A STUDY ON THE DYNKIN DIAGRAMS AND IMAGINARY ROOTS OF INDEFINITE QUASI-HYPERBOLIC KAC-MOODY ALGEBRA $QHA_5^{(2)}$

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Abstract: In this paper, the particular class of indefinite type of quasi hyperbolic Kac-Moody algebra $QHA_5^{(2)}$ is considered. We obtain, the complete classification of the Dynkin diagrams associated with the indefinite type of quasi hyperbolic Kac-Moody algebra $QHA_5^{(2)}$. Some of the properties of imaginary roots such as strictly imaginary, purely imaginary and isotropic roots are studied.

AMS Subject Classification: 17B67

Key Words: Kac-Moody algebras, indefinite, quasi hyperbolic, strictly imaginary and purely imaginary roots

1. Introduction

The subject Kac-Moody Lie algebras was first introduced and developed by Kac [3] and Moody [8] in 1968. Further, this subject has relevance and applications in various fields of mathematics and mathematical physics.

Kac [3], introduced the concepts of strictly imaginary roots for Kac-Moody
algebras. Casperson [2], obtained the complete classification of Kac-Moody algebras possessing strictly imaginary property. Using the techniques of Benkart et al. [1], Kang [4, 5, 6, 7] had computed the structure and root multiplicities for \( HA_1^{(1)} \), \( HA_2^{(2)} \) and \( HA_n^{(1)} \). The purely imaginary roots and a new class of extended-hyperbolic Kac-Moody algebra was introduced by Sthanumoorthy and Uma Maheswari in [9]. The root multiplicities for particular classes of \( EHA_1^{(1)} \) and \( EHA_2^{(2)} \) were obtained Sthanumoorthy et al. [10, 11, 12, 13].

Uma Maheswari in [14], introduced a new class of indefinite, non-hyperbolic type of Kac-Moody algebra called quasi-hyperbolic Kac-Moody algebra. Some particular classes of indefinite type quasi hyperbolic Kac-Moody algebras \( QHG_2 \), \( QHA_2^{(1)} \), \( QHA_4^{(2)} \), \( QHA_5^{(2)} \) and \( QHA_7^{(2)} \) were realized as graded Kac-Moody algebras of quasi hyperbolic type and homology modules upto level three and the structure of the components of the maximal ideals upto level four were also determined by Uma Maheswari and Krishnaveni in [15, 16, 17, 18, 19]. In [20], the complete classification of the Dynkin diagrams and some properties of real and imaginary roots for the associated quasi affine Kac Moody algebras \( QAC_2^{(1)} \) was obtained. The complete classification of the Dynkin diagrams associated with the quasi hyperbolic Kac-Moody algebra \( QHA_2^{(1)} \) was obtained and the properties purely imaginary and strictly imaginary roots were studied in [21].

In this work, we consider the particular class indefinite quasi-hyperbolic Kac-Moody algebra \( QHA_5^{(2)} \) whose associated symmetrizable and indecomposable GCM is

\[
\begin{pmatrix}
2 & 0 & -1 & 0 & -a_1 \\
0 & 2 & -1 & 0 & -b_1 \\
-1 & -1 & 2 & -1 & -c_1 \\
0 & 0 & -2 & 2 & -d_1 \\
-a_2 & -b_2 & -c_2 & -d_2 & 2
\end{pmatrix},
\]

where \( a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \) are non-negative integers. The main aim of this work is to give a complete classification of the Dynkin diagrams associated with \( QHA_5^{(2)} \) and to study the properties of imaginary roots.

## 2. Preliminaries

In this section, we recall some necessary concepts of Kac-Moody algebras. ([3, 9, 14]).

