

NUMERICAL STUDY OF THREE-PARAMETER MATRIX EIGENVALUE PROBLEM BY GRADIENT METHOD

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Abstract: In this paper, a Convergent Gradient Method, which was first developed to solve generalised eigenvalue problem of the form $Ax = \lambda Bx$, will be used to find approximate eigenvalues and corresponding eigenvectors of Multiparameter Matrix Eigenvalue Problems (MMEPs). To apply Gradient Method, we will reduce MMEPs into a single matrix pencil containing the block diagonal matrices. Although the Method can be extended to Multiparameter case, but the whole work is done by considering three-parameter case only to relax computational cost. Computational efficiency of this method will be discussed by comparing the results with Kronecker Product Method as proposed by Atkinson in 1960, analytically with the help of numerical examples.

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1. Introduction

The MMEPs are the more generalization concept of classical one-parameter standard eigenvalue problems of the form $Ax = \lambda x$ and the generalized eigenvalue problems of the form $Ax = \lambda Bx$, where the problem is to find a k-tuple of values $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in C^k$ and non-zero vector $x_i \in C^{n_i}$ for $i = 1, 2, \dots, k$

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such that

$$(A_i - \sum_{j=1}^k \lambda_j B_{ij})x_i = 0, \quad i = 1, 2, \dots, k, \quad (1)$$

where $\lambda_i \in R^i$; $i = 1, 2, 3, \dots, k$ are spectral parameters and A_i, B_{ij} ; $j = 1, 2, 3, \dots, k$ are self-adjoint, bounded linear operators that act in Hilbert Spaces $H_i, x_i \in H_i$. The k -tuple $\lambda \in C^k$ is called an eigenvalue and the tensor product $x = x_1 \otimes x_2 \otimes x_3 \otimes \dots \otimes x_k$ is the corresponding (right) eigenvector. Similarly left eigenvector can also be defined. This type of problems arises in various physical situations, particularly where the separation of variables technique is used to solve a discretised boundary value problem. In 1960s, mainly due to Atkinson's works [[4], [5]], a new stage began in the investigation of MMEPs and he has introduced an abstract algebraic setting (1) from the viewpoint of determinantal operators on the tensor product space. He showed that a non-singular MEP is equivalent to the associated system of generalized eigenvalue problems $Ax = \lambda Bx$. Many theoretical results as well as numerical methods for non-singular MMEPs are based on this relation.

Numerical solution of MMEPs for matrices arises in the discretization of Multiparameter Sturm-Liouville eigenvalue problems in ordinary differential equations. Only a few numerical methods are available for such a problems, particularly for two-parameter case. For example continuation method [2], Jacobi-Davidson's method [6], Bisection Method [9] etc. The connection between stability of invariant subspace of algebraic MMEPs and the stability of invariant subspaces of commuting matrices are found in [8], which is very important for numerical calculation.

The problem defined by the equation (1) is relatively new one in numerical analysis. Its solution is very complicated due to the lack of spectral theory for the case $k \geq 2$ as comparable to the theory for $k = 1$. Atkinson showed that the separation of spectral parameters is possible directly for the case $k = 2$ in tensor product space, which can not be done for the case $k \geq 3$. There is no standard method for treating such problem for $k \geq 3$ in the existing literature. In this paper an attempt will be done to study three-parameter matrix eigenvalue problems numerically with the help of Gradient Method presented in [7] to enrich the Multiparameter cases.

The rest of the paper is organized as follows: Section 2 contains some basic preliminaries. In Section 3 a brief description of three-parameter problem and in section 4 a convergent Gradient Method is presented. Similarly in Section 5 the efficiency of the method is tested with the help of numerical examples and finally in section 6 a conclusion and the future scope of study is drawn.

