COMMON RANDOM FIXED POINTS OF COMPATIBLE MAPPINGS OF SOME TYPES FOR RANDOM MAPPINGS IN MULTIPLICATIVE METRIC SPACES

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Abstract: In this paper, we introduce the notions of compatible mappings of type \((R)\), type \((K)\) and type \((E)\) in framework of random fixed points in multiplicative metric spaces and using these notions we prove common random fixed point theorems for these mappings.

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1. Introduction and Preliminaries

It is well known that the set of positive real numbers \(\mathbb{R}_+\) is not complete accord-
ing to the usual metric. To overcome this problem, in 2008, Bashirov et al. [1] studied the multiplicative calculus and defined a new distance so called multiplicative distance. By using this idea, Özavsar and Çevikel [7] introduced the concept of multiplicative metric spaces and studied some topological properties in such spaces.

**Definition 1.1.** ([1]) Let $X$ be a nonempty set. A multiplicative metric is a mapping $d : X \times X \rightarrow \mathbb{R}_+$ satisfying the following conditions:

(i) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) \cdot d(z, y)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

Then the mapping $d$ together with $X$, that is, $(X, d)$ is a multiplicative metric space.

**Example 1.2.** ([7]) Let $\mathbb{R}_n^+$ be the collection of all $n$-tuples of positive real numbers. Let $d^* : \mathbb{R}_n^+ \times \mathbb{R}_n^+ \rightarrow \mathbb{R}$ be defined as follows:

$$d^*(x, y) = \left| \frac{x_1}{y_1} \right|^* \cdot \left| \frac{x_2}{y_2} \right|^* \cdots \left| \frac{x_n}{y_n} \right|^*,$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}_n^+$ and $| \cdot |^* : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined by

$$|a|^* = \begin{cases} a & \text{if } a \geq 1, \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$$

Then it is obvious that all conditions of a multiplicative metric are satisfied. Therefore $(\mathbb{R}_n^+, d^*)$ is a multiplicative metric space.

**Remark 1.3.** ([8]) We note that multiplicative metrics and metric spaces are independent.

**Definition 1.4.** ([2, 7]) Let $(X, d)$ be a multiplicative metric space. Then a sequence $\{x_n\}$ in $X$ is said to be

(1) a **multiplicative convergent** to $x$ if for every multiplicative open ball $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}, \epsilon > 1$, there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n \geq N$, that is, $d(x_n, x) \rightarrow 1$ as $n \rightarrow \infty$.

(2) a **multiplicative Cauchy sequence** if for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N$, that is, $d(x_n, x_m) \rightarrow 1$ as $n, m \rightarrow \infty$.

(3) We call a multiplicative metric space **complete** if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.  

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In 2012, Özavsar and Çevikel [7] gave the concept of multiplicative contraction mapping and proved some fixed point theorem of such mappings on a complete multiplicative metric spaces.

**Definition 1.5.** Let $f$ be a mapping of a multiplicative metric space $(X, d)$ into itself. Then $f$ is said to be a multiplicative contraction if there exists a real number $\lambda \in [0, 1)$ such that
\[
d(fx, fy) \leq d^{\lambda}(x, y)
\]
for all $x, y \in X$.

## 2. Definitions and Propositions

Now we introduce the following concepts.

**Definition 2.1.** Let $(X, d)$ be a multiplicative metric space and $F : \mathbb{R} \times X \rightarrow X$ be a mapping, where $X$ is a nonempty set. Then a mapping $g : \mathbb{R} \rightarrow X$ is said to be a random fixed point of the mapping $F$ if $F(t, g(t)) = g(t)$ for all $t \in \mathbb{R}$.

Next we introduce the notions of compatible mappings and its variants for random mappings in multiplicative metric spaces as follows:

**Definition 2.2.** Let $(X, d)$ be a multiplicative metric space. Two mappings $A$ and $B : \mathbb{R} \times X \rightarrow X$ are called
\begin{itemize}
  \item[(1)] compatible for each $t \in \mathbb{R}$ ([5, 6]) if
  \[
  \lim_{n \to \infty} d(A(t, B(t, g_n(t))), B(t, A(t, g_n(t)))) = 1,
  \]
  whenever $\lim_{n \to \infty} A(t, g_n(t)) = \lim_{n \to \infty} B(t, g_n(t))$, $t \in \mathbb{R}$, where $\{g_n\}$ is a sequence of mappings;
  \item[(2)] compatible of type $(P)$ for each $t \in \mathbb{R}$ ([3]) if
  \[
  \lim_{n \to \infty} d(A(t, A(t, g_n(t))), B(t, B(t, g_n(t)))) = 1,
  \]
  whenever $\lim_{n \to \infty} A(t, g_n(t)) = \lim_{n \to \infty} B(t, g_n(t))$, $t \in \mathbb{R}$, where $\{g_n\}$ is a sequence of mappings;
  \item[(3)] compatible of type $(R)$ for each $t \in \mathbb{R}$ if
  \[
  \lim_{n \to \infty} d(A(t, B(t, g_n(t))), B(t, A(t, g_n(t)))) = 1
  \]
  and
  \[
  \lim_{n \to \infty} d(A(t, A(t, g_n(t))), B(t, B(t, g_n(t)))) = 1,
  \]
\end{itemize}
whenever \( \lim_{n \to \infty} A(t, g_n(t)) = \lim_{n \to \infty} B(t, g_n(t)), \, t \in \mathbb{R} \), where \( \{g_n\} \) is a sequence of mappings;

(4) compatible of type \((K)\) for each \( t \in \mathbb{R} \) if
\[
\lim_{n \to \infty} d(A(t, A(t, g_n(t))), B(t, g(t))) = 1
\]
and
\[
\lim_{n \to \infty} d(B(t, B(t, g_n(t))), A(t, g(t))) = 1,
\]
whenever \( \lim_{n \to \infty} A(t, g_n(t)) = \lim_{n \to \infty} B(t, g_n(t)) = g(t), \, t \in \mathbb{R} \), where \( \{g_n\} \) is a sequence of mappings;

(5) compatible of type \((E)\) for each \( t \in \mathbb{R} \) if
\[
\lim_{n \to \infty} A(t, A(t, g_n(t))) = \lim_{n \to \infty} A(t, B(t, g_n(t))) = B(t, g(t))
\]
and
\[
\lim_{n \to \infty} B(t, B(t, g_n(t))) = \lim_{n \to \infty} B(t, A(t, g_n(t))) = A(t, g(t)),
\]
whenever \( \lim_{n \to \infty} A(t, g_n(t)) = \lim_{n \to \infty} B(t, g_n(t)) = g(t), \, t \in \mathbb{R} \), where \( \{g_n\} \) is a sequence of mappings.

**Remark 2.3.** Obviously, compatible mappings of type \((R)\) are compatible mappings and compatible mappings of type \((P)\).

**Definition 2.4.** ([5]) Let \((X, d)\) be a multiplicative metric space and \(A\) and \(B : \mathbb{R} \times X \to X\) be mappings. Then the mapping \(A\) is called jointly continuous for each \( t \in \mathbb{R} \) if
\[
\lim_{n \to \infty} A(t, A(t, g_n(t))) = \lim_{n \to \infty} A(t, B(t, g_n(t))) = A(t, g(t)),
\]
whenever \( \lim_{n \to \infty} A(t, g_n(t)) = \lim_{n \to \infty} B(t, g_n(t)) = g(t) \in X, \, t \in \mathbb{R} \), where \( \{g_n\} \) is a sequence of mappings.

**Definition 2.5.** Let \((X, d)\) be a multiplicative metric space and \(A\) and \(B : \mathbb{R} \times X \to X\) be mappings. Then the mappings \(A\) and \(B\) are called reciprocally continuous for each \( t \in \mathbb{R} \) if
\[
\lim_{n \to \infty} d(A(t, B(t, g_n(t))), A(t, g(t))) = 1
\]
and
\[
\lim_{n \to \infty} d(B(t, A(t, g_n(t))), B(t, g(t))) = 1,
\]
whenever \( \lim_{n \to \infty} A(t, g_n(t)) = \lim_{n \to \infty} B(t, g_n(t)) = g(t), \, t \in \mathbb{R} \), where \( \{g_n\} \) is a sequence of mappings.
Now we give the property related to compatible mappings of type \((E)\) for random mappings.