**Definition 1.** [3] A realization of a matrix \( A = (a_{ij})_{i,j=1}^n \) of rank \( l \), is a triple \( (H, \pi, \pi^\vee) \), \( H \) is a \( 2n - l \) dimensional complex vector space, \( \pi = \)
\{\alpha_1, \ldots, \alpha_n\} and \pi^v = \{\alpha^v_1, \ldots, \alpha^v_n\} are linearly independent subsets of \(H^*\) and \(H\) respectively, satisfying \(\alpha_j(\alpha^v_i) = a_{ij}\) for \(i, j = 1, \ldots, n\). \(\pi\) is called the root basis. Elements of \(\pi\) are called simple roots. The root lattice generated by \(\pi\) is \(Q = \sum_{i=1}^{n} z\alpha_i\).

**Definition 2.** [3] The Kac-Moody algebra \(g(A)\) associated with a GCM \(A = (a_{ij})_{i,j=1}^{n}\) is the Lie algebra generated by the elements \(e_i, f_i, i = 1, 2, \ldots, n\) and \(H\) with the following defining relations:

\[
\begin{align*}
[h, h'] & = 0, \quad h, h' \in H, \\
[e_i, f_j] & = \delta_{ij}\alpha_i^v, \\
[h, e_j] & = \alpha_j(h)e_j, \\
[h, f_j] & = -\alpha_j(h)f_j, \quad i, j \in N, \\
(ade_j)^{1-a_{ij}}e_j & = 0, \\
(adf_i)^{1-a_{ij}}f_j & = 0, \quad \forall i \neq j, i, j \in N.
\end{align*}
\]

The Kac-Moody algebra \(g(A)\) has the root space decomposition \(g(A) = \bigoplus_{\alpha \in Q} g_{\alpha}(A)\) where \(g_{\alpha}(A) = \{x \in g(A) / [h, x] = \alpha(h)x, \text{ for all } h \in H\}\). An element \(\alpha, \alpha \neq 0\) in \(Q\) is called a root if \(g_{\alpha} \neq 0\). Let \(Q = \sum_{i=1}^{n} z_+\alpha_i\). \(Q\) has a partial ordering “≤” defined by \(\alpha \leq \beta\) if \(\beta - \alpha \in Q_+\), where \(\alpha, \beta \in Q\).

**Definition 3.** [14] Let \(A = (a_{ij})_{i,j=1}^{n}\) be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram \(S(A)\) to be Quasi Hyperbolic (QH) type if \(S(A)\) has a proper connected sub diagram of hyperbolic type with \((n-1)\) vertices. The GCM \(A\) is of QH type if \(S(A)\) is of QH type. We then say that the Kac-Moody algebra \(g(A)\) is of QH type.

**Definition 4.** [3] A root \(\alpha \in \Delta\) is called real, if there exist a \(w \in W\) such that \(w(\alpha)\) is a simple root, and a root which is not real is called an imaginary root. An imaginary root \(\gamma\) is said to be strictly imaginary if for every real root \(\alpha\), either \(\alpha + \gamma\) or \(\alpha - \gamma\) is a root. An imaginary root \(\alpha\) is called an isotropic if \((\alpha, \alpha) = 0\). A generalized Cartan matrix \(A\) has the property SIM (more briefly: \(A \in SIM\)) if \(\Delta^{sim}(A) = \Delta^{im}(A)\).

**Definition 5.** [9] Let \(\alpha \in \Delta^{im}_+\) is said to be purely imaginary if for any \(\beta \in \Delta^{im}_+, \alpha + \beta \in \Delta^{im}_+\). A GCM \(A\) satisfies the purely imaginary property if \(\Delta^{pim}_+(A) = \Delta^{im}_+(A)\). If \(A\) satisfies the purely imaginary property then the Kac-Moody algebra \(g(A)\) has the purely imaginary property.
3. Dynkin Diagrams Associated with the Indefinite Type of Quasi-Hyperbolic Kac-Moody Algebra $QHA_5^{(2)}$

In this section, we first prove the classification theorem wherein connected, non-isomorphic Dynkin diagrams associated with $QHA_5^{(2)}$ are completely classified. Next, we study some of the properties of roots for specific families in the class $QHA_5^{(2)}$.

