce from (28) by means of standard calculations the following equalities

$$\frac{Q(-\nu, \nu)}{(1+2\nu)^4\nu^3} = 1, \quad \left(\frac{d}{dw}\frac{Q(w, \nu)}{4(1+2\nu)^3\nu^2}\right)\Big|_{w=-\nu} = 2+3\nu, \quad (64)$$

$$\left(\left(\frac{d}{dw}\right)^2\frac{Q(w, \nu)}{4\nu^3(1+2\nu)^4}\right)\Big|_{w=-\nu} = \frac{3}{(1+2\nu)^2} + \frac{18}{\nu^2} - \frac{16}{\nu(1+2\nu)}, \quad (65)$$

$$\left(\frac{d}{dw}\right)^3\frac{Q(w, \nu)}{48\nu+24} = (119\nu^3 + 228\nu^2 + 144\nu + 30). \quad (66)$$

In view (53), (64) and (48), if $\kappa^2 = \kappa$, then

$$\left(\frac{d}{dw}\frac{Q^*(w-\kappa, \nu)}{4\nu^3(1+2\nu)^4(2+2\nu)^4}\right)\Big|_{w=-\nu-1+\kappa} = -H_1^\wedge(\nu). \quad (67)$$

In view of (45), (63), $\mathfrak{W}_1(f_3, z, \nu)$, for $\nu \in \mathbb{N}$, belongs to the null-space of the operator $(\delta + \nu)^4$ on the δ -invariant linear subspace $z^{-\nu}\mathfrak{C}^{(0)}$ of $z^{-\nu}\mathfrak{C}^{(1)}$; hence $\mathfrak{W}_1(f_3, z, \nu) = \gamma_3 z^{-\nu}$ with $\gamma_3 \in \mathbb{C}$. In view of (16), (48), (50), (62), (65),

$$\gamma_3 z^{-\nu} = -\nu^{11}\frac{\partial}{\partial t}(R(t, \nu-1))^4\Big|_{t=\nu} z^{-\nu} - \frac{\partial(-(R(t, \nu))^4)}{\partial t}\Big|_{t=\nu+1} \times \quad (68)$$

$$zQ^*(\delta, \nu)z^{-\nu-1} - (R(\nu+1, \nu))^4 z\frac{\partial(Q^*(w, \nu))}{\partial w}\Big|_{w=\delta} z^{-\nu-1} =$$

$$4\nu^3(\nu!)^8 z^{-\nu} (-S_1(\nu, \nu-1) + S_1(\nu+1, \nu) + H_1^\wedge(\nu)) / (2\nu!)^4 = 0.$$

Therefore $\mathfrak{W}_1(f_3^\vee, z, \nu) = 0$ for $\nu \in \mathbb{N}$. So, (47) holds for $k \in \{2, 3\}$, $\nu \in \mathbb{N}$. In view of (43), (52), (18), (21), (63),

$$\mathfrak{W}_1(f_5, z, \nu) = \sum_{k=0}^2 \mathfrak{W}_1((\log(z))^k f_{6-2k}, z, \nu) / k! = \quad (69)$$

$$h_{5,0}(z) + \log(z)h_{5,1}(z), \text{ with}$$

$$h_{5,1}(z) = \mathfrak{W}_1(f_4, z, \nu) - \frac{\partial T^*(z, w, \nu)}{\partial w} \Big|_{w=\delta} f_2(z, \nu) = \mathfrak{W}_1(f_3, z, \nu) = 0, \quad (70)$$

$$h_{5,0}(z) = \mathfrak{W}_1(f_6, z, \nu) - \sum_{k=1}^2 \frac{1}{2} \left(\left(\frac{\partial}{\partial w} \right)^k \frac{T^*(z, w, \nu)}{k!} \right) \Big|_{w=\delta} f_{6-2k}(z, \nu).$$

In view of (69) and (70), $\mathfrak{W}_1(f_5, z, \nu)z^{-\nu} \in \mathfrak{C}^{(0)}$. Hence, in view of (45), $\mathfrak{W}_1(f_5, z, \nu)$ belongs to the null-space of the operator $(\delta + \nu)^4$ on the δ -invariant linear subspace $z^{-\nu}\mathfrak{C}^{(0)}$ of the space $z^{-\nu}\mathfrak{C}^{(2)}$ and, if $\nu \in \mathbb{N}$ then $\mathfrak{W}_1(f_5, z, \nu) = \gamma_5 z^{-\nu}$ with $\gamma_5 \in \mathbb{C}$. In view of (50), (62), (48), (19), (53), (43), coefficient at $z^{-\nu}$ in $\mathfrak{W}_1(f_6, z, \nu)$ is equal to

$$(\nu^3(\nu!)^8(8((S_1(\nu, \nu-1))^2 - (S_1(\nu+1, \nu))^2) + 2H_2^\vee(\nu)))/((2\nu!)^4) = \quad (71)$$

$$\begin{aligned} & (\nu^3(\nu!)^8(8H_1^\wedge(\nu)(2S_1(\nu+1, \nu) + H_1^\wedge(\nu)) + 2H_2^\wedge(\nu)))/((2\nu!)^4) = \\ & \nu^3(\nu!)^8(16H_1^\wedge(\nu)S_1(\nu+1, \nu) + 8(H_1^\wedge(\nu))^2 + 2H_2^\wedge(\nu))/((2\nu!)^4. \end{aligned}$$

In view of (67), (50), (62) and (48),

$$\text{coefficient at } z^{-\nu} \text{ in } -z \left(\frac{\partial}{\partial w} Q^*(w, \nu) \right) \Big|_{w=\delta} f_4(z, \nu) \text{ is equal to}$$

$$-4\nu^3(1+2\nu)^4(2\nu+2)^4 H_1^\vee(\nu)(4(\nu!)^8 S_1(\nu+1, \nu))/((2\nu+2)!)^4. \quad (72)$$

