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A FAMILY OF HIGH-ORDER MULTIPLE FINITE DIFFERENCE METHODS FOR THE DIRECT SOLUTION OF THE GENERAL SECOND-ORDER INITIAL VALUE PROBLEM

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Abstract: We derive high-order multiple finite difference methods by an approach based on a combination of interpolation and collocation of an approximate solution at selected interpolation and collocation nodes. Their stability properties are provided.

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1. Introduction

Physical problems can be translated into mathematical equations, such as differential equations, which emerge almost in every branch of knowledge. Most times, these special equations do not have solutions in closed form, where the only information concerning the solution is that it is known to exist and to be unique. In this study, we consider the general second-order ordinary differential equation of the form

$$y''(x) = f(x, y, y'), \quad y(0) = y_0, \quad y'(0) = \delta_0, \tag{1}$$

where f is a continuous function which arises frequently in physical problems.

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It is popular to relax (1) into a system of first-order differential equations and afterwards, diverse methods are employed in obtaining approximate solutions to the system of first-order initial value problems. This technique has considerably been explored and can be seen in [4], [11], [15], [16], [17], [19], [20], [20] and [21] to mention a few. In as much as there has been favorable outcome with this approach, there are some disadvantages related with it. Computer programs involved in this technique are cumbersome particularly when including instructions that perform the task of providing starting values for the method resulting in longer computer time and human effort.

Within the past decade, researchers have been mindful about solving (1) directly without firstly relaxing it to a system of first-order differential equations. With clarity, Lambert [16] specified a pure backing via the study of asymptotic errors which recognizes that the direct application of a particular linear multistep method (Numerov type) to (1) achieves better than the application of a regular linear multistep method to a relaxed identical first-order system. Notably, attention has been given [5], [6], [17], [18], [22], [23], [24], [25] and [27] to solving the special form of (1) set as $y'' = f(x, y)$ which commonly occurs in systems without dissipation.

Diverse schemes have been presented in literature to solve (1) directly without relaxing it to an equivalent first-order system [1], [2], [3], [4], [7], [9] and [26]. Awoyemi's approach [1, 2, 3] in solving (1) directly required that the starting values be generated firstly from the Taylor's series algorithm and then applied as Predictor-Corrector methods. While the methods produced good result and saves computer time, the demerit associated with them is that the Taylor's series algorithm involves higher-order partial derivatives that are overlong. Self-starting methods are simply methods that do not require starting values from some other algorithm to be implemented with, and as such, they do not posses the demerits associated with methods proposed in [1, 2, 3] . Jator and Li [14] proposed a self-starting method of order 5 through the collocation and interpolation method for direct solution to (1). Recently, Jator [12] proposed a three-step seventh order method for the direct solution of the special second order initial value problem of the form $y''(x) = f(x, y)$ with an extension to (1). Jator et al.[13] also proposed a high-order continuous third derivative formula for (1).

Multiple finite difference methods for the numerical integration of initial value problems in ordinary differential equations has been shown to exist in literature. The appeal of formulating newer methods for solving first and higher order IVP's cannot be over emphasized as a result of the need to increase the effectiveness and efficiency of their outputs when applied in solving problems.

In this paper, we extend the results of Jator [14] to (1) by proposing a class of higher-order methods of order 7, 8 and 9 based on a combination of interpolation and collocation of an approximate solution and its second derivative at selected interpolation and collocation nodes. Also, theorems to reinforce the accuracy of the derived methods are established. Numerical evidences are provided to buttress the desirability of the proposed methods.

2. Derivation of the Method

In this section, we apply the interpolation and collocation approach to specify the class of methods that is of interest to us by choosing the accurate number of interpolation points r and collocation points s. The procedure leads to a system of $(r + s)$ equations involving $(r + s)$ unknown coefficient, which are determined by the matrix inversion approach. We consequently approximate the solution of (1) by seeking the continuous method $\bar{y}(x)$ of the form

$$\bar{y}(x) = \sum_{p=0}^{r-1} \alpha_p(x) y_{n+p} + h^2 \sum_{q=0}^{s-1} \beta_q(x) f_{n+q}, \tag{2}$$

where $x \in [a, b]$. Suppose that $k \geq 2$ denotes the step number of the method (2) that is applied to obtain the solution of (1). Taking this into account, we seek a solution on

$$\pi_N : a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots < x_N = b,$$

$$h = x_{n+1} - x_n, \quad n = 0, 1, \cdots, N,$$

where π_N is a partition of $[a, b]$ and h is the constant step size of the partiton of π_N. The number of interpolation points r and the number of distinct collocation points s are chosen to satisfy $2 \leq r \leq k$ and $0 < s \leq k + 1$, respectively. The k-step multistep collocation method of the form (2) is constructed by imposing the conditions

$$\bar{y}(x_{n+p}) = y_{n+p}, \quad p = 0, 1, 2, \ldots, r - 1, \tag{3}$$

$$\bar{y}''(x_{n+q}) = f_{n+q}, \quad q = 0, 1, 2, \ldots, s - 1, \tag{4}$$

Equation (3) assumes that the solution $y(x)$ can be obtained by interpolation at the interpolation points $x = x_{n+p}$ with $p = 0, 1, \ldots, r-1$, likewise equation (4) the differential equation collocated at the collocation points $x = x_{n+q}$

with $q = 0, 1, \ldots, s-1$. Equation (3 - 4) form a system of $(r+s)$ equations in $(r+s)$ unknowns α_p $(p = 0, 1, \ldots, r-1)$ and β_q $(q = 0, 1, \ldots, s-1)$.

Suppose that the approximate solution to (1) is an interpolation polynomial of the form

$$\bar{y}(x) = \sum_{j=0}^{r+s-1} V_j P_j(x). \tag{5}$$

Note that equation (5) can be written as

$$\bar{y}(x) = \sum_{j=0}^{r+s-1} V_j P_j(x) = \sum_{j=0}^{r-1} V_j P_j(x) + \sum_{j=r}^{r+s-1} V_j P_j(x).$$

Consider the sum $\sum_{j=r}^{r+s-1} V_j P_j(x)$ and set $j = u + r$. When $j = r, u = 0$ and also when $j = r + s - 1$, $u = s - 1$. Thus

$$\sum_{j=r}^{r+s-1} V_j P_j(x) = \sum_{u=0}^{s-1} V_{u+r} P_{u+r}(x) = \sum_{j=0}^{s-1} V_{j+r} P_{j+r}(x).$$

We then put (5) in the form,

$$\bar{y}(x) = \sum_{j=0}^{r+s-1} V_j P_j(x) = \sum_{j=0}^{r-1} V_j P_j(x) + \sum_{j=0}^{s-1} V_{j+r} P_{j+r}(x) \tag{6}$$

$$= \sum_{p=0}^{r-1} \alpha_p(x) y_{n+p} + h^2 \sum_{q=0}^{s-1} \beta_q(x) f_{n+q}. \tag{7}$$