**Theorem 6. (Classification Theorem)** For the indefinite type of quasi-hyperbolic Kac-Moody algebra $QHA_5^{(2)}$, whose associated symmetrizable and indecomposable GCM is

$$
\begin{pmatrix}
2 & 0 & -1 & 0 & -a_1 \\
0 & 2 & -1 & 0 & -b_1 \\
-1 & -1 & 2 & -1 & -c_1 \\
0 & 0 & -2 & 2 & -d_1 \\
-a_2 & -b_2 & -c_2 & -d_2 & 2
\end{pmatrix},
$$

where $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ are non-negative integers, the number of connected non-isomorphic Dynkin diagrams corresponding to these GCM of $QHA_5^{(2)}$ is 161.

**Proof.** We consider the Dynkin diagram associated to the affine Kac-Moody algebra $A_5^{(2)}$. The Dynkin diagram of the indefinite type of quasi-hyperbolic Kac-Moody algebra $QHA_5^{(2)}$ is obtained by adding a fifth vertex, which is connected to the to the Dynkin diagram $A_5^{(2)}$, generally represented by $\sim$, where $\sim$ can be one of the 9 possible edges: $\sim$ $\xrightarrow{\sim}$ $\xleftarrow{\sim}$ $\xrightarrow{\sim}$ $\xleftarrow{\sim}$ $\xrightarrow{\sim}$ $\xleftarrow{\sim}$ $\xrightarrow{\sim}$ $\xleftarrow{\sim}$ $\xrightarrow{\sim}$

**Case (i):** An edge is added from the fifth vertex to any one of the four vertices of the Dynkin diagram of affine Kac-Moody algebra $A_5^{(2)}$. This can be done in the following three ways:

![Diagrams](attachment:diagrams.png)

Consider the diagram (1):
The number of ways of joining the fifth vertex to the vertex 1 is 9.
We discuss the nature of the Dynkin diagrams case by case.

(i) When \( \rightsquigarrow \) represents the edge \( \rightarrow \), the Dynkin diagram is \( H^{(5)}_{11} \) which is of hyperbolic type.

(ii) The number of Dynkin diagrams joining the 5\(^{th} \) vertex to 1 by \( \rightsquigarrow \xrightarrow{\Leftrightarrow} \xleftarrow{\Leftrightarrow} \) which do not belong to quasi hyperbolic type is 6.
After excluding 7 Dynkin diagrams based on the above nature from 9, we get 2 connected, non isomorphic Dynkin diagrams which are given as follows:

Consider the diagram (2):
The number of ways of joining the fifth vertex to the vertex 3 is 9.
We discuss the nature of the Dynkin diagrams case by case.

(i) When \( \rightsquigarrow \) represents the edge \( \rightarrow \), the Dynkin diagram is \( H^{(5)}_{9} \) which is of hyperbolic type.

(ii) The number of Dynkin diagrams corresponding to the edges \( \rightarrow \xrightarrow{\Leftrightarrow} \xleftarrow{\Leftrightarrow} \) which do not belong to quasi hyperbolic is 4.
Based on the above nature, we have excluded 5 Dynkin diagrams from 9, we get 4 connected, non isomorphic Dynkin diagrams. They are given as follows:

Consider the diagram (3):
The number of ways of joining the fifth vertex to the vertex 4 is 9.
We discuss the nature of the Dynkin diagrams case by case.

(i) When \( \rightsquigarrow \) represents the edge \( \rightarrow \), the Dynkin diagram is \( H^{(5)}_{17} \) which is of hyperbolic type.
(ii) The number of edges corresponding to the edges \( \overset{\downarrow}{\mathbin{\uparrow}} \overset{\downarrow}{\mathbin{\uparrow}} \) which do not belong to quasi hyperbolic type is 6.

Except these 7 Dynkin diagrams from 9, we get 2 connected, non isomorphic Dynkin diagrams. They are given as follows:

Excluding the above 19 Dynkin diagrams from the total of 27, we get 8 connected, non isomorphic Dynkin diagrams in \( QHA_5^{(2)} \).

**Case (ii):** When two edges are added from the fifth vertex to any two vertices of the Dynkin diagram of affine Kac-Moody algebra \( A_5^{(2)} \), we get four different forms and they are:

Consider the diagram (4):

The number of ways, the fifth vertex can be joined to the vertices 1 and 2 is 81 \((= 9^2)\).

