STRONG AND WEAK VERTEX DOMINATION ON S-VALUED GRAPHS

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Abstract: In [5], we introduced the notion of $S$-valued graphs, where $S$ is a semiring. In our earlier paper, [2], we introduced the notion of vertex domination on $S$-valued graphs. In this paper we introduce the notion of Strong and Weak domination on $S$-valued graphs and prove simple properties.

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1. Introduction

In [3], Jonathan Golan introduced the notion of $S$-valued graph where $S$ is a semiring. This motivated us to study the notion of semiring valued graphs in our paper [5]. The fastest growing area within Graph theory is the study of...
domination. The topic of domination was given formal mathematical definition by Berge [1] in 1958 and Ore [4] in 1962. Motivated by the domination theory in graphs, in our earlier paper [2], we introduced the notion of vertex domination on $S$-valued graphs. In this paper, we introduce the notions of strong and weak vertex domination on a $S$-valued graph $G^S$, and discuss some simple but elegant results.

2. Preliminaries

In this section, we recall some basic definitions that are needed for our work.

**Definition 2.1.** (see [3]) A semiring $(S, +, \cdot)$ is an algebraic system with a non-empty set $S$ together with two binary operations $+$ and $\cdot$ such that

1. $(S, +, 0)$ is a monoid.
2. $(S, \cdot)$ is a semigroup.
3. For all $a, b, c \in S$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.
4. $0 \cdot x = x \cdot 0 = 0 \ \forall \ x \in S$.

**Definition 2.2.** Let $(S, +, \cdot)$ be a semiring. A Canonical Pre-order $\preceq$ in $S$ defined as follows: for $a, b \in S$, $a \preceq b$ if and only if, there exists $c \in S$ such that $a + c = b$.

**Definition 2.3.** A set $D \subseteq V$ of vertices in a graph $G = (V, E)$ is called a vertex dominating set in $G$ if every vertex $v \in V$ is either an element in $D$ or is adjacent to an element in $V - D$.

A set $D \subseteq V$ is a Dominating vertex set of $G$, if $\forall v \in V - D$, $N(v) \cap D \neq \emptyset$.

**Definition 2.4.** A dominating set $D$ is a minimal vertex dominating set if no proper subset of $D$ is a vertex dominating set in $G$.

**Definition 2.5.** A set $D \subseteq V$ is a minimal dominating set of the graph $G = (V, E)$ if $D$ is a dominating set and $\forall v \in D$, either $v$ has no neighbour in $D$ or there exists neighbour $u \in V - D$ of $v$ such that $u$ has no neighbour in $D \setminus \{v\}$.

**Definition 2.6.** A dominating set $D$ is a strong dominating set if for every vertex $u \in V - D$ there is a vertex $v \in D$ with $\text{deg}(v) \geq \text{deg}(u)$ and $u$ is adjacent to $v$. 

Definition 2.7. A dominating set $D$ is a weak dominating set if for every vertex $u \in V - D$ there is a vertex $v \in D$ with $\text{deg}(v) \leq \text{deg}(u)$ and $u$ is adjacent to $v$.

Definition 2.8. A set $D \subseteq V$ is an Independent set of $G$ if $u, v \in D$, $N(u) \cap \{v\} = \emptyset$.

Definition 2.9. A set $D \subseteq V$ is an Independent dominating set of $G$ if $D$ is both an independent and a dominating set.

Definition 2.10. (see [5]) Let $G = (V, E \subset V \times V)$ be a given graph with $V,E \neq \emptyset$. For any semiring $(S, +, \cdot)$, a semiring-valued graph (or a $S$-valued graph), $G^S$, is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma : V \rightarrow S$ and $\psi : E \rightarrow S$ is defined to be

$$\psi(x, y) = \begin{cases} \min \{\sigma(x), \sigma(y)\}, & \text{if } \sigma(x) \preceq \sigma(y) \text{ or } \sigma(y) \preceq \sigma(x), \\ 0, & \text{otherwise,} \end{cases}$$

for every unordered pair $(x, y)$ of $E \subset V \times V$. We call $\sigma$, a $S$-vertex set and $\psi$, a $S$-edge set of $G^S$.