Applying the interpolation (3) at x_{n+p} in equation (6) for $p = 0, 1, \ldots, r-1$ gives

$$\bar{y}(x_n) = \sum_{j=0}^{r-1} V_j P_j(x_n) + \sum_{j=0}^{s-1} V_{j+r} P_{j+r}(x_n) = y_n, \tag{8}$$

$$\bar{y}(x_{n+1}) = \sum_{j=0}^{r-1} V_j P_j(x_{n+1}) + \sum_{j=0}^{s-1} V_{j+r} P_{j+r}(x_{n+1}) = y_{n+1}, \tag{9}$$

$$\bar{y}(x_{n+2}) = \sum_{j=0}^{r-1} V_j P_j(x_{n+2}) + \sum_{j=0}^{s-1} V_{j+r} P_{j+r}(x_{n+2}) = y_{n+2}, \tag{10}$$

$$\vdots \quad = \quad \vdots \tag{11}$$

$$\bar{y}(x_{n+r-1}) = \sum_{j=0}^{r-1} V_j P_j(x_{n+r-1}) + \sum_{j=0}^{s-1} V_{j+r} P_{j+r}(x_{n+r-1}) = y_{n+r-1}. \tag{12}$$

Note that differentiating equation (6) gives

$$\bar{y}''(x) = \sum_{j=0}^{r+s-1} V_j P_j''(x) = \sum_{j=0}^{r-1} V_j P_j''(x) + \sum_{j=0}^{s-1} V_{j+r} P_{j+r}''(x) = f(x, y, y'). \tag{13}$$

Applying equation (4) at x_{n+q} in equation (13) for $q = 0, 1, \ldots, s-1$ gives

$$\bar{y}''(x_n) = \sum_{j=0}^{r-1} V_j P_j''(x_n) + \sum_{j=0}^{s-1} V_{j+r} P_{j+r}''(x_n) = f_n, \tag{14}$$

$$\bar{y}''(x_{n+1}) = \sum_{j=0}^{r-1} V_j P_j''(x_{n+1}) + \sum_{j=0}^{s-1} V_{j+r} P_{j+r}''(x_{n+1}) = f_{n+1}, \tag{15}$$

$$\bar{y}''(x_{n+2}) = \sum_{j=0}^{r-1} V_j P_j''(x_{n+2}) + \sum_{j=0}^{s-1} V_{j+r} P_{j+r}''(x_{n+2}) = f_{n+2}, \tag{16}$$

$$\vdots$$

$$\bar{y}''(x_{n+s-1}) = \sum_{j=0}^{r-1} V_j P_j''(x_{n+s-1}) + \sum_{j=0}^{s-1} V_{j+r} P_{j+r}''(x_{n+s-1}) = f_{n+s-1}. \tag{17}$$

The above equations form a list of $(r+s)$ equations in $(r+s)$ unknowns coefficients

$$V_0, V_1, V_2, V_3, \ldots\ldots, V_{r+s-1}.$$

In matrix form, the system of equations can be written as

$$(\sqcup) \wedge^{=} \Upsilon,$$

where

$$\sqcup =$$

$$\begin{bmatrix} P_0(x_n) & P_1(x_n) & \cdots & P_{r-1}(x_n) & P_r(x_n) & \cdots & P_{r+s-1}(x_n) \\ P_0(x_{n+1}) & P_1(x_{n+1}) & \cdots & P_{r-1}(x_{n+1}) & P_r(x_{n+1}) & \cdots & P_{r+s-1}(x_{n+1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ P_0(x_{n+r-1}) & P_1(x_{n+r-1}) & \cdots & P_{r-1}(x_{n+r-1}) & P_r(x_{n+r-1}) & \cdots & P_{r+s-1}(x_{n+r-1}) \\ P_0''(x_n) & P_1''(x_n) & \cdots & P_{r-1}''(x_n) & P_r''(x_n) & \cdots & P_{r+s-1}''(x_n) \\ P_0''(x_{n+1}) & P_1''(x_{n+1}) & \cdots & P_{r-1}''(x_{n+1}) & P_r''(x_{n+1}) & \cdots & P_{r+s-1}''(x_{n+1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ P_0''(x_{n+s-1}) & P_1''(x_{n+s-1}) & \cdots & P_{r-1}''(x_{n+s-1}) & P_r''(x_{n+s-1}) & \cdots & P_{r+s-1}''(x_{n+s-1}) \end{bmatrix},$$

$$\wedge = \left[V_0, V_1, \cdots, V_{r-1}, V_r, \cdots, \cdots, V_{r+s-1}\right]^T$$

and

$$\Upsilon = \left[y_n, y_{n+1}, \cdots, y_{n+r-1}, f_n \cdots, \cdots, f_{n+s-1}\right]^T.$$

It is vital to note that $P_j(x) = x^j$ and $P_j''(x) = j(j-1)x^{j-2}$, consequently $P_j(x_n) = x_n^j$ and $P_j''(x_n) = j(j-1)x_n^{j-2}$. We also made the substitution $x_n = x_0 + nh$, in the system of equations.The solution to the above matrix system gives $V_0, V_1, V_2, \ldots, V_{r+s-1}$ which are then substituted back into (5) and (13) to obtain any member k of the class of method.

3. Particular of Methods

In this section, we use (6) to obtain particular MFDMs by specifying r , s and k. We highlight that the main method are derived by evaluating (6) at $x = x_{n+p}$. We review details of specific methods next.

Case $k = 6$. Choosing $r = 2$, $s = 7$ and $k = 6$, and evaluating (6) at $x = x_{n+p}$, $p = r, r+1, r+2, \cdots k$, that is at $x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}$ and x_{n+6}, we generate the following main method and two additional methods.

$$\begin{aligned} y_{n+2} =& \frac{h^2}{60480}(4315f_n + 53994f_{n+1} - 2307f_{n+2} + 7948f_{n+3} \\ & - 4827f_{n+4} + 1578f_{n+5} - 221f_{n+6}) - y_n + 2y_{n+1}, \qquad (18) \\ y_{n+3} =& \frac{h^2}{20160}(2803f_n + 37950f_{n+1} + 14913f_{n+2} + 7108f_{n+3} \\ & - 3147f_{n+4} + 990f_{n+5} - 137f_{n+6}) - 2y_n + 3y_{n+1}, \qquad (19) \\ y_{n+4} =& \frac{h^2}{10080}(2089f_n + 28878f_{n+1} + 16383f_{n+2} + 13828f_{n+3} \\ & - 1257f_{n+4} + 654f_{n+5} - 95f_{n+6}) - 3y_n + 4y_{n+1}, \qquad (20) \end{aligned}$$

$$\begin{aligned} y_{n+5} =& \frac{h^2}{6048}(1669f_n + 23250f_{n+1} + 15207f_{n+2} + 15004f_{n+3} \\ &+ 4371f_{n+4} + 1074f_{n+5} - 95f_{n+6}) - 4y_n + 5y_{n+1}, \quad (21) \\ y_{n+6} =& \frac{h^2}{4032}(1375f_n + 19554f_{n+1} + 13401f_{n+2} + 15004f_{n+3} \\ &+ 6177f_{n+4} + 4770f_{n+5} + 199f_{n+6}) - 5y_n + 6y_{n+1}. \quad (22) \end{aligned}$$

For the reason that the MFDM's are designed to simultaneously provide the values of the five unknowns: $y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}$ and y_{n+6} for the solution of (1) at a block of points $x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}$ and x_{n+6}, additional equations are required. These additional equations and derivatives can be obtained by evaluating the derivative of (5) given as

$$\bar{y}(x)' = \sum_{j=0}^{r+s-1} V_j P_j'(x) = \delta(x), \quad \bar{y}(x_0)' = \delta_0. \quad (23)$$