**Proposition 2.6.** Let \((X, d)\) be a multiplicative metric space and \(A\) and \(B : \mathbb{R} \times X \to X\) be compatible mappings of type \((E)\). Let one of \(A\) and \(B\) are jointly continuous. Suppose that \(\lim_{n \to \infty} A(t, g_n(t)) = \lim_{n \to \infty} B(t, g_n(t)) = g(t)\) for some \(t \in \mathbb{R}\). Then

(a) \(A(t, g(t)) = B(t, g(t))\) and

\[
\lim_{n \to \infty} A(t, A(t, g_n(t))) = \lim_{n \to \infty} B(t, B(t, g_n(t)))
\]

and

\[
\lim_{n \to \infty} A(t, B(t, g_n(t))) = \lim_{n \to \infty} B(t, A(t, g_n(t)))
\]

(b) If there exists \(u \in X\) such that \(A(t, g(u)) = B(t, g(u)) = g(t)\), then \(A(t, B(t, g(u))) = B(t, A(t, g(u)))\).

**Proof.** (a) Let \(\{g_n\}\) be a sequence in \(X\) such that \(\lim_{n \to \infty} A(t, g_n(t)) = \lim_{n \to \infty} B(t, g_n(t)) = g(t), \ t \in \mathbb{R}\). Then by compatible mappings of type \((E)\), we have

\[
\lim_{n \to \infty} A(t, A(t, g_n(t))) = \lim_{n \to \infty} A(t, B(t, g_n(t))) = B(t, g(t))
\]

and

\[
\lim_{n \to \infty} B(t, B(t, g_n(t))) = \lim_{n \to \infty} B(t, A(t, g_n(t))) = A(t, g(t))
\]

If \(A\) is jointly continuous, then we get

\[
\lim_{n \to \infty} A(t, A(t, g_n(t))) = \lim_{n \to \infty} A(t, (B(t, g_n(t))) = A(t, g(t))
\]

implies that \(A(t, g(t)) = B(t, g(t))\). Also,

\[
\lim_{n \to \infty} A(t, A(t, g_n(t))) = \lim_{n \to \infty} B(t, B(t, g_n(t)))
\]

\[
= \lim_{n \to \infty} A(t, B(t, g_n(t))) = \lim_{n \to \infty} B(t, A(t, g_n(t))).
\]

Similarly, if \(B\) is jointly continuous then we get the same result.

(b) Next, suppose that \(A(t, g(u)) = B(t, g(u)) = g(t)\) for some \(u \in X\). Then

\[
A(t, B(t, g(u))) = A(t, (B(t, g(u))) = A(t, g(t))
\]

and

\[
B(t, A(t, g(u))) = B(t, (A(t, g(u)))) = B(t, g(t)).
\]

From (a), we have \(A(t, g(t)) = B(t, g(t))\). Hence

\[
A(t, B(t, g(u))) = B(t, A(t, g(u))).
\]

This completes the proof. \(\square\)
3. Main Results

Now, we give the random fixed point theorem of compatible mappings of type (R) for random mappings.

**Theorem 3.1.** Let \((X, d)\) be a complete multiplicative metric space and \(A, B, S\) and \(T : \mathbb{R} \times X \rightarrow X\) be mappings satisfying the following conditions;

\[(C_1)\quad S(t, X) \subset B(t, X)\quad \text{and}\quad T(t, X) \subset A(t, X);\]

\[(C_2)\quad d(S(t, x(t)), T(t, y(t))) \leq \left[\max\{d(A(t, x(t)), B(t, y(t))), d(A(t, x(t)), S(t, x(t))),
\quad d(B(t, y(t)), T(t, y(t))), d(S(t, x(t)), B(t, y(t))),
\quad d(A(t, x(t)), T(t, y(t)))\}\right]^{\lambda}\]

for all \(x, y \in X\) and \(t \in \mathbb{R}\), where \(\lambda \in (0, 1/2)\);

\[(C_3)\quad \text{one of } A, B, S\text{ and } T\text{ is jointly continuous}.\]

Assume that the pairs \(A, S\) and \(B, T\) are compatible of type (R). Then \(A, B, S\) and \(T\) have a unique common random fixed point.

**Proof.** It follows from Remark 2.3, [5, Theorem 3.1] or [3, Theorem 3.4]. \(\square\)

Next we give the random fixed point theorem of compatible mappings of type (K) for random mappings.

**Theorem 3.2.** Let \((X, d)\) be a complete multiplicative metric space and \(A, B, S\) and \(T : \mathbb{R} \times X \rightarrow X\) be mappings satisfying conditions \((C_1)\) and \((C_2)\).