2. Basic Preliminaries

Definition 1. F be a real functional defined on the Hilbert space H . The frechet derivative of F at $x \in H$ is a linear operator, denoted by $\Delta F(x)$ such that

$$\lim_{h \rightarrow 0} \frac{\|F(x+h) - F(x) - \Delta F(h)\|}{\|h\|} = 0.$$

For H with real inner product $\langle x, y \rangle$ we can write the linear operator $\Delta F(h) = \langle \Delta F(x), h \rangle$

Definition 2. A point $x \in H$ is called stationary point of F if $\Delta F(x) = 0$. A subset $E \subset H$ is called stationary set of F if $\Delta F(x) = 0, \forall x \in E$.

3. Three-Parameter Matrix Eigenvalue Problem and its Abstract Settings by Atkinson

Three-parameter matrix problem is a particular case of (1) for $i = 3$

$$\begin{aligned} (A_1 - \lambda_1 B_{11} - \lambda_2 B_{12} - \lambda_3 B_{13})x_1 &= 0, \\ (A_2 - \lambda_1 B_{21} - \lambda_2 B_{22} - \lambda_3 B_{23})x_2 &= 0, \\ (A_3 - \lambda_1 B_{31} - \lambda_2 B_{32} - \lambda_3 B_{33})x_3 &= 0. \end{aligned} \tag{2}$$

An eigenvalue is defined to be a 3-tuple $(\lambda_1, \lambda_2, \lambda_3)$ of complex numbers for which there exists a non-zero decomposable element $u = x_1 \otimes x_2 \otimes x_3$. The necessary definiteness condition is found in [4] and is given by

$$D(u_1, u_2, u_3) = \begin{vmatrix} (B_{11}u_1, u_1) & (B_{12}u_1, u_1) & (B_{13}u_1, u_1) \\ (B_{21}u_2, u_2) & (B_{22}u_2, u_2) & (B_{23}u_2, u_2) \\ (B_{31}u_3, u_3) & (B_{32}u_3, u_3) & (B_{33}u_3, u_3) \end{vmatrix} \geq 0, \tag{3}$$

where $(B_{ij}u_i, u_i)$ are the auxilary inner product. Problem (2) is called right definite if conditions (3) are satisfied and can be reduced into the three generalized one parameter problems in the tensor product space given by

$$\begin{aligned} \Delta_1 u &= \lambda_1 \Delta_0 u, \Delta_2 u = \lambda_2 \Delta_0 u, \Delta_3 u = \lambda_3 \Delta_0 u, \\ \Delta_0^{-1} \Delta_1 u &= \lambda_1 u, \Delta_0^{-1} \Delta_2 u = \lambda_2 u, \Delta_0^{-1} \Delta_3 u = \lambda_3 u, \end{aligned} \tag{4}$$

where:

$$\Delta_0 = \begin{vmatrix} \tilde{B}_{11} & \tilde{B}_{12} & \tilde{B}_{13} \\ \tilde{B}_{21} & \tilde{B}_{22} & \tilde{B}_{23} \\ \tilde{B}_{31} & \tilde{B}_{32} & \tilde{B}_{33} \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} \tilde{A}_1 & \tilde{B}_{12} & \tilde{B}_{13} \\ \tilde{A}_2 & \tilde{B}_{22} & \tilde{B}_{23} \\ \tilde{A}_3 & \tilde{B}_{32} & \tilde{B}_{33} \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} \tilde{B}_{11} & \tilde{A}_1 & \tilde{B}_{13} \\ \tilde{B}_{21} & \tilde{A}_2 & \tilde{B}_{23} \\ \tilde{B}_{31} & \tilde{A}_3 & \tilde{B}_{33} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} \tilde{B}_{11} & \tilde{B}_{12} & \tilde{A}_1 \\ \tilde{B}_{21} & \tilde{B}_{22} & \tilde{A}_2 \\ \tilde{B}_{31} & \tilde{B}_{32} & \tilde{A}_3 \end{vmatrix}$$