In view of (53), (65), (67), if $\kappa^2 = \kappa$, then

$$\frac{1}{4\nu^3(1+2\nu)^4(2+2\nu)^4} \left(\frac{d}{dw} \right)^2 Q^*(w - \kappa, \nu) \Big|_{w=-\nu-1+\kappa} = \quad (73)$$

$$\begin{aligned} & (3/(1+2\nu)^2 + 18/\nu^2 - 16/\nu(1+2\nu)) - 8(2+3\nu)/\nu(1+2\nu)(2+2\nu) + \\ & 3/(2+2\nu)^2 = 3/(2+2\nu)^2 + 3/(1+2\nu)^2 + 18/\nu^2 - 16/\nu(1+2\nu) - \\ & 16/\nu(2+2\nu) + 8/(2\nu+1)(2+2\nu) = (4(H_1^\wedge(\nu))^2 + H_2^\wedge(\nu)). \end{aligned}$$

In view of (62), (73), coefficient at $z^{-\nu}$ in $\frac{1}{2} \left(\left(\frac{\partial}{\partial w} \right)^2 Q^*(w, \nu) \right) \Big|_{w=\delta} f_2(z, \nu)$ is equal to $2\nu^3(\nu!)^8((2\nu!)^{-4}(4(H_1^\wedge(\nu))^2 + H_2^\wedge(\nu)))$; in view of (71), (72), (69), we have $\gamma_5 = 0$. So, (47) holds for $k \in \{1, 2, 3, 5\}$, $\nu \in \mathbb{N}$. Let $k = 7$, $\nu \in \mathbb{N}$. Since (47) holds for $k = 2$ it follows from (43), (52), (18), (23), that

$$\mathfrak{W}_1(f_7, z, \nu) = \sum_{k=0}^3 \left(((\log^k(z))/k!) f_{8-2k}, z, \nu \right) = \quad (74)$$

$$h_{7,0}(z) + (\log(z))h_{7,1}(z) + ((\log^2(z))/2)h_{7,2}(z),$$

where, in view of (63), (70),

$$h_{7,3-i}(z) = - \sum_{k=1}^i \left(\left(\frac{\partial}{\partial w} \right)^k \frac{T^*(z, w, \nu)}{k!} \right) \Big|_{w=\delta} f_{2+2i-2k}(z, \nu) + \quad (75)$$

$$\mathfrak{W}_1(f_{2+2i}, z, \nu) = h_{5,2-i} = 0 \text{ for } i = 1, 2.$$

$$h_{7,0}(z) = \mathfrak{W}_1(f_8, z, \nu) - \sum_{k=1}^3 \left(\left(\frac{\partial}{\partial w} \right)^k \frac{T^*(z, w, \nu)}{k!} \right) \Big|_{w=\delta} f_{8-2k}(z, \nu). \quad (76)$$

In view of (74), (75), $\mathfrak{W}_1(f_7, z, \nu) \in \mathfrak{C}^{(0)}$. In view of (45), $\mathfrak{W}_1(f_7, z, \nu)$ belongs to the null-space of the operator $(\delta + \nu)^4$ on the δ -invariant linear subspace $z^{-\nu}\mathfrak{C}^{(0)}$ of the space $z^{-\nu}\mathfrak{C}^{(3)}$. Consequently, $\mathfrak{W}_1(f_7, z, \nu) = \gamma_7 z^{-\nu}$ with $\gamma_7 \in \mathbb{C}$.

We conclude from (50), (62), (48), (22), (53) and (43) that coefficient at $z^{-\nu}$ in $-3((2\nu!)^4)\mathfrak{W}_1(f_8, z, \nu)/(2\nu^3(\nu!)^8)$ is equal to

$$16((S_1(\nu, \nu - 1))^3 - (S_1(\nu + 1, \nu))^3) + \quad (77)$$

$$\begin{aligned} & 12(S_1(\nu, \nu - 1)S_2(\nu, \nu - 1) - S_1(\nu + 1, \nu)S_2(\nu + 1, \nu)) + H_3^\wedge(\nu) = \\ & 16(3H_1^\wedge(\nu)S_1^2(\nu + 1, \nu) + 3(H_1^\wedge(\nu))^2S_1(\nu + 1, \nu) + (H_1^\wedge(\nu))^3) + \\ & 12(H_1^\wedge(\nu)S_2(\nu + 1, \nu) + H_1^\wedge(\nu)H_2^\wedge(\nu) + S_1(\nu + 1, \nu)H_2^\wedge(\nu)) + H_3^\wedge(\nu), \end{aligned}$$

In view of (54), (67) and (50), coefficient at $z^{-\nu}$ in

$$\begin{aligned} & -3((2\nu!)^4) \left(\frac{\partial}{\partial w} T^*(z, w, \nu) \right) \Big|_{w=\delta} \frac{f_6(z, \nu)}{(2\nu^3(\nu!)^8)} \text{ is equal to} \\ & 12H_1^\wedge(\nu)(4S_1^2(\nu + 1, \nu) + S_2(\nu + 1, \nu)). \end{aligned} \quad (78)$$

Hence, $-3(2\nu!)^4 \left(\mathfrak{W}_1(f_8, z, \nu) - \left(\frac{\partial}{\partial w} T^*(z, w, \nu) \right) \Big|_{w=\delta} f_6(z, \nu) \right) / (2\nu^3(\nu!)^8)$

has coefficient at $z^{-\nu}$ equal to result of subtraction (78) from (77), i.e.

$$16(3(H_1^\wedge(\nu))^2 S_1(\nu+1, \nu) + (H_1^\wedge(\nu))^3) + \quad (79)$$

$$12(H_1^\wedge(\nu)H_2^\wedge(\nu) + S_1(\nu+1, \nu)H_2^\wedge(\nu)) + H_3^\wedge(\nu),$$