Definition 2.11. Consider the $S$-valued graph $G^S = (V, E \subset V \times V, \sigma, \psi)$. The open neighbourhood of $v_i$ in $G^S$ is defined as the set

$$N^S_S(v_i) = \{(v_j, \sigma(v_j)), \text{ where } (v_i, v_j) \in E, \psi(v_i, v_j) \in S.)\}.$$

Definition 2.12. The closed neighbourhood of $v_i$ in $G^S$ is defined to be the set $N^S_S[v_i] = N^S_S(v_i) \cup \{(v_i, \sigma(v_i))\}$

Definition 2.13. A vertex $v \in D$ of $G^S$ is said to be an $S$-isolate vertex if $N^S_S(v) \subseteq V - D$.

Definition 2.14. A vertex $v$ in $G^S$ is said to be a weight dominating vertex if $\sigma(u) \preceq \sigma(v)$, $\forall u \in N^S_S[v]$.

Definition 2.15. A subset $D \subseteq V$ is said to be a weight dominating vertex set if for each $v \in D$, $\sigma(u) \preceq \sigma(v)$, $\forall u \in N^S_S[v]$.

3. Strong Domination on $S$-Valued Graphs

In this section, we introduce the notion of strong dominating set of $G^S$ and prove some properties.
Definition 3.1. A subset $D \subseteq V$ is said to be a strong weight dominating vertex set if

1. $D$ is a weight dominating vertex set.
2. For each vertex $v \in D$, $deg_S(u) \preceq deg_S(v) \forall u \in N_S[v]$.

Example 3.2. Let $(S = \{0,a,b,c\},+,\cdot)$ be a semiring with the following Cayley Tables:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>\cdot</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
</tbody>
</table>

Let $\preceq$ be a canonical pre-order in $S$, given by

$0 \preceq 0, \ 0 \preceq a, \ 0 \preceq b, \ 0 \preceq c, \ a \preceq a, \ b \preceq b, \ c \preceq c, \ c \preceq a, \ c \preceq b$

Consider $G^S = (V,E,\sigma,\psi)$. where,

$\sigma : V \to S$ by $\sigma(v_1) = \sigma(v_3) = a ; \sigma(v_2) = b ; \sigma(v_4) = c.$

and $\psi : E \to S$ by

$\psi(v_1,v_2) = \psi(v_2,v_3) = b ; \psi(v_3,v_4) = \psi(v_1,v_4) = c$ and $\psi(v_1,v_3) = a$

In graph $G^S$, $D = \{v_1,v_3\}$ is a weight dominating vertex set of $G^S$. Here

$deg_S(v_1) = \{\sigma(v_2) + \sigma(v_3) + \sigma(v_4) \ , \ 3\} = \{(b + a + c \ 3)\} = \{(a, \ 3)\}.$
Then $\deg_S(v_2) \preceq \deg_S(v_1)$; $\deg_S(v_3) \preceq \deg_S(v_1)$; $\deg_S(v_4) \preceq \deg_S(v_1)$.

Similarly,

$$
\deg_S(v_3) = \{(\sigma(v_3) + \sigma(v_1) + \sigma(v_2), 3)\} = \{(c + a + b, 3)\} = \{(a, 3)\}.
$$

Then $\deg_S(v_1) \preceq \deg_S(v_3)$; $\deg_S(v_2) \preceq \deg_S(v_3)$; $\deg_S(v_4) \preceq \deg_S(v_3)$.

Hence $D = \{v_1, v_3\}$ is a Strong weight dominating vertex set of $G^S$.

**Definition 3.3.** If $D$ is a strong weight dominating vertex set of $G^S$, then the scalar cardinality of $D$, denoted by $|D|_S$, is defined by

$$
|D|_S = \sum_{v \in D} \sigma(v).
$$

**Definition 3.4.** A subset $D \subseteq V$ is said to be a minimal strong weight dominating vertex set of $G^S$ if

1. $D$ is a strong weight dominating vertex set.