Assume that the pairs \(A, S\) and \(B, T\) are compatible of type (K), the pairs \(A, S\) and \(B, T\) are reciprocally continuous. Then \(A, B, S\) and \(T\) have a unique common random fixed point.

**Proof.** Let \(g_0 : \mathbb{R} \rightarrow X\) be an arbitrary mapping. By \((C_1)\), there exists \(g_1 : \mathbb{R} \rightarrow X\) such that \(B(t, g_1(t)) = S(t, g_0(t)), t \in \mathbb{R}\) and for this \(g_1 : \mathbb{R} \rightarrow X\) we can choose \(g_2 : \mathbb{R} \rightarrow X\) such that \(T(t, g_1(t)) = A(t, g_2(t)), t \in \mathbb{R}\), and so on. By the method of induction we can define a sequence \(\{y_n(t)\}, t \in \mathbb{R}\), of mappings as follows:

\[y_{2n+1}(t) = B(t, g_{2n+1}(t)) = S(t, g_{2n}(t)),\]
\[y_{2n}(t) = A(t, g_{2n}(t)) = T(t, g_{2n-1}(t)), \quad t \in \mathbb{R},\]

\(n = 0, 1, 2, ...\). From the proof of [5, Theorem 3.1], \(\{y_n(t)\}\) is a multiplicative Cauchy sequence and since \(X\) is complete so \(\{y_n(t)\}\) converges to a point \(z(t) \in \)
$X$ as $n \to \infty$. Also subsequences of $\{y_n(t)\}$ also converges to a point $z(t) \in X$, that is

\[
B(t, g_{2n+1}(t)) \to z(t), \quad S(t, g_{2n}(t)) \to z(t), \\
A(t, g_{2n}(t)) \to z(t), \quad T(t, g_{2n+1}(t)) \to z(t), \quad t \in \mathbb{R}
\]
as $n \to \infty$.

Since the pairs $A, S$ and $B, T$ are compatible of type $(K)$. Then we have

\[
A(t, A(t, g_{2n+2}(t))) \to S(t, z(t)), \quad S(t, S(t, g_{2n}(t))) \to A(t, z(t)), \\
B(t, B(t, g_{2n+1}(t))) \to T(t, z(t)), \quad T(t, T(t, g_{2n+1}(t))) \to B(t, z(t))
\]
as $n \to \infty$.

Now claim that $A(t, z(t)) = B(t, z(t))$.

Putting $x(t) = S(t, g_{2n}(t))$ and $y(t) = T(t, g_{2n+1}(t))$ in $(C_2)$, we have

\[
d(S(t, S(t, g_{2n}(t))), T(t, T(t, g_{2n+1}(t)))) \\
\leq \left[ \max \{d(A(t, S(t, g_{2n}(t))), B(t, T(t, g_{2n+1}(t)))), \\
d(A(t, S(t, g_{2n}(t))), S(t, S(t, g_{2n}(t)))), \\
d(B(t, T(t, g_{2n+1}(t))), T(t, T(t, g_{2n+1}(t)))), \\
d(S(t, S(t, g_{2n}(t))), B(t, T(t, g_{2n+1}(t)))), \\
d(A(t, S(t, g_{2n}(t))), T(t, T(t, g_{2n+1}(t)))) \right]^\lambda.
\]

Letting $n \to \infty$ and using reciprocal continuity of the pairs $A, S$ and $B, T$, we have

\[
d(A(t, z(t)), B(t, z(t))) \\
\leq \left[ \max \{d(A(t, z(t)), B(t, z(t))), d(A(t, z(t)), A(t, z(t))), \\
d(B(t, z(t)), B(t, z(t))), d(A(t, z(t)), B(t, z(t))), \\
d(A(t, z(t)), B(t, z(t))) \right]^\lambda \\
= d^\lambda(A(t, z(t)), B(t, z(t))),
\]
which implies that $d(A(t, z(t)), B(t, z(t))) = 1$ and hence $A(t, z(t)) = B(t, z(t))$.

Next we claim that $S(t, z(t)) = B(t, z(t))$.