In view of (54), (73), (50), (62), (48), coefficient at $z^{-\nu}$ in

$$-3((2\nu)!)^4 \left(\left(\frac{\partial}{\partial w} \right)^2 T^*(z, w, \nu) \right) \Big|_{w=\delta} \frac{f_4(z, \nu)}{(4\nu^3(\nu!)^8)} \text{ is equal to}$$

$$(48(H_1^\wedge(\nu))^2 + 12H_2^\wedge(\nu))S_1(\nu+1, \nu). \quad (80)$$

In view of (79), (80), (73), coefficient at $z^{-\nu}$ in

$$\mathfrak{W}_1(f_8, z, \nu) - \sum_{k=1}^2 \left(\left(\frac{\partial}{\partial w} \right)^k \frac{T^*(z, w, \nu)}{k!} \right) \Big|_{w=\delta} f_{8-2k}(z, \nu)$$

is equal to

$$-\nu^3(\nu!)^8(32(H_1^\wedge(\nu))^3 + 24H_1^\wedge(\nu)H_2^\wedge(\nu) + 2H_3^\wedge(\nu))/(3(2\nu)!)^4. \quad (81)$$

In view of (66) and (53),

$$\left(\left(\frac{d}{dw} \right)^3 \frac{8Q^*(w, \nu)}{6(2\nu+1)(2\nu+2)} \right) \Big|_{w=-\nu-1} = \quad (82)$$

$$\begin{aligned} & \left(\left(\frac{d}{dw} \right)^3 \frac{8Q(w, \nu)(w-\nu-2)^4}{6(2\nu+1)(2\nu+2)} \right) \Big|_{w=-\nu} = (119\nu^3 + 228\nu^2 + 144\nu + 30) \times \\ & 32(2\nu+2)^3 - 64\nu(2\nu+1)(2\nu+2)^2(18+56\nu+43\nu^2) + 192\nu^2(2\nu+1)^2 \times \\ & (2\nu+2)(2+3\nu) - 32\nu^3(2\nu+1)^3 = 4(\theta+1)^3 \times \\ & (119(\theta-1)^3 + 456(\theta-1)^2 + 576(\theta-1) + 240) - 8(\theta-1)\theta(\theta+1)^2 \times \\ & (43(\theta-1)^2 + 112(\theta-1) + 72) + 24(\theta-1)^2\theta^2(\theta+1)(3\theta+1) - 4(\theta-1)^3\theta^3 = \\ & 476(\theta^2-1)^3 + 1824(\theta+1)(\theta^2-1)^2 + 2304(\theta+1)^2(\theta^2-1) + 960(\theta+1)^3 - \\ & 344(\theta^2-\theta)(\theta^2-1)^2 - 896\theta(\theta^2-1)^2 - 576(\theta^2-\theta)(\theta+1)^2 + \\ & 24(\theta^4-\theta^2)(3\theta^2-2\theta-1) - 4(\theta-1)^3\theta^3 = (476-344+72-4)\theta^6 + \\ & (1824+344-896-48+12)\theta^5 + (-1428+1824+2304+688-96-12-576)\theta^4 + \end{aligned}$$

$$\begin{aligned}
& (-3648 + 4608 + 960 - 688 + 1792 - 576 + 48 + 4)\theta^3 + \\
& (1428 - 3648 + 2880 - 344 + 576 + 24)\theta^2 + \\
& (1824 - 4608 + 2880 + 344 - 896 + 576)\theta + (-476 + 1824 - 2304 + 960) = \\
& 200\theta^6 + 1236\theta^5 + 2704\theta^4 + 2500\theta^3 + 1374\theta^2 + 120\theta + 4 = \\
& -2(16(H_1^\vee(\nu))^3 + 12H_1^\vee(\nu)H_2^\vee(\nu) + H_3^\vee(\nu))/3.
\end{aligned}$$

Hence, coefficient at $z^{-\nu}$ in

$$\left(\left(\frac{\partial}{\partial w} \right)^3 \frac{T^*(z, w, \nu)}{3!} \right) \Big|_{w=\delta} f_2(z, \nu)$$

is equal to

$$\begin{aligned}
& -((2\nu + 1)(2\nu + 2)/8)((\nu!)^8/((2\nu + 2)!)^4)) \times \\
& 2(16(H_1^\vee(\nu))^3 + 12H_1^\vee(\nu)H_2^\vee(\nu) + H_3^\vee(\nu))/3 = \\
& -((\nu)^3(\nu!)^8)/(3(2\nu)!)^4)(32(H_1^\wedge(\nu))^3 + 24H_1^\wedge(\nu)H_2^\wedge(\nu) + 2H_3^\wedge(\nu)).
\end{aligned} \tag{83}$$

In view of (76), (81), (83), $\gamma_7 = 0$.

So, (47) holds for $i \in \{1, 2\}$, $k \in \{1, 2, 3, 5, 7\}$, $\nu \in (2 - i, +\infty) \cap \mathbb{Z}_0$. If $i = 1$, $2\nu \in (-\infty, -1 - i] \cap \mathbb{Z}$, then $\nu_1 = -\nu - 2 \in [i - 1, +\infty) \cap \mathbb{Z}$, and $\mathfrak{W}_i(f_k^\vee, z, \nu) = \mathfrak{W}_{3-i}(f_k^\vee, z, \nu_1) = 0$. So, (47) holds for $k \in \{1, 2, 3, 5, 7\}$. $\square$

3. 2 Solution of an Auxiliary Difference Equation

Let $j \in [1, 9] \cap \mathbb{N}$, $\bar{y}_j^*(\nu) = \sum_{k=1}^8 y_{j,k}^*(\nu) \bar{e}_{8,k}$ is the solution of difference equation

$$\bar{y}_{j+1}^*(\nu) = \bar{y}_j^*(\nu)(E_8 + N_8) + p_{8-j}^*(\nu) \bar{R}(\nu) \tag{84}$$

with $j \in [1, 8] \cap \mathbb{N}$, with $\bar{R}(\nu)$ defined in (29) and with initial values

$$\bar{y}_1^*(\nu) = \bar{v}_1^*(z; \nu) = \sum_{\eta=0}^7 q_\eta^*(\nu) \bar{e}_{8,\eta+1}. \tag{85}$$