2. No proper subset of $D$ is a strong weight dominating vertex set.

**Definition 3.5.** The cardinality of the minimal Strong weight dominating vertex set $D \subseteq V$ is called the Strong weight domination vertex number of $G^S$. It is denoted by $\gamma_{SV}^S(G^S)$. That is

$$
\gamma_{SV}^S(G^S) = (|D|_S, |D|).
$$

**Example 3.6.** Consider the semiring $(S = \{0, a, b, c\}, +, \cdot)$ with canonical pre-order given in Example 3.2.

Consider $G^S = (V, E, \sigma, \psi)$, where

$$
\sigma : V \rightarrow S \text{ by } \sigma(v_1) = \sigma(v_3) = a; \sigma(v_2) = b; \sigma(v_4) = c.
$$

and $\psi : E \rightarrow S$ by

$$
\psi(v_1, v_2) = \psi(v_2, v_3) = b; \psi(v_3, v_4) = \psi(v_1, v_4) = c \text{ and } \psi(v_1, v_3) = a.
$$

From the above figure, In graph $G^S$, $D_1 = \{v_1\}$; $D_2 = \{v_3\}$ and $D_3 = \{v_1, v_3\}$ all weight dominating vertex sets of $G^S$.

$$
\deg_S(v_1) = \{(\sigma(v_2) + \sigma(v_3) + \sigma(v_4), 3)\} = \{(b + a + c, 3)\} = \{(a, 3)\},
$$

$\deg_S(v_2) \preceq \deg_S(v_1)$; $\deg_S(v_3) \preceq \deg_S(v_1)$; $\deg_S(v_4) \preceq \deg_S(v_1)$.  

Similarly

\[
\deg_S(v_3) = \{(\sigma(v_4) + \sigma(v_1) + \sigma(v_2), 3)\} = \{(c + a + b, 3)\},
\]

\[
\deg_S(v_1) \preceq \deg_S(v_3); \quad \deg_S(v_2) \preceq \deg_S(v_3); \quad \deg_S(v_4) \preceq \deg_S(v_3).
\]

Hence \(D_1 = \{v_1\} \quad D_2 = \{v_3\} \quad D_3 = \{v_1, v_3\}\) all Strong weight dominating vertex sets of \(G_S\) with \(|D_1| = 1 = |D_2|\) and \(|D_3| = 2\). Further, the scalar cardinality of \(D_1\) is given by \(|D_1|_S = \sigma(v_1) = a\); \(D_2\) is \(|D_2|_S = \sigma(v_3) = a\) and \(|D_3|_S = (\sigma(v_1) + \sigma(v_3)) = (a + a) = a\).

Thus, the strong domination number of \(G_S\), is given by

\[
\gamma_{SV}^S(G_S) = (|D|_S, |D|) = (a, 1)
\]

**Remark 3.7.** Minimal Strong weight dominating vertex set in a \(S\)-valued graph, \(G_S\), need not be, in general, unique, as seen in Example 3.6.

**Definition 3.8.** A subset \(D \subseteq V\) is said to be a maximal strong weight dominating vertex set of \(G_S\) if

1. \(D\) is a strong weight dominating vertex set, and
2. there is no strong weight dominating vertex subset \(D' \subset V\) such that \(D \subset D' \subset V\).

**Example 3.9.** In Example 3.6 \(D_3 = \{v_1, v_3\}\) is a maximal Strong weight dominating set with \(|D_3| = 2\).

**Remark 3.10.** Maximal strong weight dominating vertex set in a \(S\)-valued graph, \(G_S\), need not be, in general, unique.

**Definition 3.11.** A subset \(D \subseteq V\) is said to be an independent strong weight dominating vertex set if

1. \(D\) is a strong weight dominating vertex set.
2. No two vertices of \(D\) are adjacent.

**Example 3.12.** Consider the semiring \((S = \{0, a, b, c\}, +, \cdot)\) with canonical pre-order given in Example 4.2.
Consider $G^S$:

![Figure 2](image)

where $\sigma : V \to S$ is defined by

$\sigma(v_1) = \sigma(v_3) = \sigma(v_4) = \sigma(v_8) = b; \sigma(v_2) = \sigma(v_6) = a; \sigma(v_5) = c = \sigma(v_7)$,

and $\psi : E \to S$ by

$\psi(v_1, v_2) = \psi(v_1, v_8) = \psi(v_2, v_8) = \psi(v_2, v_6) = \psi(v_3, v_6) = \psi(v_4, v_6) = b$

$\psi(v_2, v_7) = \psi(v_3, v_7) = \psi(v_4, v_5) = \psi(v_5, v_6) = \psi(v_6, v_7) = c$

In graph $G^S$, $D = \{v_2, v_6\}$ is a weight dominating vertex set of $G^S$.