I. We fix the edge 5 to 1 by \( \overset{\downarrow}{\mathbin{\uparrow}} \) and vary the edge 5 to 2 by \( \overset{\downarrow}{\mathbin{\uparrow}} \) as in

We discuss the nature of the resulting Dynkin diagrams:
Table 1

<table>
<thead>
<tr>
<th>Possibilities</th>
<th>Quasi Hyperbolic (QH)</th>
<th>Not Quasi Hyperbolic (Not QH)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>√</td>
<td></td>
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<tr>
<td></td>
<td>√</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>2</td>
</tr>
</tbody>
</table>

II. We fix the edge 5 to 1 by \( \sim \) and vary the edge 5 to 2 by \( \sim \) as in

![Diagram](image)

We discuss the nature of the resulting Dynkin Diagrams:

III. We fix the edge 5 to 1 by \( \rightarrow \) and vary the edge 5 to 2 by \( \sim \) as in

![Diagram](image)

We discuss the nature of the resulting Dynkin Diagrams:
Table 2

<table>
<thead>
<tr>
<th>Possibilities of $\sim$</th>
<th>Hyperbolic type (H)</th>
<th>QH</th>
<th>Not QH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-$</td>
<td></td>
<td>$\checkmark$</td>
<td></td>
</tr>
<tr>
<td>$-$</td>
<td>$\checkmark$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rhd$</td>
<td>$\checkmark$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Leftarrow$</td>
<td>$\checkmark$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Uparrow$</td>
<td>$\checkmark$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\leftarrow$</td>
<td>$\checkmark$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Uparrow$</td>
<td></td>
<td></td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$\Leftarrow$</td>
<td></td>
<td></td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$\Uparrow$</td>
<td></td>
<td></td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 3

<table>
<thead>
<tr>
<th>Possibilities of $\sim$</th>
<th>QH</th>
<th>Not QH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-$</td>
<td>$\checkmark$</td>
<td></td>
</tr>
<tr>
<td>$\rhd$</td>
<td>$\checkmark$</td>
<td></td>
</tr>
<tr>
<td>$\Leftarrow$</td>
<td>$\checkmark$</td>
<td></td>
</tr>
<tr>
<td>$\Uparrow$</td>
<td>$\checkmark$</td>
<td></td>
</tr>
<tr>
<td>$\leftarrow$</td>
<td>$\checkmark$</td>
<td></td>
</tr>
<tr>
<td>$\Uparrow$</td>
<td></td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$\Leftarrow$</td>
<td></td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$\Uparrow$</td>
<td></td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$\leftarrow$</td>
<td></td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>Total</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

IV. We fix the edge 5 to 1 by $\Leftarrow$ and vary the edge 5 to 2 by
as in

![Dynkin Diagram](image)

We discuss the nature of the resulting Dynkin Diagrams:

Table 4

<table>
<thead>
<tr>
<th>Possibilities of $\sim$</th>
<th>QH</th>
<th>Not QH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\subseteq$</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>$\subseteq$</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>$\supseteq$</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>$\supseteq$</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>$\not\subseteq$</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>$\not\supseteq$</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>Total</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

V. We fix the edge 5 to 1 by $\subseteq$ and vary the edge 5 to 2 by $\sim$ as in

![Dynkin Diagram](image)

We discuss the nature of the resulting Dynkin Diagrams:
Table 5

<table>
<thead>
<tr>
<th>Possibilities of $\sim$</th>
<th>QH</th>
<th>Not QH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Rightarrow$</td>
<td>√</td>
<td></td>
</tr>
<tr>
<td>$\leftarrow$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td></td>
<td>√</td>
</tr>
<tr>
<td>$\Leftarrow$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Leftarrow$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

VI. We fix the edge 5 to 1 by $\Rightarrow$ and vary the edge 5 to 2 by $\sim$ as in

![Diagram]

We discuss the nature of the resulting Dynkin Diagrams:

Table 6

<table>
<thead>
<tr>
<th>Possibilities of $\sim$</th>
<th>QH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Rightarrow$</td>
<td>√</td>
</tr>
<tr>
<td>$\leftarrow$</td>
<td>√</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>√</td>
</tr>
<tr>
<td>$\Leftarrow$</td>
<td>√</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>√</td>
</tr>
<tr>
<td>$\Leftarrow$</td>
<td>√</td>
</tr>
<tr>
<td>Total</td>
<td>4</td>
</tr>
</tbody>
</table>

VII. We fix the edge 5 to 1 by $\Leftarrow$ and vary the edge 5 to 2 by $\sim$ as in

![Diagram]

We discuss the nature of the resulting Dynkin Diagrams:
Table 7

<table>
<thead>
<tr>
<th>Possibilities of $\sim$</th>
<th>QH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Rightarrow$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>Total</td>
<td>2</td>
</tr>
</tbody>
</table>

VIII. We fix the edge 5 to 1 by $\Rightarrow$ and vary the edge 5 to 2 by $\sim$ as in

![Dynkin Diagram](image1)

We discuss the nature of the resulting Dynkin Diagrams:

Table 8

<table>
<thead>
<tr>
<th>Possibilities of $\sim$</th>
<th>Not QH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Rightarrow$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$\Leftarrow$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>Total</td>
<td>2</td>
</tr>
</tbody>
</table>

IX. We fix the edge 5 to 1 by $\Leftarrow$ and vary the edge 5 to 2 by $\sim$ as in

![Dynkin Diagram](image2)

We discuss the nature of the resulting Dynkin Diagrams:
Table 9

<table>
<thead>
<tr>
<th>Possibilities of ( \sim )</th>
<th>Not QH</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \equiv )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
</tr>
</tbody>
</table>

We get 21 quasi hyperbolic type of Dynkin diagrams, 1 hyperbolic type and 22 Dynkin diagrams are not of quasi hyperbolic type for the diagram (4).

Consider the diagram (5)

![Diagram 5](image)

The number of ways, the fifth vertex can be joined to the vertices 1 and 3 is 81. If we fix the edge between the vertices 5 and 1 by \( \sim \) and varying the edge between the vertices 5 and 3 by \( \sim \), then we get 37 quasi hyperbolic type of Dynkin diagrams and 43 Dynkin diagrams are not of quasi hyperbolic type.

Consider the diagram (6)

![Diagram 6](image)

The number of ways, the fifth vertex can be joined to the vertices 1 and 4 is 81. By the same way, fixing the edge between the vertices 5 and 1 by \( \sim \) and varying the edge between the vertices 5 and 4 by \( \sim \), we get 23 quasi hyperbolic type of Dynkin diagrams and 57 Dynkin diagrams are not of quasi hyperbolic type.

Consider the diagram (7)

![Diagram 7](image)

The number of ways, the fifth vertex can be joined to the vertices 3 and 4 is 81.
Similarly, by fixing the edge between the vertices 5 and 3 by \( \sim \) and varying the edge between the vertices 5 and 4 by the same nine possibilities, the resulting Dynkin diagrams are not of quasi hyperbolic type.

In this case (ii), we get \( 21 + 37 + 23 = 81 \) connected, non isomorphic Dynkin diagrams in \( QH A_5^{(2)} \).

**Case (iii):** When three edges are joined from the from the fifth vertex to any three vertices of the Dynkin diagram of affine Kac-Moody algebra \( A_5^{(2)} \), we get three different forms and they are:

\[
\begin{align*}
\text{(8)} & \quad \begin{array}{c}
\begin{array}{c}
\quad \\
\quad \\
\quad \end{array}
\end{array} \\
\text{(9)} & \quad \begin{array}{c}
\begin{array}{c}
\quad \\
\quad \\
\quad \end{array}
\end{array} \\
\text{(10)} & \quad \begin{array}{c}
\begin{array}{c}
\quad \\
\quad \\
\quad \end{array}
\end{array}
\end{align*}
\]

Consider the diagram (8): The possible connected Dynkin diagrams for the diagram (8) is \( 729 \) (\( = 9^3 \)).