To start the IVP, the initial conditions at $x = 0$ and (with $n = 0$) are imposed on (23) to get

$$\begin{aligned} \delta_0 h = -\frac{h^2}{120960}&(28549f_0 + 57750f_1 - 51453f_2 + 42484f_3 \\ &- 23109f_4 + 7254f_5 - 995f_6) - y_0 + y_1. \quad (24) \end{aligned}$$

We evaluate (23) at x_{n+p} , $p = r+1, r+2, \cdots k$ to obtain $\bar{y}'(x_{n+p}) = \delta_{n+p}$. Thus, we have for the derivatives

$$\begin{aligned} h\delta_{n+1} =& \frac{h^2}{120960}(9625f_n + 72474f_{n+1} - 41469f_{n+2} + 32524f_{n+3} \\ &- 17313f_{n+4} + 5370f_{n+5} - 731f_{n+6}) - y_n + y_{n+1}, \\ h\delta_{n+2} =& \frac{h^2}{40320}(2633f_n + 40910f_{n+1} + 17503f_{n+2} + 4f_{n+3} \\ &- 905f_{n+4} + 398f_{n+5} - 63f_{n+6}) - y_n + y_{n+1}, \\ h\delta_{n+3} =& \frac{h^2}{120960}(8441f_n + 117210f_{n+1} + 114147f_{n+2} + 75020f_{n+3} \\ &- 16257f_{n+4} + 4410f_{n+5} - 571f_{n+6}) - y_n + y_{n+1}, \\ h\delta_{n+4} =& \frac{h^2}{120960}(8059f_n + 120426f_{n+1} + 100605f_{n+2} + 150028f_{n+3} \\ &+ 45381f_{n+4} - 1110f_{n+5} - 29f_{n+6}) - y_n + y_{n+1}, \end{aligned}$$

$$\begin{aligned}
h\delta_{n+5} =& \frac{h^2}{40320}(2867f_n + 38750f_{n+1} + 38401f_{n+2} + 39172f_{n+3} \\
& + 46453f_{n+4} + 16382f_{n+5} - 585f_{n+6}) - y_n + y_{n+1}, \\
h\delta_{n+6} =& \frac{h^2}{120960}(6875f_n + 128874f_{n+1} + 74781f_{n+2} + 192524f_{n+3} \\
& + 46437f_{n+4} + 179370f_{n+5} + 36419f_{n+6}) - y_n + y_{n+1}.
\end{aligned}$$

Case $k = 7$. Choosing $r = 2$, $s = 8$ and $k = 7$, and evaluating (6) at $x = x_{n+p}$, $p = r, r+1, r+2, \cdots k$, we generate the following methods.

$$\begin{aligned}
y_{n+2} =& \frac{h^2}{60480}(4125f_n + 55324f_{n+1} - 6297f_{n+2} + 14598f_{n+3} \\
& - 11477f_{n+4} + 5568f_{n+5} - 1551f_{n+6} + 190f_{n+7}) - y_n + 2y_{n+1}, \quad (25)\\
y_{n+3} =& \frac{h^2}{60480}(8060f_n + 116293f_{n+1} + 37410f_{n+2} + 33539f_{n+3} - \\
& 21656f_{n+4} + 10299f_{n+5} - 2854f_{n+6} + 349f_{n+7}) - \\
& 2y_n + 3y_{n+1}, \quad (26)\\
y_{n+4} =& \frac{h^2}{30240}(6013f_n + 88412f_{n+1} + 43815f_{n+2} + 50374f_{n+3} \\
& - 12661f_{n+4} + 7296f_{n+5} - 2063f_{n+6} + 254f_{n+7}) \\
& - 3y_n + 4y_{n+1}, \quad (27)\\
y_{n+5} =& \frac{h^2}{30240}(7996f_n + 118693f_{n+1} + 68706f_{n+2} + 87235f_{n+3} \\
& + 9640f_{n+4} + 12699f_{n+5} - 2918f_{n+6} + 349f_{n+7}) \\
& - 4y_n + 5y_{n+1}, \quad (28)\\
y_{n+6} =& \frac{h^2}{60480}(19927f_n + 298196f_{n+1} + 186357f_{n+2} + 249490f_{n+3} \\
& + 68225f_{n+4} + 86208f_{n+5} - 1901f_{n+6} + 698f_{n+7}) \\
& - 5y_n + 6y_{n+1}, \quad (29)\\
y_{n+7} =& \frac{h^2}{8640}(3436f_n + 51065f_{n+1} + 34410f_{n+2} + 44719f_{n+3} \\
& + 18824f_{n+4} + 20103f_{n+5} + 8194f_{n+6} + 689f_{n+7}) \\
& - 6y_n + 7y_{n+1}. \quad (30)
\end{aligned}$$

To start the IVP, the initial conditions at $x = 0$ and (with $n = 0$) are imposed on (23) to get

$$\delta_0 h = -\frac{h^2}{1814400}(416173f_0 + 950684f_1 - 1025097f_2 + 1059430f_3 \\ - 768805f_4 + 362112f_5 - 99359f_6 + 12062f_7) \\ - y_0 + y_1. \tag{31}$$

We also evaluate (23) at x_{n+p}, $p = r+1, r+2, \cdots k$, to obtain $\bar{y}^{\prime}(x_{n+p}) = \delta_{n+p}$. Thus, we have for the derivatives

$$h\delta_{n+1} = \frac{h^2}{1814400}(135812f_n + 1147051f_{n+1} - 801858f_{n+2} + 787565 \\ f_{n+3} - 559400f_{n+4} + 260373f_{n+5} - 70906f_{n+6} + 8563f_{n+7}) \\ - y_n + y_{n+1},$$

$$h\delta_{n+2} = \frac{h^2}{1814400}(115187f_n + 1864036f_{n+1} + 718377f_{n+2} + \\ 115610f_{n+3} - 156155f_{n+4} + 87168f_{n+5} - 25921f_{n+6} + \\ 3298f_{n+7})y_n + y_{n+1},$$

$$h\delta_{n+3} = \frac{h^2}{1814400}(120452f_n + 1801291f_{n+1} + 1582782f_{n+2} + \\ 1341005f_{n+3} - 459560f_{n+4} + 195573f_{n+5} - 51706f_{n+6} + \\ 6163f_{n+7}) - y_n + y_{n+1},$$

$$h\delta_{n+4} = \frac{h^2}{1814400}(117587f_n + 1829476f_{n+1} + 1439817f_{n+2} + \\ 2365850f_{n+3} + 565285f_{n+4} + 52608f_{n+5} - 23521f_{n+6} + \\ 3298f_{n+7}) - y_n + y_{n+1},$$

$$h\delta_{n+5} = \frac{h^2}{1814400}(120452f_n + 1803691f_{n+1} + 1548222f_{n+2} + \\ 2062445f_{n+3} + 1790680f_{n+4} + 917013f_{n+5} - 86266f_{n+6} + \\ 8563f_{n+7}) - y_n + y_{n+1},$$

$$h\delta_{n+6} = \frac{h^2}{1814400}(115187f_n + 1848676f_{n+1} + 1375017f_{n+2} + \\ 2465690f_{n+3} + 1118725f_{n+4} + 2437248f_{n+5} + 630719f_{n+6} - \\ 12062f_{n+7}) - y_n + y_{n+1},$$

$$h\delta_{n+7} = \frac{h^2}{1814400}(135812f_n + 1678411f_{n+1} + 1997502f_{n+2} + \\ 1137485f_{n+3} + 2965720f_{n+4} + 610293f_{n+5} + 2728454f_{n+6} +$$