Lemma 2.1 *Let $j \in [1, 8] \cap \mathbb{Z}$, $k \in [0, 9 - j] \cap \mathbb{Z}$. Then*

$$\bar{y}_{j+k}^*(\nu) = \bar{y}_j^*(\nu)(E_8 + N_8)^k + \bar{R}(\nu) \sum_{\kappa=1}^k p_{9-j-\kappa}^*(\nu)(E_8 + N_8)^{k-\kappa}. \tag{86}$$

Proof. For $k = 0$ the equality (86) is evident, for $k = 1$ it follows from (84). Let $1 \leq k \leq 8 - j$ and (86) holds for this k . Then $j + k \in [1, 8] \cap \mathbb{Z}$, and we apply (84), for $j + k$ instead j . Then we have

$$\begin{aligned} \bar{y}_{j+k+1}^*(\nu) &= \bar{y}_{j+k}^*(\nu)(E_8 + N_8) + p_{8-j-k}^*(\nu)\bar{R}(\nu) = \bar{y}_j^*(\nu)(E_8 + N_8)^{k+1} + \\ &\bar{R}(\nu) \sum_{\kappa=1}^k p_{9-j-\kappa}^*(\nu)(E_8 + N_8)^{k+1-\kappa} + p_{8-j-k}^*(\nu)\bar{R}(\nu)(E_8 + N_8)^0 = \\ &\bar{y}_j^*(\nu)(E_8 + N_8)^{k+1} + \bar{R}(\nu) \left(\sum_{\kappa=1}^{k+1} p_{\alpha, 2, 9-j-\kappa}^*(\nu)(E_8 + N_8)^{k+1-\kappa}, \right). \end{aligned}$$

Hence (86) holds for all $k \in [1, 9 - i] \cap \mathbb{Z}$. $\square$ If we take in (86) $j = 1$, $i = j + k$, and apply (85) then we see that $\bar{y}_i^*(\nu)$ is equal to

$$\left(\sum_{\sigma=1}^8 q_{\sigma-1}^*(\nu) \bar{e}_{8,\sigma}(E_8 + N_8)^{i-1} \right) + \bar{R}(\nu) \sum_{\kappa=1}^{i-1} p_{8-\kappa}^*(\nu)(E_8 + N_8)^{i-1-\kappa}.$$

Therefore, in view of (31),

$$y_{i,k}^*(\nu) = v_{i,k}^*(\nu) \text{ for } i \in [1, 8] \cap \mathbb{Z}, k \in [1, 8] \cap \mathbb{Z}. \quad (87)$$

In view of (85), $y_{1,j}^*(\nu) = q_{j-1}^*(\nu)$ for $j \in [1, 8] \cap \mathbb{Z}$. In view of (84), (29),

$$y_{i+1,j}^*(\nu) = y_{i,j}^*(\nu) + y_{i,j-1}^*(\nu) + p_{8-i}^*(\nu)r_k^*(\nu) \text{ if } \{i, j\} \subset [1, 8] \cap \mathbb{Z}. \quad (88)$$

In view of (27), (29),

$$\begin{aligned} p_0^*(\nu) &= (\tau - 1)^7 \tau^4 = \tau^{11} - 7\tau^{10} + 21\tau^9 - 35\tau^8 + 35\tau^7 - 21\tau^6 + \\ &7\tau^5 - \tau^4, p_1^*(\nu) = -4(2\tau - 1)(\tau - 1)^6 \tau^3 = -8\tau^{10} + 52\tau^9 - 144\tau^8 + \\ &220\tau^7 - 200\tau^6 + 108\tau^5 - 32\tau^4 + 4\tau^3, p_2^*(\nu) = 2\tau^2(\tau - 1)^5(2\tau - 1) \times \\ &(8\tau - 3) = 32\tau^9 - 188\tau^8 + 466\tau^7 - 630\tau^6 + 500\tau^5 - 232\tau^4 + 58\tau^3 - \\ &6\tau^2, p_3^*(\nu) = -4(\tau - 1)^4(2\tau - 1)\tau(11\tau^2 - 7\tau + 1) = -88\tau^8 + 452\tau^7 - \\ &964\tau^6 + 1100\tau^5 - 720\tau^4 + 268\tau^3 - 52\tau^2 + 4\tau, p_4^*(\nu) = (2\tau - 1)(\tau - 1)^3 \times \\ &(96\tau^3 - 78\tau^2 + 18\tau - 1) = 192\tau^7 - 828\tau^6 + 1446\tau^5 - 1310\tau^4 + 655\tau^3 - \\ &177\tau^2 + 23\tau - 1, p_5^*(\nu) = -4(\tau - 1)^2(2\tau - 1)(3\tau - 1)(15\tau^2 - 9\tau + 1) = \\ &-4(90\tau^6 - 309\tau^5 + 414\tau^4 - 275\tau^3 + 95\tau^2 - 16\tau + 1), p_6^*(\nu) = 2(\tau - 1) \times \end{aligned} \quad (89)$$

$$(2\tau - 1)(152\tau^3 - 155\tau^2 + 50\tau - 5) = 2(304\tau^5 - 766\tau^4 + 717\tau^3 - 315\tau^2) + \\ 130\tau - 10, p_7^*(\nu) = -4(2\tau - 1)(119\tau^3 - 129\tau^2 + 45\tau - 5) = -952\tau^4 + \\ 4(377\tau^3 - 219\tau^2 + 55\tau - 5).$$