Now

$\deg_S(v_2) = (\sigma(v_1) + \sigma(v_3) + \sigma(v_7) + \sigma(v_8), 4) = (a + b + c + b, 4) = (a, 4)$,

then

$\deg_S(v_1) \leq \deg_S(v_2)$; $\deg_S(v_8) \leq \deg_S(v_2)$;

$\deg_S(v_3) \leq \deg_S(v_2)$; $\deg_S(v_7) \leq \deg_S(v_2)$,

$\deg_S(v_6) = (\sigma(v_3) + \sigma(v_4) + \sigma(v_5) + \sigma(v_7), 4) = (c + b + b + c, 4) = (b, 4)$,

$\deg_S(v_7) \leq \deg_S(v_6)$; $\deg_S(v_3) \leq \deg_S(v_6)$;

$\deg_S(v_4) \leq \deg_S(v_6)$; $\deg_S(v_5) \leq \deg_S(v_6)$.

Hence $D = \{v_2, v_6\}$ is a strong weight dominating vertex set of $G^S$ and also $v_2$ and $v_6$ are not adjacent.

Therefore $D = \{v_2, v_6\}$ is an independent strong weight dominating set of $G^S$. 

Lemma 3.13. A star $G^S$ will have a strong weight dominating vertex set if its pole has the maximum weight.

Proof. Let $G^S$ be a star. Let the pole $\{v_1\}$ have the maximum weight then $\deg_S(v_1)$ is maximum than all the vertices in $G^S$.

Therefore $\{v_1\}$ is a strong weight dominating vertex set of the $S$-valued graph $G^S$.

Lemma 3.14. A wheel $G^S$ will have a strong weight dominating vetrex set if the pole has the maximum weight.

Proof. Let $G^S$ be a wheel. If the pole $\{v_1\}$ have the maximum weight then $\deg_S(v_1)$ is maximum than all other vertices in $G^S$.

Therefore $\{v_1\}$ is a strong weight dominating vertex set.

Theorem 3.15. In a $S$−Star or a $S$−Wheel of $G^S$, the Strong weight dominating vertex set is unique.

Proof. Follows from lemma 5.11 and 5.13.

4. Weak Domination on $S$-Valued Graphs

In this section, we introduce the notion of Weak dominating set of $G^S$ and prove some properties.

Definition 4.1. A subset $D \subseteq V$ is said to be a weak weight dominating vertex set of $G^S$ if

1. $D$ is a weight dominating vertex set.
2. For each vertex $v \in D$, $\deg_S(v) \leq \deg_S(u) \forall u \in N_S[v]$.

Example 4.2. Let $(S = \{0, a, b, c\}, +, \cdot)$ be a semiring with the following Cayley Tables:

\[
\begin{array}{cccc}
+ & 0 & a & b & c \\
0 & 0 & a & b & c \\
a & a & a & a & a \\
b & b & a & b & b \\
c & c & a & b & c \\
\end{array}
\quad \quad
\begin{array}{cccc}
\cdot & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & a & a \\
b & b & 0 & b & b \\
c & 0 & b & b & b \\
\end{array}
\]

Let $\preceq$ be a canonical pre-order in $S$, given by

\[0 \preceq 0, \ 0 \preceq a, \ 0 \preceq b, \ 0 \preceq c, \ a \preceq a, \ b \preceq b, \ b \preceq a, \ c \preceq c, \ c \preceq a, \ c \preceq b.\]
Consider the $S$-valued graph $G^S = (V, E, \sigma, \psi)$.

$\sigma : V \to S$ and $\psi : E \to S$ are respectively defined by

$\sigma(v_1) = a, \sigma(v_2) = b = \sigma(v_4)$ and $\sigma(v_3) = c = \sigma(v_5)$

$\psi(v_1, v_2) = \psi(v_2, v_4) = \psi(v_4, v_5) = b$

and

$\psi(v_1, v_5) = \psi(v_1, v_3) = \psi(v_2, v_5) = \psi(v_3, v_4) = c$

In graph $G^S$, $D = \{v_1, v_4\}$ is a weight dominating set of $G^S$.

Here

$\text{degs}(v_1) = \{(\sigma(v_2) + \sigma(v_3) + \sigma(v_5), 3)\} = \{(b + c + c, 3)\} = \{(b, 3)\}.$

Then $\text{degs}(v_1) \preceq \text{degs}(v_2); \text{degs}(v_1) \preceq \text{degs}(v_3); \text{degs}(v_1) \preceq \text{degs}(v_5)$, and

$\text{degs}(v_4) = \{(\sigma(v_5) + \sigma(v_2) + \sigma(v_3), 3)\} = \{(c + b + c, 3)\} = \{(b, 3)\},$

$\text{degs}(v_4) \preceq \text{degs}(v_5); \text{degs}(v_4) \preceq \text{degs}(v_2); \text{degs}(v_4) \preceq \text{degs}(v_3).$

Here, we receive for the all $\text{degs}(v) \preceq \text{degs}(u)$.