$539923 f_{n+7}) - y_n + y_{n+1}.$

Case $k = 8$. Choosing $r = 2$, $s = 9$ and $k = 8$, and evaluating (6) at $x = x_{n+p}$, where $p = r, r+1, r+2, \cdots k$, we generate the following main method and two additional methods.

$$y_{n+2} = \frac{h^2}{3628800}(237671 f_n + 3398072 f_{n+1} - 653032 f_{n+2} + 1426304 f_{n+3} - 1376650 f_{n+4} + 884504 f_{n+5} - 368272 f_{n+6} + 90032 f_{n+7} - 9829 f_{n+8}) - y_n + 2y_{n+1}, \quad (32)$$

$$y_{n+3} = \frac{h^2}{1209600}(155171 f_n + 2374092 f_{n+1} + 579388 f_{n+2} + 1008404 f_{n+3} - 855150 f_{n+4} + 543604 f_{n+5} - 225892 f_{n+6} + 55212 f_{n+7} - 6029 f_{n+8}) - 2y_n + 3y_{n+1}, \quad (33)$$

$$y_{n+4} = \frac{h^2}{604800}(115821 f_n + 1803752 f_{n+1} + 752008 f_{n+2} + 1256064 f_{n+3} - 563950 f_{n+4} + 394504 f_{n+5} - 165552 f_{n+6} + 40592 f_{n+7} - 4439 f_{n+8}) - 3y_n + 4y_{n+1}, \quad (34)$$

$$y_{n+5} = \frac{h^2}{362880}(92405 f_n + 2(726346 f_{n+1} + 362578 f_{n+2} + 622726 f_{n+3} - 66305 f_{n+4} + 175510 f_{n+5} - 67166 f_{n+6} + 16282 f_{n+7}) - 3547 f_{n+8}) - 4y_n + 5y_{n+1}, \quad (35)$$

$$y_{n+6} = \frac{h^2}{241920}(76859 f_n + 1215576 f_{n+1} + 665656 f_{n+2} + 1157504 f_{n+3} + 73470 f_{n+4} + 504376 f_{n+5} - 87376 f_{n+6} + 25584 f_{n+7} - 2849 f_{n+8}) - 5y_n + 6y_{n+1}, \quad (36)$$

$$y_{n+7} = \frac{h^2}{172800}(65871 f_n + 1044092 f_{n+1} + 608428 f_{n+2} + 1053924 f_{n+3} + 177050 f_{n+4} + 561604 f_{n+5} + 84108 f_{n+6} + 36572 f_{n+7} - 2849 f_{n+8}) - 6y_n + 7y_{n+1}, \quad (37)$$

$$y_{n+8} = \frac{h^2}{129600}(57281 f_n + 918152 f_{n+1} + 542888 f_{n+2} + 992384 f_{n+3} + 177050 f_{n+4} + 623144 f_{n+5} + 149648 f_{n+6} + 162512 f_{n+7} + 5741 f_{n+8}) - 7y_n + 8y_{n+1}, \quad (38)$$

To start the IVP the initial conditions at $x = 0$ (with $n = 0$) are imposed on (23) to obtain

$$\begin{aligned}\delta_0 h = &-\frac{h^2}{7257600}(1624505f_0 + 4124232f_1 - 5225624f_2 + 6488192f_3 \\ &- 5888310f_4 + 3698920f_5 - 1522672f_6 + 369744f_7 - 40187f_8) \\ &- y_0 + y_1. \end{aligned} \tag{39}$$

Additional derivatives are obtained by evaluating (23) at $x_{n+p},\ p = r + 1, r+2, \cdots k$ to obtain $\bar{y}'(x_{n+p}) = \delta_{n+p}$. Thus, we have for the derivatives

$$\begin{aligned}
h\delta_{n+1} =&\frac{h^2}{7257600}(515529f_n + 4809956f_{n+1} - 3983564f_{n+2}+ \\
&4702524f_{n+3} - 4177930f_{n+4} + 2593756f_{n+5} - 1059756f_{n+6} \\
&+ 256004f_{n+7} - 27719f_{n+8}) - y_n + y_{n+1}, \\
h\delta_{n+2} =&\frac{h^2}{7257600}(447623f_n + 7561144f_{n+1} + 2(1253004f_{n+2} \\
&+ 598720f_{n+3} - 771685f_{n+4} + 541836f_{n+5} - 235592f_{n+6}+ \\
&59096f_{n+7}) - 13125f_{n+8}) - y_n + y_{n+1}, \\
h\delta_{n+3} =&\frac{h^2}{7257600}(462217f_n + 7361892f_{n+1} + 5782580f_{n+2} \\
&+ 6461116f_{n+3} - 3209610f_{n+4} + 1879388f_{n+5} - 755372f_{n+6} \\
&+ 181380f_{n+7} - 19591f_{n+8}) - y_n + y_{n+1}, \\
h\delta_{n+4} =&\frac{h^2}{7257600}(455751f_n + 7434680f_{n+1} + 2(2675276f_{n+2}+ \\
&5140416f_{n+3} + 619675f_{n+4} + 513932f_{n+5} - 251400f_{n+6} \\
&+ 64984f_{n+7}) - 14597f_{n+8}) - y_n + y_{n+1}, \\
h\delta_{n+5} =&\frac{h^2}{7257600}(460745f_n + 7383268f_{n+1} + 5603124f_{n+2} \\
&+ 9429308f_{n+3} + 5688310f_{n+4} + 4847580f_{n+5} - 934828f_{n+6} \\
&+ 202756f_{n+7} - 21063f_{n+8}) - y_n + y_{n+1}, \\
h\delta_{n+6} =&\frac{h^2}{7257600}(454279f_n + 7446456f_{n+1} + 5318936f_{n+2}+ \\
&10225024f_{n+3} + 4022070f_{n+4} + 10111256f_{n+5} + 2341744f_{n+6} \\
&+ 3504f_{n+7} - 6469f_{n+8}) - y_n + y_{n+1}, \\
h\delta_{n+7} =&\frac{h^2}{7257600}(468873f_n + 7308644f_{n+1} + 5907508f_{n+2}+ \\
&8714940f_{n+3} + 6656630f_{n+4} + 6606172f_{n+5} + 8831316f_{n+6}
\end{aligned}$$

$$+ 2754692f_{n+7} - 74375f_{n+8}) - y_n + y_{n+1},$$

$$h\delta_{n+8} = \frac{h^2}{7257600}(400967f_n + 7934392f_{n+1} + 3325080f_{n+2} + 15007616f_{n+3} - 3409610f_{n+4} + 17796888f_{n+5} - 377872f_{n+6} + 11688880f_{n+7} + 2065659f_{n+8}) - y_n + y_{n+1}.$$

Any member of the class of derived MFDM are applied to simultaneously provide values for $y_1, \cdots, y_k$, where $k = Max[r-1, s-1]$. In general, for a k-step method, the initial conditions are required at x_{n+k}, $n = 0, k, 2k, 3k, \ldots, N-k$, using the computed values $y(x_{n+k}) = y_{n+k}$ and $\delta(x_{n+k}) = \delta_{n+k}$ over subintervals

$$[x_0, x_k], [x_k, x_{2k}], \ldots, [x_{N-k}, x_N]$$

which do not overlap.