In view of (28), $Q^{(h)}(1, w + 1, \nu) = 704\nu^6 + 3134\nu^5 + 5846\nu^4 + 5827\nu^3 +$
 $3264\nu^2 + 972\nu + 120 + 4(455\nu^5 + 1659\nu^4 + 2415\nu^3 + 1750\nu^2 + 630\nu + 90)w +$
 $2(800\nu^4 + 2237\nu^3 + 2324\nu^2 + 1062\nu + 180)w^2 + 4(119\nu^3 + 228\nu^2 + 144\nu + 30)w^3 =$
 $4(119\tau^3 - 129\tau^2 + 45\tau - 5)w^3 + 2(800\tau^4 - 963\tau^3 + 413\tau^2 - 75\tau + 5)w^2 +$
 $4(455\tau^5 - 616\tau^4 + 329\tau^3 - 91\tau^2 + 14\tau - 1)w + 704\tau^6 - 1090\tau^5 + 736\tau^4 -$
 $297\tau^3 + 79\tau^2 - 13\tau + 1, Q(w, \nu) = 1408\tau^7 - 2884\tau^6 + 2562\tau^5 - 1330\tau^4 +$
 $455\tau^3 - 105\tau^2 + 15\tau - 1 + 4(910\tau^6 - 1687\tau^5 + 1274\tau^4 - 511\tau^3 + 119\tau^2 - 16\tau + 1)w +$
 $2(1600\tau^5 - 2726\tau^4 + 1789\tau^3 - 563\tau^2 + 85\tau - 5)w^2 + 952\tau^4 w^3 +$
 $4(-377\tau^3 + 219\tau^2 - 55\tau + 5)w^3 = \sum_{k=0}^3 q_k(\nu)w^k.$ In view of (85), (89),

$$y_{1,1}^*(\nu) = q_0^*(\nu) = \tau^4 q_0(\nu) = 1408\tau^{11} - 2884\tau^{10} + 2562\tau^9 - 1330\tau^8 + \quad (90)$$

$$455\tau^7 - 105\tau^6 + 15\tau^5 - \tau^4, y_{1,2}^*(\nu) = q_1^*(\nu) = -4\tau^3 q_0(\nu) + \tau^4 q_1(\nu) =$$

 $4(-498\tau^{10} + 1197\tau^9 - 1288\tau^8 + 819\tau^7 - 336\tau^6 + 89\tau^5 - 14\tau^4 + \tau^3),$
 $y_{1,3}^*(\nu) = 6\tau^2 q_0 - 4\tau^3 q_1(\nu) + \tau^4 q_2(\nu) = -6\tau^2 + 74\tau^3 - 384\tau^4 +$
 $2(498\tau^5 - 465\tau^6 - 717\tau^7 + 2118\tau^8 + 2118\tau^8 - 1456\tau^9), y_{1,4}^*(\nu) =$
 $-4\tau q_0(\nu) + 6\tau^2 q_1(\nu) - 4\tau^3 q_2(\nu) + \tau^4 q_3(\nu) = 4\tau - 36\tau^2 + 76\tau^3 +$
 $4(94\tau^4 - 665\tau^5 + 1723\tau^6 - 2163\tau^7 + 1090\tau^8), y_{1,5}^*(\nu) = q_0(\nu) - 4 \times$
 $\tau q_1(\nu) + 6\tau^2 q_2(\nu) - 4\tau^3 q_3(\nu) + \tau^4 q_4(\nu) = -1 - \tau + 91\tau^2 - 509\tau^3 +$
 $970\tau^4 + 142\tau^5 - 2572\tau^6 + 2240\tau^7, y_{1,6}^*(\nu) = q_1(\nu) - 4\tau q_2(\nu) + 6\tau^2 q_3(\nu) =$
 $4(1 - 6\tau - 21\tau^2 + 285\tau^3 - 990\tau^4 + 1503\tau^5 - 862\tau^6), y_{1,7}^*(\nu) = q_2(\nu) - 4\tau q_3(\nu)$
 $= -10 + 90\tau - 246\tau^2 + 74\tau^3 + 580\tau^4 - 608\tau^5, y_{1,8}^*(\nu) = q_3(\nu) = -p_7^*(\nu).$

Consecutive calculations according to (88),(29) give the following results:

$$y_{2,1}^*(\nu) = -952\tau^{12} + 2916\tau^{11} - 3760\tau^{10} + 2782\tau^9 - 1350\tau^8 + 455\tau^7 - \quad (91)$$