Therefore $D = \{v_1, v_4\}$ is a weak weight dominating vertex set of $G^S$.

**Definition 4.3.** If $D$ is a weak weight dominating vertex set of $G^S$, then the scalar cardinality of $D$, denoted by $|D|_S$, is defined by

$|D|_S = \sum_{v \in D} \sigma(v).$
Definition 4.4. A subset \( D \subseteq V \) is said to be a minimal weak weight dominating vertex set of \( G^S \), if

1. \( D \) is a weak weight dominating vertex set.
2. No proper subset of \( D \) is a weak weight dominating vertex set.

Definition 4.5. The cardinality of the minimal weak weight dominating vertex set \( D \subseteq V \) is called the weak weight domination vertex number of \( G^S \). It is denoted by \( \gamma_{WV}^S(G^S) \). That is,

\[
\gamma_{WV}^S(G^S) = (|D|_{S}, |D|)
\]

Example 4.6. Let \( (S = \{0, a, b, c\}, +, \cdot) \) be a semiring with canonical pre-order given in Example 5.2.
Consider the \( S \)-valued graph \( G^S = (V, E, \sigma, \psi) \).

\[
\begin{align*}
\sigma(v_1) &= \sigma(v_4) = \sigma(v_5) = b; \quad \sigma(v_2) = a \text{ and } \sigma(v_3) = \sigma(v_6) = c, \\
\psi(v_1, v_2) &= \psi(v_4, v_5) = b
\end{align*}
\]

and

\[
\begin{align*}
\psi(v_1, v_3) &= \psi(v_1, v_6) = \psi(v_2, v_3) = \psi(v_3, v_5) = \psi(v_3, v_6) \\
&= \psi(v_3, v_4) = \psi(v_5, v_6) = \psi(v_4, v_5) = c.
\end{align*}
\]

In graph \( G^S \), \( D_1 = \{v_2, v_4, v_5\} \) is a weight dominating vertex set of \( G^S \).
Here
\[ \text{deg}_S(v_2) = (\sigma(v_1) + \sigma(v_3), 2) = (b + c, 2) \leq \text{deg}_S(v_1), \]
and \( \text{deg}_S(v_2) \leq \text{deg}_S(v_3). \)
Similarly,
\[ \text{deg}_S(v_4) = (\sigma(v_3) + \sigma(v_4) + \sigma(v_5), 3) = (c + b + c, 4) = (b, 3). \]
Then
\[ \text{deg}_S(v_4) \leq \text{deg}_S(v_3); \quad \text{deg}_S(v_4) \leq \text{deg}_S(v_5); \quad \text{deg}_S(v_4) \leq \text{deg}_S(v_6). \]
Again, we have
\[ \text{deg}_S(v_5) = (\sigma(v_3), \sigma(v_4) + \sigma(v_6), 2) = (c + b + c, 3) = (b, 3), \]
then \( \text{deg}_S(v_5) \leq \text{deg}_S(v_3); \text{deg}_S(v_5) \leq \text{deg}_S(v_4); \text{deg}_S(v_5) \leq \text{deg}_S(v_6); \) Here all \( \text{deg}_S(v) \leq \text{deg}_S(u). \)
\[ \Rightarrow D_1 = \{v_2, v_4, v_5\} \text{ is a weak dominating vertex set of } G^S. \]
Similarly, \( D_2 = \{v_2, v_4\} \) is a weak dominating vertex set of \( G^S. \)
Moreover, \( D_3 = \{v_2, v_5\} \) is also a weak dominating vertex set of \( G^S. \)
Therefore \( D_2 = \{v_2, v_4\} \) and \( D_3 = \{v_2, v_5\} \) are minimal weak dominating vertex set of \( G^S \) with \( |D_2| = 2 = |D_3| \)

**Definition 4.7.** A set \( D \subseteq V \) is said to be a maximal weak weight dominating vertex set if
1. \( D \) is a weak weight dominating vertex set.
2. \(|D|\) is maximal among all the weak weight dominating vertex set.