4. Analysis of the Method

4.1. Order of Accuracy

In this section, we amplify the concepts in the linear difference operator associated with the formulation of the methods defined by

$$L[y(x); h] = \sum_{j=0}^{k}[\alpha_j y(x + jh) - h^2\beta_j y^{''}(x + jh)]. \tag{40}$$

Expanding $y(x+jh)$ and $y^{''}(x+jh)$ as Taylor's series about x and colleting like terms in (40) gives

$$L[y(x); h] = C_0 y(x) + C_1 h y^{'}(x) + \cdots + C_p h^p y^p + \cdots$$

where

$$C_0 = \sum_{j=0}^{k}\alpha_j, C_1 = \sum_{j=0}^{k}(j\alpha_j),$$

$$C_2 = \sum_{j=0}^{k}(\frac{1}{2!}j^2\alpha_j - \beta_j),$$

$$C_q = \frac{1}{q!}\sum_{j=0}^{k}(j^q\alpha_j - \frac{1}{(q-2)!}j^{q-2}\beta_j).$$

Also by amplifying the linear difference operators associated with the formulation of the starters (26)(33) and(41) we obtain

$$L[y(x);h]=\sum_{j=0}^{k}[\alpha_j y(x+jh)-h\gamma_j y^{'}(x+jh)-h^2\beta_j y^{''}(x+jh)],$$

and applying the same concept, the order for the starters (26),(36)and (50) are obtained from $C_0=\sum_{j=0}^{k}(\alpha_j)$, $C_1=\sum_{j=0}^{k}(j\alpha_j-\gamma_j)$, $C_2=\sum_{j=0}^{k}(j^2\frac{1}{2!}\alpha_j-j\gamma_j-\beta_j),\cdots,C_q=\sum_{j=0}^{k}(j^q\frac{1}{q!}\alpha_j-j^{q-1}\frac{1}{(q-1)!}\gamma_j-j^{q-2}\frac{1}{(q-2)!}\beta_j)$.

Definition 1. The family of methods are said to be of order p if C_0, $C_1,\cdots,C_p=0$ but $C_{p+2}\neq 0$ and C_{p+2} is called the error constant.

Therefore, C_{p+2} is the error constant with $C_{p+2}h^{p+2}y^{p+2}(x_n)$ as the principal part of the local truncation error [LTE]. Below is a display of the orders and error constants of the self starting methods.

$K=6$: Order $p=(7,7,7,7,7,7)$. Error Constant

$$C_{p+2}=(\frac{19}{6048},\frac{349}{60480},\frac{127}{15120},\frac{349}{30240},\frac{349}{30240},\frac{6031}{907200})^T.$$

$K=7$: Order $p=(8,8,8,8,8,8,8)$. Error Constant

$$C_{p+2}=(-\frac{9829}{3628800},-\frac{6029}{1209600},-\frac{4439}{604800},-\frac{3547}{362880},-\frac{407}{34560},-\frac{2849}{172800},-\frac{5741}{1036800})^T.$$

$K=8$: Order $p=(9,9,9,9,9,9,9,9)$. Error Constant

$$C_{p+2}=(\frac{407}{172800},\frac{3953}{907200},\frac{1669}{259200},\frac{773}{90720},\frac{1091}{103680},\frac{1669}{129600},\frac{1669}{1296000},\frac{1129981}{239500800})^T.$$

4.2. Stability Analysis

We also rewrite the class of k-step methods as a matrix finite difference equation of the form

$$\sum_{j=0}^{n+1}A_k^{(j)}Y_{z+j}+h^2\sum_{j=0}^{n+1}B_k^{(j)}F_{z+j}+h\sum_{j=0}^{n+1}C_k^{(j)}\delta_{z+j}=0, \quad (41)$$

where

$$Y_{z+1} = (y_{n+1}, \cdots, y_{n+k-1}, y_{n+k})^T,$$

$$Y_z = (y_{n-k+1}, \cdots, y_{n-1}, y_n)^T,$$

$$F_{z+1} = (f_{n+1}, \cdots, f_{n+k-1}, f_{n+k})^T,$$

$$F_z = (f_{n-k+1}, \cdots, \delta_{n-1}, \delta_n)^T,$$

$$\delta_z = (\delta_{n-k+1}, \cdots, \delta_{n-1}, \delta_n)^T.$$

Also, where $A_k^{(1)}, A_k^{(0)}, B_k^{(1)}, B_k^{(0)}, C_k^{(0)}$ and the null $C_k^{(1)}$ are $N \times N$ matrices of the same dimension for each member k of the class of methods.

4.2.1. Zero Stability

Zero stability is concerned with the stability of the difference system (41) in the limit as $h \to 0$. Thus as $h \to 0$, our class of methods tend to the difference system

$$\sum_{j=0}^{n+1} A_k^{(j)} Y_{z+j} = 0, \tag{42}$$

where

$$A_6^{(1)} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 & 0 \\ -5 & 0 & 0 & 0 & 1 & 0 \\ -6 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_6^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_7^{(1)} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ -5 & 0 & 0 & 0 & 1 & 0 & 0 \\ -6 & 0 & 0 & 0 & 0 & 1 & 0 \\ -7 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_7^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_8^{(1)} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -7 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -8 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } A_8^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Definition 2. A first characteristic polynomial is defined as

$$\rho(R) = DET[RA_k^{(1)} - A_k^{(0)}]. \tag{43}$$

Definition 3. A Multiple Finite Difference Method is said to be zero stable if the roots $R_j, j = 0(1)k$ of the first characteristic polynomial $\rho(R)$ satisfies $\mid R_j \mid \leq 1$ and for roots with $\mid R_j \mid = 1$ the multiplicity must not exceed two.

Hence the first characteristic polynomials for our class of methods is:

For $r = 2$ and $s = 7$:

$$\rho(R) = R^5(1 + r) = 0, R_1\,, R_2,\ R_3,\ R_4,\ R_5 = 0,\ \mid R_6 \mid = 1.$$

For $r = 2$ and $s = 8$:

$$\rho(R) = -R^6(1 + R) = 0, R_1\,, R_2,\ R_3,\ R_4,\ R_5 =,\ R_6 = 0,\ \mid R_7 \mid = 1.$$

For $r = 2$ and $s = 9$:

$$\rho(R) = R^7(R + 1) = 0,\ R_1,\ R_2,\ R_3,\ R_4,\ R_5,\ R_6,\ R_7 = 0\,, \mid R_8 \mid = 1.$$

Hence the class of method's are zero stable.