$$\begin{aligned}
& 105\tau^6 + 15\tau^5 - \tau^4, y_{2,2}^*(\nu) = 1408\tau^{11} - 4876\tau^{10} + 7350\tau^9 - 6482\tau^8 + \\
& 3731\tau^7 - 1449\tau^6 + 371\tau^5 - 57\tau^4 + 4\tau^3, y_{2,3}^*(\nu) = -6\tau^2 + 78\tau^3 - 440\tau^4 + 8 \\
& 1352\tau^5 - 2194\tau^6 + 962\tau^7 + 2588\tau^8 - 4156\tau^9 + 1816\tau^{10}, y_{2,4}^*(\nu) = -2912\tau^9 + \\
& 8596\tau^8 - 10086\tau^7 + 5962\tau^6 - 1664\tau^5 - 8\tau^4 + 150\tau^3 - 42\tau^2 + 4\tau, y_{2,5}^*(\nu) = \\
& -1352\tau^8 + 2636\tau^7 - 936\tau^6 - 1198\tau^5 + 1226\tau^4 - 433\tau^3 + 55\tau^2 + 3\tau - 1, \\
& y_{2,6}^*(\nu) = 2240\tau^7 - 6020\tau^6 + 6154\tau^5 - 2990\tau^4 + 631\tau^3 + 7\tau^2 - 25\tau + 3, \\
& y_{2,7}^*(\nu) = 360\tau^6 - 628\tau^5 + 124\tau^4 + 334\tau^3 - 250\tau^2 + 66\tau - 6, y_{2,8}^*(\nu) = -p_6^*(\nu), \\
& y_{3,1}^*(\nu) = 608\tau^{13} - 2484\tau^{12} + 4350\tau^{11} - 4390\tau^{10} + 2912\tau^9 - 1360\tau^8 + 455\tau^7 - \\
& 105\tau^6 + 15\tau^5 - \tau^4, y_{3,2}^*(\nu) = -952\tau^{12} + 4324\tau^{11} - 8636\tau^{10} + 10132\tau^9 - \\
& 7832\tau^8 + 4186\tau^7 - 1554\tau^6 + 386\tau^5 - 58\tau^4 + 4\tau^3, y_{3,3}^*(\nu) = -1024\tau^{11} + \\
& 3068\tau^{10} - 2542\tau^9 - 1374\tau^8 + 4173\tau^7 - 3603\tau^6 + 1723\tau^5 - 497\tau^4 + 82\tau^3 - \\
& 6\tau^2, y_{3,4}^*(\nu) = 1816\tau^{10} - 7068\tau^9 + 11184\tau^8 - 9124\tau^7 + 3768\tau^6 - 312\tau^5 - \\
& 448\tau^4 + 228\tau^3 - 48\tau^2 + 4\tau, y_{3,5}^*(\nu) = 736\tau^9 - 1948\tau^8 + 1154\tau^7 + 1246\tau^6 - \\
& 2082\tau^5 + 1158\tau^4 - 283\tau^3 + 13\tau^2 + 7\tau - 1, y_{3,6}^*(\nu) = -1352\tau^8 + 4876\tau^7 - \\
& 6956\tau^6 + 4956\tau^5 - 1764\tau^4 + 198\tau^3 + 62\tau^2 - 22\tau + 2, y_{3,7}^*(\nu) = 192\tau^7 + \\
& 468\tau^6 - 210\tau^5 - 346\tau^4 + 445\tau^3 - 203\tau^2 + 41\tau - 3, y_{3,8}^*(\nu) = -p_5^*(\nu), \\
& y_{4,1}^*(\nu) = -360\tau^{14} + 1844\tau^{13} - 4140\tau^{12} + 5450\tau^{11} - 4770\tau^{10} + 2976\tau^9 - \\
& 1364\tau^8 + 455\tau^7 - 105\tau^6 + 15\tau^5 - \tau^4, y_{4,2}^*(\nu) = 608\tau^{13} - 3436\tau^{12} + \\
& 8674\tau^{11} - 13026\tau^{10} + 13044\tau^9 - 9192\tau^8 + 4641\tau^7 - 1659\tau^6 + 401\tau^5 - \\
& 59\tau^4 + 4\tau^3, y_{4,3}^*(\nu) = 488\tau^{12} - 1644\tau^{11} + 1056\tau^{10} + 3190\tau^9 - 7686\tau^8 + \\
& 8103\tau^7 - 5141\tau^6 + 2109\tau^5 - 555\tau^4 + 86\tau^3 - 6\tau^2, y_{4,4}^*(\nu) = -1024\tau^{11} + \\
& 4884\tau^{10} - 9610\tau^9 + 9810\tau^8 - 4951\tau^7 + 165\tau^6 + 1411\tau^5 - 945\tau^4 + 310\tau^3 - \\
& 54\tau^2 + 4\tau, y_{4,5}^*(\nu) = -344\tau^{10} + 1084\tau^9 - 700\tau^8 - 1370\tau^7 + 2734\tau^6 - 2010\tau^5 + \\
& 686\tau^4 - 55\tau^3 - 35\tau^2 + 11\tau - 1, y_{4,6}^*(\nu) = 736\tau^9 - 3300\tau^8 + 6030\tau^7 - \\
& 5710\tau^6 + 2874\tau^5 - 606\tau^4 - 85\tau^3 + 75\tau^2 - 15\tau + 1, y_{4,7}^*(\nu) = 88\tau^8 - 260\tau^7 + \\
& 136\tau^6 + 346\tau^5 - 590\tau^4 + 387\tau^3 - 125\tau^2 + 19\tau - 1, y_{4,8}^*(\nu) = -p_4^*(\nu),
\end{aligned}$$