and

$$u = x_1 \otimes x_2 \otimes x_3,$$

where the induced operators \tilde{A}_i and \tilde{B}_{ij} in tensor product space are the operators induced by A_i and B_{ij} respectively in $H_i, i = 1, 2, 3$. The induced operators are defined by $\tilde{A}_1 = A_1 \otimes I_2 \otimes I_3, \tilde{A}_2 = I_1 \otimes A_2 \otimes I_3, \tilde{A}_3 = I_1 \otimes I_2 \otimes A_3, \tilde{B}_{1i} = B_{1i} \otimes I_2 \otimes I_3, \tilde{B}_{2i} = I_1 \otimes B_{2i} \otimes I_3$ and $\tilde{B}_{3i} = I_1 \otimes I_2 \otimes B_{3i}$ where $I_i, i = 1, 2, 3$ are the identity operators in $H_i, i = 1, 2, 3$. MMEP is right definite $\Leftrightarrow \Delta_0$ is nonsingular. The operators $G_i = \Delta_0^{-1} \Delta_i, i = 1, 2, 3$ commute. If MMEP is right definite then

1. Eigen 3-tuple are real.
2. There exist n^3 linearly independent eigenvectors.
3. Eigenvectors of distinct eigenvalues are Δ_0 -orthogonal, i.e., $(x_1 \otimes x_2 \otimes x_3)^T \Delta_0 (x_1 \otimes x_2 \otimes x_3) = 0$.

4. The Gradient Method for Matrix Pencils

The method was first developed to solve generalized eigenvalue problem of the form $Ax = \lambda Bx$ [Presented in [3]] and later generalized to solve Matrix Eigenvector-Eigentuple Problems of the form $Ax = \sum_{i=1}^{i=k} \lambda_i B_i x$, [presented in [1], [7] by Blum etc.]. First we consider the matrix pencil of the form

$$(A - \sum_{i=1}^{i=k} \lambda_i B_i)x = 0, \quad (5)$$

where A, B_i are linear operators in Hilbert Spaces H . A vector $x \in H, x \neq 0$ satisfying (5) for a given $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is called eigenvector of the Matrix Eigenvector-Eigentuple Problems. Let for each eigenvector x the set of vectors $\{B_1 x, B_2 x, \dots, B_k x\}$ is linearly independent and this assumption is essential to remove the degenerate case. Define $\alpha_i(x) = \langle Ax, B_i x \rangle, \alpha(x) = (\alpha_1(x), \alpha_2(x), \dots, \alpha_k(x)), \beta_{ij}(x) = \langle B_i x, B_j x \rangle$ and

$$\beta(x) = (\beta_{ij}(x)), \quad (6)$$

where $1 \leq i, j \leq k$. Let $\alpha(x)$ be the column vector with elements $\alpha_i(x)$ and Let $\beta^{-1}(x)$ exist. Let $\wedge(x)$ be defined by the equation

$$\wedge(x) = \beta^{-1}(x)\alpha(x). \tag{7}$$

Consider the functional

$$F(x) = \frac{1}{2} \left\| Ax - \sum_{i=1}^{i=n} \lambda_i B_i x \right\|^2, \tag{8}$$

defined for all $x \in H$ such that $\beta^{-1}(x)$ exists. And λ_i are the components of the vector $\wedge(x)$ in (7). It can be proved that the eigenvectors of (1) are the stationary points of F and conversely. We define

$$\Delta F(x) = (A - \sum_{i=1}^{i=n} \lambda_i B_i)^* (A - \sum_{i=1}^{i=n} \lambda_i B_i)x. \tag{9}$$

We now define the Gradient Method from [3]

$$h(x) = \begin{cases} -2F(x)\Delta F(x)/\|\Delta F(x)\|^2 & \text{if } \Delta F(x) \neq 0, \\ 0 & \text{if } \Delta F(x) = 0, \end{cases} \tag{10}$$

and

$$x_{k+1} = x_k + h(x_k), \quad k = 0, 1, 2, \dots \tag{11}$$

To find the solution of (5) we have to choose initial gauss x_0 on the unit sphere and then compute $\wedge(x_0)$, $F(x)$ and $\Delta F(x)$ from the equations (7), (8) and (9) respectively. Then after calculating $h(x_0)$, the next iteration x_1 will be determined using the equations (10) and (11). Since in general x_1 may not be on the unit sphere and hence by normalizing x_1 the procedure will be repeated. The iteration (11) will be continued in this way such that x_m and its corresponding k-tuple $\wedge(x_m)$ is the approximate solution of the problem (5). This iterative procedure can be used to solve three-parameter eigenvalue problems (1) by reducing the same into a single matrix pencil, which is discussed in the following section.