Among these, there are no Dynkin diagrams of hyperbolic type. The only possible connected, non isomorphic Dynkin diagrams of quasi hyperbolic types are 9. The remaining Dynkin diagrams are not of quasi- hyperbolic type.

Consider the diagram(9): The possible connected Dynkin diagrams for the diagram (9) is \( 729 \) (\( = 9^3 \)).

The following Table 10 shows the number of possible connected quasi hyperbolic type of Dynkin diagrams:

<table>
<thead>
<tr>
<th>5-1</th>
<th>5-4</th>
<th>5-2</th>
<th>Quasi Hyperbolic</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>-</td>
<td>( \sim )</td>
<td>9</td>
</tr>
<tr>
<td>( \sqcup )</td>
<td>-</td>
<td>( \sim )</td>
<td>9</td>
</tr>
<tr>
<td>( \sqcap )</td>
<td>-</td>
<td>( \sim )</td>
<td>9</td>
</tr>
<tr>
<td>( \sqcup )</td>
<td>-</td>
<td>( \sim )</td>
<td>9</td>
</tr>
<tr>
<td>( \sqcap )</td>
<td>-</td>
<td>( \sim )</td>
<td>9</td>
</tr>
<tr>
<td>( \sqcup )</td>
<td>-</td>
<td>( \sim )</td>
<td>9</td>
</tr>
<tr>
<td>( \sqcap )</td>
<td>-</td>
<td>( \sim )</td>
<td>9</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
<td><strong>45</strong></td>
</tr>
</tbody>
</table>
Here we get, 45 connected, non isomorphic Dynkin diagrams of quasi hyperbolic type.

Consider the diagram (10): The possible connected Dynkin diagrams for the diagram (10) is 729.

Among these there are no hyperbolic Dynkin diagrams. The only possible connected quasi hyperbolic type of Dynkin diagrams are obtained by adding the fifth vertex to 1 by \( \circ \), 5 to 3 by \( \circ \) and 5 to 4 can be joined by any of the nine possibilities.

Here we get only 9 quasi hyperbolic type of Dynkin diagrams.

Remaining Dynkin diagrams are not of quasi hyperbolic type.

Hence, in this case (iii), we get \( (9 + 45 + 9) = 63 \) connected, non isomorphic Dynkin diagrams in \( QH A_5^{(2)} \).

**Case (iv):** In this case, the fifth vertex is connected to all the four vertices of the Dynkin diagram of affine Kac-Moody algebra \( A_5^{(2)} \).

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\node (1) at (0,0) [circle,draw] {1};
\node (2) at (1,0) [circle,draw] {2};
\node (3) at (2,0) [circle,draw] {3};
\node (4) at (3,0) [circle,draw] {4};
\node (5) at (4,0) [circle,draw] {5};
\draw (1) -- (2);
\draw (2) -- (3);
\draw (3) -- (4);
\draw (4) -- (5);
\end{tikzpicture}}
\end{array}
\]

The number of ways of joining the fifth vertex to all the four vertices is \( 9^4 = 6561 \). Among these there are no hyperbolic Dynkin diagrams. The only possible connected quasi hyperbolic type of Dynkin diagrams are obtained by adding the fifth vertex to 1 by \( \circ \), 5 to 2 by \( \circ \), 5 to 3 by \( \circ \) and 5 to 4 can be joined by any of the nine possibilities. Here we get only 9 connected, non isomorphic quasi hyperbolic type of Dynkin diagrams.

Hence, in this case (iv), we get 9 connected, non isomorphic Dynkin diagrams associated with \( QH A_5^{(2)} \).

From the above four cases we get, a total of 161 connected, non isomorphic Dynkin diagrams in \( QH A_5^{(2)} \).

\[\square\]

4. Properties of Imaginary Roots

**Proposition 7.** Consider the indefinite type of quasi hyperbolic Kac-Moody algebra \( QH A_5^{(2)} \), whose associated symmetrizable and indecomposable
A STUDY ON THE DYNKIN DIAGRAMS AND IMAGINARY...