$$\begin{aligned}
y_{5,1}^*(\nu) &= 192\tau^{15} - 1188\tau^{14} + 3290\tau^{13} - 5450\tau^{12} + 6105\tau^{11} - 4947\tau^{10} + \\
&2999\tau^9 - 1365\tau^8 + 455\tau^7 - 105\tau^6 + 15\tau^5 - \tau^4, y_{5,2}^*(\nu) = -360\tau^{14} + \\
&2452\tau^{13} - 7576\tau^{12} + 14124\tau^{11} - 17796\tau^{10} + 16020\tau^9 - 10556\tau^8 + 5096\tau^7 - \\
&1764\tau^6 + 416\tau^5 - 60\tau^4 + 4\tau^3, y_{5,3}^*(\nu) = -160\tau^{13} + 364\tau^{12} + 1246\tau^{11} - \\
&6730\tau^{10} + 13614\tau^9 - 16170\tau^8 + 12652\tau^7 - 6796\tau^6 + 2510\tau^5 - 614\tau^4 + 90\tau^3 - \\
&6\tau^2, y_{5,4}^*(\nu) = 488\tau^{12} - 2668\tau^{11} + 5940\tau^{10} - 6420\tau^9 + 2124\tau^8 + \\
&3152\tau^7 - 4976\tau^6 + 3520\tau^5 - 1500\tau^4 + 396\tau^3 - 60\tau^2 + 4\tau, y_{5,5}^*(\nu) = \\
&128\tau^{11} - 428\tau^{10} + 150\tau^9 + 1250\tau^8 - 2391\tau^7 + 1837\tau^6 - 461\tau^5 - \\
&265\tau^4 + 255\tau^3 - 89\tau^2 + 15\tau - 1, y_{5,6}^*(\nu) = -344\tau^{10} + 1820\tau^9 - \\
&4000\tau^8 + 4660\tau^7 - 2976\tau^6 + 864\tau^5 + 80\tau^4 - 140\tau^3 + 40\tau^2 - 4\tau, \\
&y_{5,7}^*(\nu) = -32\tau^9 + 100\tau^8 - 14\tau^7 - 334\tau^6 + 600\tau^5 - 488\tau^4 + 210\tau^3 - \\
&46\tau^2 + 4\tau, y_{5,8}^*(\nu) = -p_3^*(\nu), y_{6,1}^*(\nu) = -88\tau^{16} + 644\tau^{15} - 2152\tau^{14} + \\
&4390\tau^{13} - 6170\tau^{12} + 6373\tau^{11} - 4999\tau^{10} + 3003\tau^9 - 1365\tau^8 + 455\tau^7 - \\
&105\tau^6 + 15\tau^5 - \tau^4, y_{6,2}^*(\nu) = 192\tau^{15} - 1548\tau^{14} + 5742\tau^{13} - 13026\tau^{12} + \\
&20229\tau^{11} - 22743\tau^{10} + 19019\tau^9 - 11921\tau^8 + 5551\tau^7 - 1869\tau^6 + \\
&431\tau^5 - 61\tau^4 + 4\tau^3, y_{6,3}^*(\nu) = -8\tau^{14} + 484\tau^{13} - 3356\tau^{12} + 10970\tau^{11} - \\
&21646\tau^{10} + 28562\tau^9 - 26518\tau^8 + 17732\tau^7 - 8560\tau^6 + 2926\tau^5 - \\
&674\tau^4 + 94\tau^3 - 6\tau^2, y_{6,4}^*(\nu) = -160\tau^{13} + 852\tau^{12} - 1422\tau^{11} - \\
&790\tau^{10} + 7194\tau^9 - 14046\tau^8 + 15804\tau^7 - 11772\tau^6 + 6030\tau^5 - \\
&2114\tau^4 + 486\tau^3 - 66\tau^2 + 4\tau, y_{6,5}^*(\nu) = -40\tau^{12} + 172\tau^{11} - 272\tau^{10} + \\
&330\tau^9 - 946\tau^8 + 2369\tau^7 - 3451\tau^6 + 3083\tau^5 - 1765\tau^4 + 651\tau^3 - \\
&149\tau^2 + 19\tau - 1, y_{6,6}^*(\nu) = 128\tau^{11} - 772\tau^{10} + 1970\tau^9 - 2750\tau^8 + \\
&2269\tau^7 - 1139\tau^6 + 403\tau^5 - 185\tau^4 + 115\tau^3 - 49\tau^2 + 11\tau - 1, \\
&y_{6,7}^*(\nu) = 8\tau^{10} - 20\tau^9 - 44\tau^8 + 246\tau^7 - 430\tau^6 + 392\tau^5 - 200\tau^4 + \\
&54\tau^3 - 6\tau^2, y_{6,8}^*(\nu) = -p_2^*(\nu), y_{7,1}^*(\nu) = 32\tau^{17} - 276\tau^{16} + 1110\tau^{15} - \\
&2782\tau^{14} + 4890\tau^{13} - 6402\tau^{12} + 6431\tau^{11} - 5005\tau^{10} + 3003\tau^9 -
\end{aligned}$$

$$\begin{aligned}
& 1365\tau^8 + 455\tau^7 - 105\tau^6 + 15\tau^5 - \tau^4, y_{7,2}^*(\nu) = -88\tau^{16} + 836\tau^{15} - \\
& 3700\tau^{14} + 10132\tau^{13} - 19196\tau^{12} + 26602\tau^{11} - 27742\tau^{10} + \\
& 22022\tau^9 - 13286\tau^8 + 6006\tau^7 - 1974\tau^6 + 446\tau^5 - 62\tau^4 + 4\tau^3, \\
& y_{7,3}^*(\nu) = 64\tau^{15} - 804\tau^{14} + 4362\tau^{13} - 13862\tau^{12} + 29199\tau^{11} - \\
& 43461\tau^{10} + 47349\tau^9 - 38415\tau^8 + 23283\tau^7 - 10429\tau^6 + \\
& 3357\tau^5 - 735\tau^4 + 98\tau^3 - 6\tau^2, y_{7,4}^*(\nu) = -8\tau^{14} + 324\tau^{13} - \\
& 2504\tau^{12} + 9548\tau^{11} - 22436\tau^{10} + 35756\tau^9 - 40564\tau^8 + \\
& 33536\tau^7 - 20332\tau^6 + 8956\tau^5 - 2788\tau^4 + 580\tau^3 - \\
& 72\tau^2 + 4\tau, y_{7,5}^*(\nu) = 32\tau^{13} - 316\tau^{12} + 1546\tau^{11} - 4842\tau^{10} + \\
& 10524\tau^9 - 16384\tau^8 + 18521\tau^7 - 15259\tau^6 + 9113\tau^5 - 3879\tau^4 + \\
& 1137\tau^3 - 215\tau^2 + 23\tau - 1, y_{7,6}^*(\nu) = -40\tau^{12} + 300\tau^{11} - \\
& 1044\tau^{10} + 2300\tau^9 - 3696\tau^8 + 4638\tau^7 - 4590\tau^6 + 3486\tau^5 - \\
& 1950\tau^4 + 766\tau^3 - 198\tau^2 + 30\tau - 2, y_{7,7}^*(\nu) = -12\tau^{10} + \\
& 86\tau^9 - 274\tau^8 + 515\tau^7 - 641\tau^6 + 563\tau^5 - 361\tau^4 + 169\tau^3 - \\
& 55\tau^2 + 11\tau - 1, y_{7,8}^*(\nu) = -p_1^*(\nu), y_{8,1}^*(\nu) = -8\tau^{18} + \\
& 84\tau^{17} - 420\tau^{16} + 1330\tau^{15} - 2982\tau^{14} + 4998\tau^{13} - 6434\tau^{12} + \\
& 6435\tau^{11} - 5005\tau^{10} + 3003\tau^9 - 1365\tau^8 + 455\tau^7 - 105\tau^6 + \\
& 15\tau^5 - \tau^4, y_{8,2}^*(\nu) = 32\tau^{17} - 364\tau^{16} + 1946\tau^{15} - 6482\tau^{14} + \\
& 15022\tau^{13} - 25598\tau^{12} + 33033\tau^{11} - 32747\tau^{10} + 25025\tau^9 - \\
& 14651\tau^8 + 6461\tau^7 - 2079\tau^6 + 461\tau^5 - 63\tau^4 + 4\tau^3, y_{8,3}^*(\nu) = \\
& -56\tau^{16} + 692\tau^{15} - 3928\tau^{14} + 13614\tau^{13} - 32258\tau^{12} + \\
& 55369\tau^{11} - 71075\tau^{10} + 69355\tau^9 - 51701\tau^8 + 29289\tau^7 - \\
& 12403\tau^6 + 3803\tau^5 - 797\tau^4 + 102\tau^3 - 6\tau^2, y_{8,4}^*(\nu) = \\
& 64\tau^{15} - 812\tau^{14} + 4686\tau^{13} - 16366\tau^{12} + 38747\tau^{11} - 65897\tau^{10} + \\
& 83105\tau^9 - 78979\tau^8 + 56819\tau^7 - 30761\tau^6 + 12313\tau^5 - 3523\tau^4 + \\
& 678\tau^3 - 78\tau^2 + 4\tau, y_{8,5}^*(\nu) = -56\tau^{14} + 668\tau^{13} - 3684\tau^{12} +
\end{aligned}$$