4.1. Conversion of Three-Parameter Matrix Eigenvalue Problem into a Single Matrix Pencil

The system (2) can be written in the following form

$$\begin{aligned}
& \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix} \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} = \left\{ \lambda_1 \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & B_{21} & 0 \\ 0 & 0 & B_{31} \end{pmatrix} \right. \\
& \left. + \lambda_2 \begin{pmatrix} B_{12} & 0 & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & B_{32} \end{pmatrix} + \lambda_2 \begin{pmatrix} B_{13} & 0 & 0 \\ 0 & B_{23} & 0 \\ 0 & 0 & B_{33} \end{pmatrix} \right\} \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}. \\
& \Rightarrow AX = (\lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3) X, \tag{12}
\end{aligned}$$

where:

$$A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}, \quad B_i = \begin{pmatrix} B_{1i} & 0 & 0 \\ 0 & B_{2i} & 0 \\ 0 & 0 & B_{3i} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}.$$

Obviously the problem formulation of the equation (2) and that of the equation (12) are equivalent. Although, Gradient Method discussed in the Section 3 can't be applied directly to find a common solution of the system (2), but this conversion into single matrix pencil allow us to deal with the problem successfully.

Theorem 1. *Let $R : R^n \rightarrow R$ be the function defined by $F(x) = \frac{1}{2} \|(A - \Sigma \lambda_i B_i)x\|^2$ where A, B_i are $n \times n$ matrix. Then the Frechet derivative of F at x_0 is a function M , where*

$$M(h) = x_0^T (A - \Sigma \lambda_i B_i)^T (A - \Sigma \lambda_i B_i) h.$$

Proof. We have

$$\begin{aligned}
F(x_0 + h) &= \frac{1}{2} \|(A - \Sigma \lambda_i B_i)(x_0 + h)\|^2 \\
&= \frac{1}{2} \|(A - \Sigma \lambda_i B_i)x_0 + (A - \Sigma \lambda_i B_i)h\|^2 \\
&= \frac{1}{2} \left\{ \|(A - \Sigma \lambda_i B_i)x_0\|^2 + \|(A - \Sigma \lambda_i B_i)h\|^2 \right. \\
&\quad \left. + 2[(A - \Sigma \lambda_i B_i)x_0]^T (A - \Sigma \lambda_i B_i)h \right\} \\
&= \frac{1}{2} \|(A - \Sigma \lambda_i B_i)x_0\|^2 + \frac{1}{2} \|(A - \Sigma \lambda_i B_i)h\|^2 \\
&\quad + x_0^T (A - \Sigma \lambda_i B_i)^T (A - \Sigma \lambda_i B_i)h \\
&= F(x_0) + M(h) + \frac{1}{2} \|(A - \Sigma \lambda_i B_i)h\|^2.
\end{aligned}$$

Hence it follows the theorem. Thus, the Frechet derivative of F at x_0 is given by

$$\Delta F(x_0) = x_0^T (A - \Sigma \lambda_i B_i)^T (A - \Sigma \lambda_i B_i),$$

which is useful at the calculation part of the equation (9). □

Theorem 2. *If $\Delta F(x) = 0$ then $F(x) = 0$.*

Proof: See Lemma 3.1 pp. 252 – 253 [1].

4.2. Finding of Stationary Point by B-R Gradient Procedure

Let x_0 be the initial point in H . Let $\lambda_i(x_0)$ be the component of $\wedge(x_0)$. Define x_n for $n \geq 1$ recursively by

$$y_{n+1} = x_n + h(x_n), \quad x_{n+1} = \frac{y_{n+1}}{\|y_{n+1}\|},$$

where

$$h(x_n) = \begin{cases} \frac{-2F(x_n)\Delta F(x_n)}{\|\Delta F(x_n)\|^2} & \text{if } \Delta F(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

The sequence x_n converges to a stationary point of F .