4.2.2. Region of Absolute Stability

We obtain the region of absolute stability by simplifying (41). The $N \times N$ matrices $B_k^{(1)}, B_k^{(0)}, C_k^{(0)}$ and $C_k^{(1)}$ are given as

$$B_6^{(1)} = \begin{pmatrix} \frac{8999}{10080} & -\frac{769}{20160} & \frac{1987}{15120} & -\frac{1609}{20160} & \frac{263}{10080} & -\frac{221}{60480} \\ \frac{1265}{672} & \frac{1657}{2240} & \frac{1777}{5040} & -\frac{1049}{6720} & \frac{11}{224} & -\frac{137}{20160} \\ \frac{4813}{1680} & \frac{5461}{3360} & \frac{3457}{2520} & -\frac{419}{3360} & \frac{109}{1680} & -\frac{19}{2016} \\ \frac{3875}{1008} & \frac{5069}{2016} & \frac{3751}{1512} & \frac{1457}{2016} & \frac{179}{1008} & -\frac{95}{6048} \\ \frac{3259}{672} & \frac{1489}{448} & \frac{3751}{1008} & \frac{2059}{1344} & \frac{265}{224} & \frac{199}{4032} \\ -\frac{275}{576} & \frac{5717}{13440} & -\frac{10621}{30240} & \frac{7703}{40320} & -\frac{403}{6720} & \frac{199}{24192} \end{pmatrix},$$

$$B_6^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{863}{12096} \\ 0 & 0 & 0 & 0 & 0 & \frac{2803}{20160} \\ 0 & 0 & 0 & 0 & 0 & \frac{2089}{10080} \\ 0 & 0 & 0 & 0 & 0 & \frac{1669}{6048} \\ 0 & 0 & 0 & 0 & 0 & \frac{1375}{4032} \\ 0 & 0 & 0 & 0 & 0 & -\frac{28549}{120960} \end{pmatrix}, \; C_6^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$B_7^{(1)} = \begin{pmatrix} \frac{13831}{15120} & -\frac{2099}{20160} & \frac{811}{3360} & -\frac{11477}{60480} & \frac{29}{315} & -\frac{517}{20160} & \frac{19}{6048} \\ \frac{116293}{60480} & \frac{1247}{2016} & \frac{33539}{60480} & -\frac{2707}{7560} & \frac{3433}{20160} & -\frac{1427}{30240} & \frac{349}{60480} \\ \frac{22103}{7560} & \frac{2921}{2016} & \frac{25187}{15120} & -\frac{12661}{30240} & \frac{76}{315} & -\frac{2063}{30240} & \frac{127}{15120} \\ \frac{118693}{30240} & \frac{3817}{1680} & \frac{17447}{6048} & \frac{241}{756} & \frac{1411}{3360} & -\frac{1459}{15120} & \frac{349}{30240} \\ \frac{74549}{15120} & \frac{62119}{20160} & \frac{24949}{6048} & \frac{13645}{12096} & \frac{449}{315} & -\frac{1901}{60480} & \frac{349}{30240} \\ \frac{10213}{1728} & \frac{1147}{288} & \frac{44719}{8640} & \frac{2353}{1080} & \frac{6701}{2880} & \frac{4097}{4320} & \frac{689}{8640} \\ -\frac{33953}{64800} & \frac{341699}{604800} & -\frac{105943}{181440} & \frac{153761}{362880} & -\frac{943}{4725} & \frac{99359}{1814400} & -\frac{6031}{907200} \end{pmatrix},$$

$$B_7^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{275}{4032} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{403}{3024} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{859}{4320} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1999}{7560} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{19927}{60480} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{859}{2160} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{416173}{1814400} \end{pmatrix}, \; C_7^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$B_8^{(1)} = \begin{pmatrix} \frac{424759}{453600} & -\frac{81629}{453600} & \frac{11143}{28350} & -\frac{27533}{72576} & \frac{110563}{453600} & -\frac{23017}{226800} & \frac{5627}{226800} & -\frac{9829}{3628800} \\ \frac{9421}{4800} & \frac{144847}{302400} & \frac{252101}{302400} & -\frac{5701}{8064} & \frac{135901}{302400} & -\frac{56473}{302400} & \frac{4601}{100800} & -\frac{6029}{1209600} \\ \frac{225469}{75600} & \frac{94001}{75600} & \frac{3271}{1575} & -\frac{11279}{12096} & \frac{49313}{75600} & -\frac{3449}{12600} & \frac{2537}{37800} & -\frac{4439}{604800} \\ \frac{363173}{90720} & \frac{181289}{90720} & \frac{311363}{90720} & -\frac{13261}{36288} & \frac{17551}{18144} & -\frac{33583}{90720} & \frac{1163}{12960} & -\frac{3547}{362880} \\ \frac{16883}{3360} & \frac{83207}{30240} & \frac{9043}{1890} & \frac{2449}{8064} & \frac{63047}{30240} & -\frac{5461}{15120} & \frac{533}{5040} & -\frac{407}{34560} \\ \frac{261023}{43200} & \frac{152107}{43200} & \frac{87827}{14400} & \frac{3541}{3456} & \frac{140401}{43200} & \frac{7009}{14400} & \frac{9143}{43200} & -\frac{2849}{172800} \\ \frac{114769}{16200} & \frac{67861}{16200} & \frac{15506}{2025} & \frac{3541}{2592} & \frac{77893}{16200} & \frac{9353}{8100} & \frac{10157}{8100} & \frac{5741}{129600} \\ -\frac{8183}{14400} & \frac{653203}{907200} & -\frac{50689}{56700} & \frac{196277}{241920} & -\frac{92473}{181440} & \frac{95167}{453600} & -\frac{7703}{151200} & \frac{5741}{1036800} \end{pmatrix},$$

$$B_8^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{33953}{518400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{155171}{1209600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{12869}{67200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{18481}{72576} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{76859}{241920} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7319}{19200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{57281}{129600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{324901}{1451520} \end{pmatrix}$$

and

$$C_8^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

We proceed for the class of methods in the spirit of [7] and [12] by linearizing and leads to

$$y' = \lambda y, \tag{44}$$

for which

$$y'' = \lambda^2 y. \tag{45}$$

Equations (44) and (45) inserted in (41) yields

$$\sum_{j=0}^{k} (A^{(j)} + h^2\lambda^2 B^{(j)} + h\lambda C^{(j)}) Y_{z+j} = 0. \tag{46}$$

To obtain the characteristic polynomial, we make the substitution $m = \lambda h$ and $Y_{z+j} = w^j$,say, in (46) which then gives

$$\sum_{j=0}^{k} (A^{(j)} + m^2 B^{(j)} + mC^{(j)}) w^j = 0. \tag{47}$$

Equation (47) is, for $w^j = e^{j\theta}$, a polynomial which gives rise to root locus curve which, together, describe the stability domain.