Putting $x(t) = z(t)$ and $y(t) = T(t, g_{2n+1}(t))$ in $(C_2)$, we have

\[
d(S(t, z(t)), T(t, T(t, g_{2n+1}(t)))) \\
\leq \left[ \max \{d(A(t, z(t)), B(t, T(t, g_{2n+1}(t)))), d(A(t, z(t)), S(t, z(t))), \\
d(B(t, T(t, g_{2n+1}(t))), T(t, T(t, g_{2n+1}(t)))), \\
d(S(t, z(t)), B(t, T(t, g_{2n+1}(t)))), \\
d(A(t, z(t)), T(t, T(t, g_{2n+1}(t)))) \right]^\lambda.
\]
Letting $n \to \infty$, and using reciprocal continuity of the pairs $A, S$ and $B, T$, we have
\[
d(S(t, z(t)), B(t, z(t))) \\
\leq [\max\{d(A(t, z(t)), B(t, z(t))), d(A(t, z(t)), S(t, z(t))), d(S(t, z(t)), B(t, z(t))), d(A(t, z(t)), B(t, z(t)))\}]^\lambda \\
= d^\lambda(S(t, z(t)), B(t, z(t))).
\]
This implies that $S(t, z(t)) = B(t, z(t))$.

Again we claim that $S(t, z(t)) = T(t, z(t))$.

Putting $x(t) = z(t)$ and $y(t) = z(t)$ in $(C_2)$, we have
\[
d(S(t, z(t)), T(t, z(t))) \\
\leq [\max\{d(A(t, z(t)), B(t, z(t))), d(A(t, z(t)), S(t, z(t))), d(B(t, z(t)), T(t, z(t))), d(S(t, z(t)), B(t, z(t))), d(A(t, z(t)), T(t, z(t)))\}]^\lambda.
\]
This implies that $S(t, z(t)) = T(t, z(t))$. Hence we have
\[
A(t, z(t)) = B(t, z(t)) = S(t, z(t)) = T(t, z(t)).
\]

Next we claim that $z(t) = T(t, z(t))$.

Putting $x(t) = g_{2n}(t)$ and $y(t) = z(t)$ in $(C_2)$, we have
\[
d(S(t, g_{2n}(t)), T(t, z(t))) \\
\leq [\max\{d(A(t, g_{2n}(t)), B(t, z(t))), d(A(t, g_{2n}(t)), S(t, g_{2n}(t))), d(B(t, z(t)), T(t, z(t))), d(S(t, g_{2n}(t)), B(t, z(t))), d(A(t, g_{2n}(t)), T(t, z(t)))\}]^\lambda.
\]
Letting $n \to \infty$, we have
\[
d(z(t), T(t, z(t))) \\
\leq [\max\{d(z(t), T(t, z(t))), 1, 1, d(z(t), T(t, z(t))), d(z(t), T(t, z(t)))\}]^\lambda.
\]
This implies that $z(t) = T(t, z(t))$. Hence
\[
z(t) = A(t, z(t)) = B(t, z(t)) = S(t, z(t)) = T(t, z(t)).
\]
Therefore $z(t)$ is a common random fixed point of $A, S, B$ and $T$.

Uniqueness follows easily from $(C_2)$. Therefore $A, S, B$ and $T$ have a unique common random fixed point. This completes the proof. \qed
Finally, we give the random fixed point theorem of compatible mappings of type (E) for random mappings.

**Theorem 3.3.** Let \((X,d)\) be a complete multiplicative metric space and \(A,B,S\) and \(T : \mathbb{R} \times X \to X\) be mappings satisfying the conditions \((C_1)-(C_3)\).

Assume that the pairs \((A,S)\) and \((B,T)\) are compatible of type \((E)\). Then \(A,B,S\) and \(T\) have a unique common random fixed point.

**Proof.** From the proof of Theorem 3.2, \(\{y_n(t)\}\) is a multiplicative Cauchy sequence and since \(X\) is complete so \(\{y_n(t)\}\) converges to a point \(z(t) \in X\) as \(n \to \infty\). Also subsequences of \(\{y_n(t)\}\) also converges to a point \(z(t) \in X\), that is

\[
B(t,g_{2n+1}(t)) \to z(t), \quad S(t,g_{2n}(t)) \to z(t),
\]

\[
A(t,g_{2n}(t)) \to z(t), \quad T(t,g_{2n+1}(t)) \to z(t), \quad t \in \mathbb{R}
\]

as \(n \to \infty\).

Now \(A\) and \(S\) are compatible of type \((E)\) and one of \(A\) and \(S\) is jointly continuous. Then by Proposition 2.6 we have \(A(t,z(t)) = S(t,z(t))\). Since \(S(t,X) \subset B(t,X)\), there exists a point \(w(t) \in X\) such that \(S(t,z(t)) = B(t,w(t))\).