$$\begin{aligned}
& 12414\tau^{11} - 28478\tau^{10} + 46928\tau^9 - 57140\tau^8 + 52081\tau^7 - \\
& 35591\tau^6 + 18069\tau^5 - 6667\tau^4 + 1717\tau^3 - 287\tau^2 + 27\tau - 1, \\
& y_{8,6}^*(\nu) = 32\tau^{13} - 356\tau^{12} + 1846\tau^{11} - 5886\tau^{10} + 12824\tau^9 - 20080\tau^8 + \\
& 23159\tau^7 - 19849\tau^6 + 12599\tau^5 - 5829\tau^4 + 1903\tau^3 - 413\tau^2 + \\
& 53\tau - 3, y_{8,7}^*(\nu) = -8\tau^{12} + 92\tau^{11} - 480\tau^{10} + 1506\tau^9 - 3170\tau^8 + \\
& 4721\tau^7 - 5103\tau^6 + 4033\tau^5 - 2311\tau^4 + 935\tau^3 - 253\tau^2 + 41\tau - 3 \text{ and} \\
& y_{8,8}^*(\nu) = -p_0^*(\nu) + (\tau - 1)^{11}. \tag{92}
\end{aligned}$$

Among the results of calculations (90)-(91)-(92) we emphasize the following

$$y_{i,8}^*(\nu) = -p_{8-i}^*(\nu) \text{ for } i \in [1, 7] \cap \mathbb{Z}. \tag{93}$$

4. 3 End of the Proof of Theorem ??1

Let $\tilde{B}(z; \nu)$ be 8×8 -matrix such that $-(z-1)^{-1}z\bar{R}(\nu)$ is its last row, and all elements of its other rows are equal to 0. Let $B(z; \nu) = (N_8 + \tilde{B}(z; \nu))$. In view of (26), (37),

$$\delta X_k(z, \nu) = B(z; \nu)X_k(z; \nu) \tag{94}$$

Let $\bar{s}_i^*(\nu) = \sum_{k=1}^8 s_{i,k}^*(\nu)$, $\bar{v}_i^*(\nu)$ and $\bar{a}_i^*(z, \nu) = \sum_{k=1}^8 a_{i,k}^*(z, \nu)$ be the i -th row of the matrices respectively $S^*(\nu)$, $V^*(\nu)$ and $A^*(z, \nu)$. In view of (30), (32),

$$\bar{s}_i^*(\nu)N_8 = s_{i+1}^*(\nu) \text{ for } i \in [1, 7] \cap \mathbb{Z}, \tag{95}$$

$$a_{i,k}^*(z, \nu) = s_{i,k}^*(\nu) + zv_{i,k}^*(\nu), \bar{a}_i^*(z, \nu) = \bar{s}_i^*(\nu) + z\bar{v}_i^*(\nu)$$

with $\{i, k\} \subset [1, 8] \cap \mathbb{Z}$, and in view, of (93), if $i \in [1, 7] \cap \mathbb{Z}$, then

$$a_{i,8}^*(z, \nu) = s_{i,n}^*(z\nu) + zv_{i,n}^*(z, \nu) = p_{8-i}(1 - z). \tag{96}$$

We must prove the equality

$$\nu^{11}\delta^{i-1}f_k(z, \nu - 1) = \bar{a}_i^*(z, \nu)X_k(z, \nu) \text{ if } i \in [1, 8] \cap \mathbb{Z}, k \in \mathfrak{K}, \nu \in \mathfrak{M}_1. \tag{97}$$

Since $T^*(z, w, \nu) = \sum_{k=0}^7 (p_k^*(\nu) + zq_k^*(\nu))w^k$, the equality (97) for $i = 1$ follows from (26) and (47). Suppose that (97) holds for some $i \in [1, 7] \cap \mathbb{Z}$. Then in view of (97), (32), (94), (96), (88), (95),

$$\begin{aligned} \nu^{11} \delta^i f_k(z, \nu - 1) &= \delta(\bar{a}_i^*(z, \nu) X_k(z, \nu)) = (\delta \bar{a}_i^*(z, \nu)) X_k(z, \nu) + \\ &\bar{a}_i^*(z, \nu) \delta X_k(z, \nu) = (z \bar{y}_i^*(\nu) + \bar{a}_i^*(z, \nu)(N_8 + \tilde{B}(z; \nu))) X_k(z, \nu) = \\ &\bar{s}_i^*(\nu) N_8 X_k(z, \nu) + (z \bar{y}_i^*(\nu)(E_8 + N_8) + p_{8-i}(1-z) \tilde{B}(z; \nu)) X_k(z, \nu) = \\ &(z \bar{y}_i^*(\nu)(E_8 + N_8) + z \bar{a}_i^*(z, \nu) \tilde{B}(z; \nu) p_{8-i}(1-z)(-(z-1)^{-1} z \bar{R}(\nu))) X_k(z, \nu) + \\ &\bar{s}_{i+1}^*(\nu) X_k(z, \nu) = (z \bar{y}_{i+1}^*(\nu) + \bar{s}_{i+1}^*(\nu)) X_k(z, \nu). \square \end{aligned}$$

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