Hence, the stability polynomials for $r = 2$ $s = 7$ gives

$$\begin{aligned} \pi_7(m,w) = {} & \frac{m^{12}w^6}{196} - \frac{157m^{12}w^5}{1960} - \frac{167m^{11}w^5}{400} - \frac{131m^{10}w^6}{5600} \\ & + \frac{9973m^{10}w^5}{5600} + \frac{3167m^9w^5}{600} + \frac{89m^8w^6}{6300} - \frac{44993m^8w^5}{3600} - \frac{7961m^7w^5}{336} \\ & + \frac{67m^6w^6}{2016} + \frac{73265m^6w^5}{2016} + \frac{1793m^5w^5}{40} \\ & + \frac{41m^4w^6}{240} - \frac{10481m^4w^5}{240} - \frac{65m^3w^5}{2} + \frac{7m^2w^6}{12} + \frac{209m^2w^5}{12} \\ & + 6mw^5 + w^6 - w^5. \end{aligned} \tag{48}$$

For $r = 2$ and $s = 8$

$$\pi_8(m,w) = -\frac{1}{288}m^{14}w^7 - \frac{383m^{14}w^6}{5760} - \frac{48763m^{13}w^6}{134400} + \frac{64313m^{12}w^7}{2822400} + \frac{7111847m^{12}w^6}{4233600} + \frac{93529m^{11}w^6}{17280} - \frac{21601m^{10}w^7}{907200} - \frac{51858839m^{10}w^6}{3628800} - \frac{591061m^9w^6}{19200} - \frac{2021m^8w^7}{241920} + \frac{13352561m^8w^6}{241920} + \frac{23749m^7w^6}{288} - \frac{757m^6w^7}{15120} - \frac{3088573m^6w^6}{30240} - \frac{8281m^5w^6}{80} - \frac{161m^4w^7}{720} + \frac{60431m^4w^6}{720} + \frac{105m^3w^6}{2} - \frac{2m^2w^7}{3} - \frac{143m^2w^6}{6} - 7mw^6 - w^7 + w^6. \quad (49)$$

For $r = 2$ and $s = 9$

$$\pi_9(m,w) = \frac{m^{16}w^8}{405} - \frac{796m^{16}w^7}{14175} - \frac{210797m^{15}w^7}{661500} - \frac{329807m^{14}w^8}{15876000} + \frac{75199301m^{14}w^7}{47628000} + \frac{129589219m^{13}w^7}{23814000} + \frac{4069637m^{12}w^8}{114307200} - \frac{358371889m^{12}w^7}{22861440} - \frac{626179m^{11}w^7}{16800} - \frac{7381m^{10}w^8}{1555200} + \frac{818534707m^{10}w^7}{10886400} + \frac{557953m^9w^7}{4320} + \frac{5473m^8w^8}{403200} - \frac{228749219m^8w^7}{1209600} - \frac{1775891m^7w^7}{7560} + \frac{619m^6w^8}{8640} + \frac{423505m^6w^7}{1728} + \frac{634m^5w^7}{3} + \frac{17m^4w^8}{60} - \frac{2939m^4w^7}{20} - \frac{238m^3w^7}{3} + \frac{3m^2w^8}{4} + \frac{125m^2w^7}{4} + 8mw^7 + w^8 - w^7. \quad (50)$$

Plotting the roots of the stability polynomials (48),(49) and (50), respectively in the boundary locus sense gives the region of absolute stability of these methods. The boundary locus method is the most commonly used method for ascertaining the region of absolute stability. The expanse and shape of the region of absolute stability is vital in examining the value of a particular method and when comparing it with other methods. As associated with linear multistep methods, the curves for our class of methods are symmetric to the real axis.

The shape and expanse generated by our methods can be made to contract or expand, depending on the choice of w, where $w \in (-\infty, 0) \cup (0, \infty)$. In comparing our methods, we firstly let $w = 10{,}000$ and subsequently $w = 0.6$ in the stability polynomials (48), (49) and (50) respectively, then plotted in the boundary locus sense. From the figures 1-6 above, it is clear that our methods of order 9 generated the largest region as compared with other methods within the

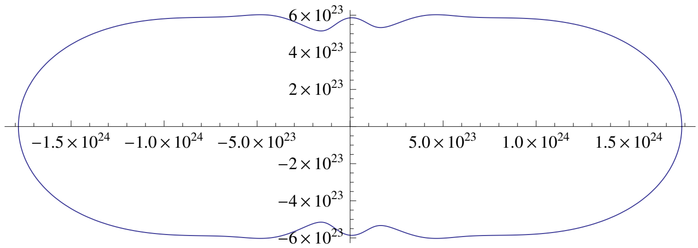

Figure 1: Region of absolute stability of method $r = 2$ and $s = 7$ as $w = 10000$

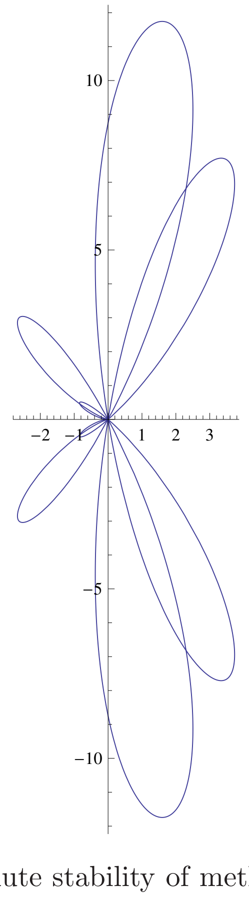

Figure 2: Region of absolute stability of method $r = 2$ and $s = 7$ with $w = 0.6$.

family and as such, signifies superiority. This trend is associated with LMM's

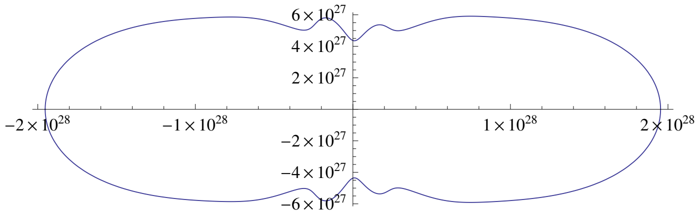

Figure 3: Region of absolute stability of method $r = 2$ and $s = 8$ as $w = 10000$

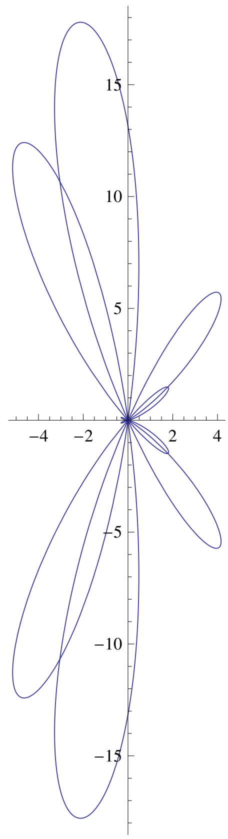

Figure 4: Region of absolute stability of method $r = 2$ and $s = 8$ as $w = 0.6$.

on the accuracy of higher order methods over lower order methods especially when comparing methods within a family.

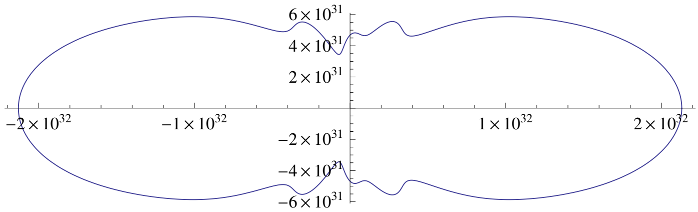

Figure 5: Region of absolute stability of method $r = 2$ and $s = 9$ as $w = 10000$

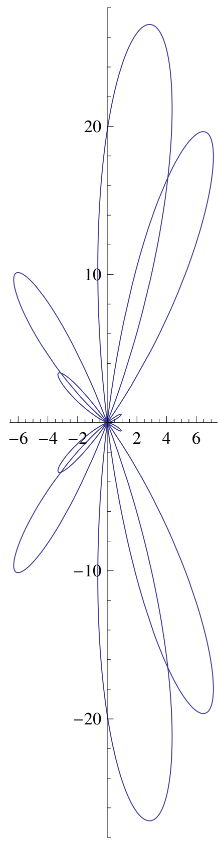

Figure 6: Region of absolute stability of method $r = 2$ and $s = 9$ as $w = 0.6$.