GCM is
\[
\begin{pmatrix}
2 & 0 & -1 & 0 & -a_1 \\
0 & 2 & -1 & 0 & -b_1 \\
-1 & -1 & 2 & -1 & -c_1 \\
0 & 0 & -2 & 2 & -d_1 \\
-a_2 & b_2 & -c_2 & -d_2 & 2
\end{pmatrix},
\]
where \(a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2\) are non-negative integers. Then the Kac-Moody algebra \(g(A)\) corresponding to \(QHA_{5}^{(2)}\) has the following properties:

(i) The imaginary roots of \(g(A)\) satisfy the purely imaginary property.

(ii) The imaginary roots of \(g(A)\) satisfy the strictly imaginary property.

Proof. (i) Since \(A\) is a connected, symmetrizable and indecomposable GCM, by using corollary 3.11 in [9], we get, \(\Delta_{\text{pim}}^{+}(A) = \Delta_{\text{im}}^{+}(A)\). Hence \(g(A)\) has purely imaginary property.

(ii) Since \(A\) is symmetrizable and indecomposable GCM, \(A\) satisfies the required condition given in the Theorem 23 in [2]. Hence \(g(A)\) has strictly imaginary property. \(\square\)

We give the decomposition of the symmetrizable GCM for a general family in \(QHA_{5}^{(2)}\).

For the indefinite type of quasi-hyperbolic Kac-Moody algebra \(QH A_{5}^{(2)}\), the associated symmetrizable and indecomposable GCM is
\[
\begin{pmatrix}
2 & 0 & -1 & 0 & -a_1 \\
0 & 2 & -1 & 0 & -b_1 \\
-1 & -1 & 2 & -1 & -c_1 \\
0 & 0 & -2 & 2 & -d_1 \\
-a_2 & b_2 & -c_2 & -d_2 & 2
\end{pmatrix},
\]
where \(a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2\) are non-negative integers. Since \(A\) is symmetrizable, \(A\) can be expressed as \(A = DB\) where
\[
D = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & a_2/a_1
\end{pmatrix},
\]
and

\[
\begin{pmatrix}
  2 & 0 & -1 & 0 & -a_1 \\
  0 & 2 & -1 & 0 & -b_1 \\
  -1 & -1 & 2 & -1 & -c_1 \\
  0 & 0 & -1 & 1 & -d_1/2 \\
  -a_1 & -b_2 & -c_2 & -d_2/2 & 2a_1/a_2
\end{pmatrix},
\]

with the conditions, \( b_2 = b_1a_2/a_1, c_2 = c_1a_2/a_1, d_2 = d_1a_2/2a_1 \).

**Example 8.** Consider the indefinite quasi-hyperbolic Kac-Moody algebra \( QHA_5^{(2)} \) whose associated symmetrizable and indecomposable GCM

\[
A = \begin{pmatrix}
  2 & 0 & -1 & 0 & -1 \\
  0 & 2 & -1 & 0 & -1 \\
  -1 & -1 & 2 & -1 & 0 \\
  0 & 0 & -2 & 2 & 0 \\
  -1 & -1 & 0 & 0 & 2
\end{pmatrix}.
\]

Since \( A \) is symmetrizable, \( A = DB \) where

\[
D = \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 2 & 0 \\
  0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

and

\[
B = \begin{pmatrix}
  2 & 0 & -1 & 0 & -1 \\
  0 & 2 & -1 & 0 & -1 \\
  -1 & -1 & 2 & -1 & 0 \\
  0 & 0 & -1 & 1 & 0 \\
  -1 & -1 & 0 & 0 & 2
\end{pmatrix}.
\]

Here \((a_1, a_1) = 2, (a_2, a_2) = 2, (a_3, a_3) = 2, (a_4, a_4) = 1, (a_5, a_5) = 2, (a_1, a_2) = (a_2, a_1) = 0, (a_1, a_3) = (a_3, a_1) = -1, (a_1, a_4) = (a_4, a_1) = 0, (a_1, a_5) = (a_5, a_1) = -1, (a_2, a_3) = (a_3, a_2) = -1, (a_2, a_4) = (a_4, a_2) = 0, (a_2, a_5) = (a_5, a_2) = -1, (a_3, a_4) = (a_4, a_3) = -1, (a_3, a_5) = (a_5, a_3) = 0, (a_4, a_5) = (a_5, a_4) = 0\).