Putting \(x(t) = z(t)\) and \(y(t) = w(t)\) in \((C_2)\), we have

\[
d(S(t,z(t)),T(t,w(t))) \\
\leq [\max\{d(A(t,z(t)),B(t,w(t))),d(A(t,z(t)),S(t,z(t))),
\]

\[
d(B(t,w(t)),T(t,w(t))),d(S(t,z(t)),B(t,w(t))),
\]

\[
d(A(t,z(t)),T(t,w(t))))]^\lambda.
\]

Letting \(n \to \infty\), we have

\[
d(S(t,z(t)),T(t,w(t))) \\
\leq [\max\{1,1,d(Z(t,z(t)),T(t,w(t))),1,d(S(t,z(t)),T(t,w(t))))]^\lambda \\
= d^\lambda(S(t,z(t)),T(t,w(t))).
\]

This implies that \(d(S(t,z(t)),T(t,w(t))) = 1\) and hence \(S(t,z(t)) = T(t,w(t))\). Thus we have

\[
A(t,z(t)) = S(t,z(t)) = T(t,w(t)) = B(t,w(t)).
\]

Putting \(x(t) = z(t)\) and \(y(t) = g_{2n+1}(t)\) in \((C_2)\), we have

\[
d(S(t,z(t)),T(t,g_{2n+1}(t))) \\
\leq [\max\{d(A(t,z(t)),B(t,g_{2n+1}(t))),d(A(t,z(t)),S(t,z(t))),
\]

\[
d(B(t,g_{2n+1}(t)),T(t,g_{2n+1}(t))),d(S(t,z(t)),B(t,g_{2n+1}(t))),
\]

\[
d(A(t,z(t)),T(t,g_{2n+1}(t))))]^\lambda.
\]
Letting $n \to \infty$ we have
\[
    d(S(t, z(t)), z(t)) \\
    \leq \left[ \max \{d(A(t, z(t)), z(t)), 1, 1, d(S(t, z(t)), z(t)), \\
    d(A(t, z(t)), z(t)) \} \right]^\lambda.
\]
This implies that $d(S(t, z(t)), z(t)) = 1$, that is, $S(t, z(t)) = z(t)$. Hence $z(t)$ is a common random fixed point of $A$ and $S$.

Again, suppose that $B$ and $T$ are compatible of type $(E)$ and one of mapping $B$ and $T$ is jointly continuous we get
\[
    B(t, w(t)) = T(t, w(t)) = A(t, z(t)) = z(t).
\]

By Proposition 2.6 we have
\[
    B(t, B(t, w(t))) = B(t, T(t, w(t))) = T(t, B(t, w(t))) = T(t, T(t, w(t))
\]
and hence $B(t, z(t)) = T(t, z(t))$.

Putting $x(t) = g_{2n}(t)$ and $y(t) = z(t)$ in $(C_2)$, we have
\[
    d(S(t, g_{2n}(t)), T(t, z(t))) \\
    \leq \left[ \max \{d(A(t, g_{2n}(t)), B(t, z(t))), d(A(t, g_{2n}(t)), S(t, g_{2n}(t))), \\
    d(B(t, z(t)), T(t, z(t))), d(S(t, g_{2n}(t)), B(t, z(t))), \\
    d(A(t, g_{2n}(t)), T(t, z(t))) \} \right]^\lambda.
\]

Letting $n \to \infty$, we get
\[
    d(z(t), T(t, z(t))) \\
    \leq \left[ \max \{d(z(t), T(t, z(t))), 1, 1, d(z(t), T(t, z(t))), \\
    d(z(t), T(t, z(t))) \} \right]^\lambda \\
    = d^\lambda(z(t), T(t, z(t))).
\]
This implies that $d(z(t), T(t, z(t))) = 1$ and hence $T(t, z(t)) = z(t)$. Thus we have $T(t, z(t)) = B(t, z(t)) = z(t)$. Hence $z(t)$ is a common random fixed point of $B$ and $T$.

Uniqueness follows easily from $(C_2)$. Therefore $A, S, B$ and $T$ have a unique common random fixed point. This completes the proof. \[\square\]

**Remark 3.4.** If we take $\mathbb{R}$ to be a singleton set, then the results can reduce into the result of Kang et al. [4].
References