5. Numerical Examples

In this section, we give three examples: a mildly stiff IVP, a nonlinear IVP and a linear non-homogeneous ODE, to test the accuracy and efficiency of

x	Jator Order $p = 6$	Our method Order $p = 7$	Our method Order $p = 8$	Our method Order $p = 9$
0.1	6.98677×10^{-12}	4.54414×10^{-13}	32.7516×10^{-15}	34.4169×10^{-16}
0.2	1.00275×10^{-12}	0.16653×10^{-13}	6.77236×10^{-15}	15.5431×10^{-16}
0.3	7.85878×10^{-12}	3.73479×10^{-13}	27.7556×10^{-15}	15.5431×10^{-16}
0.4	10.4778×10^{-12}	1.73750×10^{-13}	5.21805×10^{-15}	34.4169×10^{-16}
0.5	63.2212×10^{-12}	10.0603×10^{-13}	45.1861×10^{-15}	64.3929×10^{-16}
0.6	10.0508×10^{-12}	49.3106×10^{-13}	62.8386×10^{-15}	6.66134×10^{-16}
0.7	9.36336×10^{-12}	14.3552×10^{-13}	439.149×10^{-15}	46.6293×10^{-16}
0.8	2.64766×10^{-12}	3.85247×10^{-13}	41.3456×10^{-15}	34.2504×10^{-16}
0.9	10.6793×10^{-12}	6.59750×10^{-13}	51.2367×10^{-15}	18.8738×10^{-16}
1.0	23.2731×10^{-12}	5.83711×10^{-13}	12.4345×10^{-15}	48.8498×10^{-16}

Table 1: Absolute errors,$| \; y(x) - y \; |$, for Example 5.1, where $y(x) = e^{-x}$

the methods. We find absolute errors of the approximate solution π_N. All computations were implemented using Mathematica code in Mathematica 9.0.

Example 5.1

We consider the mildly stiff IVPwhich was also solved by Jator [10].

$$y^{''} + 1001y^{'} + 1000y = 0,\ y(0) = 1,\ y(0)^{'} = -1.$$

It is observed that for any member of the class of multiple finite difference methods, order $p + 1$ outperforms order p.The details of the numerical results are given in Table 1.

Example 5.2

We consider the non-linear IVP which was solved by Awoyemi [3] for $h =$ 0.003125.

$$y^{''} - x(y^{'})^2 = 0,\ y(0) = 1,\ y(0)^{'} = \frac{1}{2}.$$

Example 5.3

In this example, we compare between Awoyemi and Kayode [3] and our methods. Except at point $x = 0.1$, it is clear that our methods outperforms the method of order 8 in [3]. The details of the numerical results at some selected points are given in Table 2.

x	Awoyemi and Kayode Order $p = 8$ $h = 0.003125$	Our method Order $p = 7$ $h = 0.05$	Our method Order $p = 8$ $h = 0.05$	Our method Order $p = 9$ $h = 0.05$
0.1	0.66391×10^{-13}	3.11351×10^{-12}	4.41425×10^{-13}	1.70974×10^{-13}
0.2	0.20012×10^{-9}	6.65978×10^{-12}	9.48353×10^{-13}	3.69260×10^{-13}
0.3	1.72007×10^{-9}	9.83258×10^{-12}	14.5750×10^{-13}	5.71099×10^{-13}
0.4	5.89464×10^{-9}	21.7255×10^{-12}	42.9812×10^{-13}	7.49845×10^{-13}
0.5	14.4347×10^{-9}	35.7039×10^{-12}	111.318×10^{-13}	31.6347×10^{-13}
0.6	41.8664×10^{-9}	48.5893×10^{-12}	182.725×10^{-13}	60.7625×10^{-13}
0.7	53.1096×10^{-9}	130.977×10^{-12}	267.204×10^{-13}	91.8643×10^{-13}
0.8	91.1317×10^{-9}	231.336×10^{-12}	1142.32×10^{-13}	120.925×10^{-13}
0.9	149.242×10^{-9}	328.624×10^{-12}	2210.70×10^{-13}	1138.66×10^{-13}
1.0	237.189×10^{-9}	1334.65×10^{-12}	3394.29×10^{-13}	2436.48×10^{-13}

Table 2: Absolute errors,$\mid y(x) - y \mid$, for Example 5.2, where $y(x) = 1 + \frac{1}{2}\ln(\frac{(2+x)}{(2-x)})$.

x	y	Our method Order $p = 7$ $h = 0.05$	Our method Order $p = 8$ $h = 0.05$	Our method Order $p = 9$ $h = 0.05$
0.1	2.39411258	6.97387×10^{-8}	5.67239×10^{-8}	2.40776×10^{-8}
0.2	2.74814133	20.9597×10^{-8}	17.0575×10^{-8}	7.25917×10^{-8}
0.3	3.00786694	39.0274×10^{-8}	131.7701×10^{-8}	13.5528×10^{-8}
0.4	3.10176240	62.3548×10^{-8}	50.7720×10^{-8}	21.6796×10^{-8}
0.5	2.93954310	91.1316×10^{-8}	74.1770×10^{-8}	31.6693×10^{-8}
0.6	2.41183653	124.396×10^{-8}	101.492×10^{-8}	43.4577×10^{-8}
0.7	1.39155483	63.4866×10^{-8}	134.844×10^{-8}	56.9264×10^{-8}
0.8	-0.26232676	91.2310×10^{-8}	209.821×10^{-8}	70.5282×10^{-8}
0.9	-2.69777116	302.414×10^{-8}	320.761×10^{-8}	81.2743×10^{-8}
1.0	-6.05856072	589.936×10^{-8}	451588×10^{-8}	89.1783×10^{-8}

Table 3: Absolute errors,$\mid y(x) - y \mid$, for Example 5.3, where $y(x) = e^{2x}(2\text{Cos}(2x) - \frac{3}{64}\text{Sin}(2x)) + \frac{3}{32}x + \frac{3}{16}x^2 + \frac{1}{8}x^3$.

We consider the non-homogeneous ODE given by

$$y'' - 4y' + 8y = x^3,\ y(0) = 2,\ y(0)' = 4.$$

$$\text{Exact: } y(x) = e^{2x}(2\text{Cos}(2x) - \frac{3}{64}\text{Sin}(2x)) + \frac{3}{32}x + \frac{3}{16}x^2 + \frac{1}{8}x^3.$$

Although the numerical results of this problem were not compared with another method, the result were compared with the theoretical solution. The Table 3 is a display of the absolute error of our methods.

6. Conclusion

We proposed three particular methods for $k = 6, 7, 8$ for solving general second order IVP's directly without firstly relaxing to an equivalent first order system. The methods are implemented in the absence of self starting values or predictors and hence complex subroutines are avoided. We also proposed their stability properties. Hereafter, we will be focused in using the MFDM's to solve boundary value problems.

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