Let \( \beta = a_1 + a_2 + a_3 + a_4 + a_5 \) then \((\beta, \beta) = -1 < 0 \). Therefore \( \beta \) is an imaginary root. Now for every real root \( \alpha \), \( \beta + \alpha \) is also a root. Hence \( \beta \) is a strictly imaginary root.

**Example 9.** Consider the indefinite quasi-hyperbolic Kac-Moody algebra \( QHA_5^{(2)} \) whose associated symmetrizable and indecomposable GCM

\[
A = \begin{pmatrix}
  2 & 0 & -1 & 0 & -1 \\
  0 & 2 & -1 & 0 & -1 \\
  -1 & -1 & 2 & -1 & -1 \\
  0 & 0 & -2 & 2 & -d_1 \\
  -1 & -1 & -1 & -d_2/2 & 2
\end{pmatrix}.
\]
Since $A$ is symmetrizable, $A$ has the decomposition $DB$ where

$$
D = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

and

$$
B = \begin{pmatrix}
2 & 0 & -1 & 0 & -1 \\
0 & 2 & -1 & 0 & -1 \\
-1 & -1 & 2 & -1 & -1 \\
0 & 0 & -1 & 1 & -d_1/2 \\
-1 & -1 & -1 & -d_1/2 & 2
\end{pmatrix},
$$

with $d_1 = 2d_2$.

Here $(\alpha_1, \alpha_1) = 2$, $(\alpha_2, \alpha_2) = 2$, $(\alpha_3, \alpha_3) = 2$, $(\alpha_4, \alpha_4) = 1$, $(\alpha_5, \alpha_5) = 2$, $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1) = 0$, $(\alpha_1, \alpha_3) = (\alpha_3, \alpha_1) = -1$, $(\alpha_1, \alpha_4) = (\alpha_4, \alpha_1) = 0$, $(\alpha_1, \alpha_5) = (\alpha_5, \alpha_1) = -1$, $(\alpha_2, \alpha_3) = (\alpha_3, \alpha_2) = -1$, $(\alpha_2, \alpha_4) = (\alpha_4, \alpha_2) = 0$, $(\alpha_2, \alpha_5) = (\alpha_5, \alpha_2) = -1$, $(\alpha_3, \alpha_4) = (\alpha_4, \alpha_3) = -1$, $(\alpha_3, \alpha_5) = (\alpha_5, \alpha_3) = 0$, $(\alpha_4, \alpha_5) = (\alpha_5, \alpha_4) = -d_1/2$.

Let $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$, then $(\alpha, \alpha) = -(3 + d_1) < 0$ (since $d_1 > 0$) Therefore, $\alpha$ is an imaginary root. For every real root $\delta$, we can see that $\alpha + \delta$ is also a root. Hence, $\alpha$ is a strictly imaginary root. Let $\beta = \alpha_4 + \alpha_5$ then $(\beta, \beta) = 3 - d_1 = 0$ if $d_1 = 3$, hence $\beta$ is an isotropic root. Let $\gamma = \alpha_3 + \alpha_4 + \alpha_5$, then $(\gamma, \gamma) < 0$, $\gamma$ is an another imaginary root. We find that, $(\gamma + \alpha, \gamma + \alpha) = -4(d_1 + 2) < 0$. Therefore, $\alpha$ is also a purely imaginary root.

5. Conclusions

In this paper, the complete classification of the Dynkin diagrams is obtained for the indefinite quasi-hyperbolic Kac-Moody algebra $QHA_5^{(2)}$. We can extend this work further to determine the root multiplicities for $QHA_5^{(2)}$ and other families in $QHA_5^{(2)}$. 
References