**Example 4.8.** In Example 5.4 \( D_1 = \{v_2, v_4, v_5\} \) is a maximal weak dominating vertex set of \( G^S \) with \(|D_1| = 3\).

**Definition 4.9.** A subset \( D \subseteq V \) is said to be an independent weak weight dominating vertex set of \( G^S \) if
1. \( D \) is a weak weight dominating vertex set.
2. No two vertices of \( D \) are adjacent.

**Example 4.10.** Let \((S = \{0, a, b, c\}, +, \cdot)\) be a semiring with canonical pre-order given in Example 4.2.
Consider the $S$-valued graph $G^S = (V, E, \sigma, \psi)$.

In graph $G^S$, $D = \{v_1, v_3\}$ is a weight dominating vertex set of $G^S$.

Here $\text{deg}_S(v_1) = \{(\sigma(v_2) + \sigma(v_5), 2)\} = \{(b + c, 2)\}$, then $\text{deg}_S(v_1) \preceq \text{deg}_S(v_2); \text{deg}_S(v_1) \preceq \text{deg}_S(v_5)$.

Similarly, $\text{deg}_S(v_3) = \{(\sigma(v_4) + \sigma(v_5), 2)\} = \{(c + c, 2)\}$, then $\text{deg}_S(v_3) \preceq \text{deg}_S(v_4); \text{deg}_S(v_3) \preceq \text{deg}_S(v_5)$.

Therefore all $\text{deg}_S(v_i) \preceq \text{deg}_S(u)$. Hence $D = \{v_1, v_3\}$ is a weak weight dominating set. Also $v_1$ and $v_3$ are not adjacent. $\Rightarrow D$ is an independent weak weight dominating set of the $S$-valued graph $G^S$.

**Lemma 4.11.** In a complete bipartite graph $K_{m,n}$ with $V = (V_1, V_2)$, the weight dominating vertex set $V_1$ is strong weight dominating vertex set or weak weight dominating vertex set, if $|V_1| < |V_2|$ or $|V_1| > |V_2|$.

**Proof.** Let $G^S$ be a complete bipartite graph with $V = (V_1, V_2)$. Let $V_1$ be a weight dominating vertex set. If $|V_1| < |V_2|$, then $\text{deg}_S(v_2) \preceq \text{deg}_S(v_1)$ where $v_1 \in V_1$ and $v_2 \in V_2$.

Therefore $V_1$ is a strong weight dominating vertex set.

If $|V_1| > |V_2|$, then $\text{deg}_S(v_1) \preceq \text{deg}_S(v_2)$ where $v_1 \in V_1$ and $v_2 \in V_2$.

Therefore $V_1$ is a weak weight dominating vertex set.

**Lemma 4.12.** In a complete bipartite graph $K_{m,n}$ with $V = (V_1, V_2)$, the weight dominating vertex set $V_1$ is strong weight dominating vertex set as well as weak weight dominating vertex set, if $|V_1| = |V_2|$.

**Proof.** Let $G^S$ be a complete bipartite graph with $V = (V_1, V_2)$. Let $V_1$ be a weight dominating vertex set. If $|V_1| = |V_2|$, then $\text{deg}_S(v_2) \preceq \text{deg}_S(v_1)$ and
\[ \deg_S(v_1) \leq \deg_S(v_2) \] where \( v_1 \in V_1 \) and \( v_2 \in V_2 \).

Therefore \( V_1 \) is both strong and weak weight dominating vertex set.

**Lemma 4.13.** In a tree \( G^S \) a weight dominating vertex set \( V_1 \) will be strong weight dominating vertex set if all the vertices of \( V_1 \) are intermediate vertices. If the vertices of \( V_1 \) are pendent vertices then \( V_1 \) is a weak weight dominating vertex set.

**Proof.** Let \( G^S \) be a tree. Let \( V_1 \subset V \) be a weight dominating vertex set. If all the vertices of \( V_1 \) are intermediate vertices, then for every vertex \( v \in V_1 \), \( \deg_S(u) \leq \deg_S(v) \) where \( u \in N_S[v] \).

Hence \( V_1 \) is a strong weight dominating vertex set. If all the vertices of \( V_1 \) are pendent vertices of, then for every vertex \( v \in V_1 \), \( \deg_S(v) \leq \deg_S(u) \) where \( u \in N_S[v] \).

Therefore \( V_1 \) is a weak weight dominating vertex set.

**References**