4.3. Validity Conditions

To apply Gradient Method for MMEPs, it is essential to remove degeneracy case of (12). To deal with the degeneracy case, we need the assumption that the set $\{B_1X, B_2X, B_3X\}$ is linearly independent.

Theorem 3. *The set $\{B_1X, B_2X, B_3X\}$ is linearly independent if the sets $\{B_{11}x_1, B_{12}x_1, B_{13}x_1\}$; $\{B_{21}x_2, B_{22}x_2, B_{23}x_2\}$ and $\{B_{31}x_3, B_{32}x_3, B_{33}x_3\}$ are linearly independent.*

Proof. Assume that

$$\{B_{11}x_1, B_{12}x_1, B_{13}x_1\}, \quad \{B_{21}x_2, B_{22}x_2, B_{23}x_2\},$$

and

$$\{B_{31}x_3, B_{32}x_3, B_{33}x_3\}$$

are linearly independent.

Therefore

$$\alpha_1 B_{11}x_1 + \alpha_2 B_{12}x_1 + \alpha_3 B_{13}x_1 = 0,$$

$$\alpha_1 B_{21}x_2 + \alpha_2 B_{22}x_2 + \alpha_3 B_{23}x_2 = 0,$$

and

$$\alpha_1 B_{31}x_3 + \alpha_2 B_{32}x_3 + \alpha_3 B_{33}x_3 = 0 \Rightarrow \alpha_i = 0, \quad \forall i = 1, 2, 3.$$

$$\begin{aligned} \Rightarrow \alpha_1 \begin{pmatrix} B_{11}x_1 & 0 & 0 \\ 0 & B_{21}x_2 & 0 \\ 0 & 0 & B_{31}x_3 \end{pmatrix} + \alpha_2 \begin{pmatrix} B_{12}x_1 & 0 & 0 \\ 0 & B_{22}x_2 & 0 \\ 0 & 0 & B_{32}x_3 \end{pmatrix} \\ + \alpha_3 \begin{pmatrix} B_{13}x_1 & 0 & 0 \\ 0 & B_{23}x_2 & 0 \\ 0 & 0 & B_{33}x_3 \end{pmatrix} = 0, \end{aligned}$$

$\forall i = 1, 2, 3.$

If $\alpha_i = 0:$

$$\Rightarrow \alpha_1 B_1 X + \alpha_2 B_2 X + \alpha_3 B_3 X = 0, \quad \forall i = 1, 2, 3.$$

$\alpha_i = 0:$

$\Rightarrow \{B_1 X, B_2 X, B_3 X\}$ is linearly independent. \square

Hence Gradient Method can be applied to three-parameter eigenvalue problems if $\{B_{11}x_1, B_{12}x_1, B_{13}x_1\}; \{B_{21}x_2, B_{22}x_2, B_{23}x_2\}$ and $\{B_{31}x_3, B_{32}x_3, B_{33}x_3\}$ are linearly independent. Again from equation (6) we have

$$\begin{aligned} \beta(x) &= \begin{pmatrix} \langle B_1 X, B_1 X \rangle & \langle B_1 X, B_2 X \rangle & \langle B_1 X, B_3 X \rangle \\ \langle B_2 X, B_1 X \rangle & \langle B_2 X, B_2 X \rangle & \langle B_2 X, B_3 X \rangle \\ \langle B_3 X, B_1 X \rangle & \langle B_3 X, B_2 X \rangle & \langle B_3 X, B_3 X \rangle \end{pmatrix}, \\ \beta(x) &= \begin{pmatrix} Tr[(B_1 X)^T B_1 X] & Tr[(B_2 X)^T B_1 X] & Tr[(B_3 X)^T B_1 X] \\ Tr[(B_1 X)^T B_2 X] & Tr[(B_2 X)^T B_2 X] & Tr[(B_3 X)^T B_2 X] \\ Tr[(B_1 X)^T B_3 X] & Tr[(B_2 X)^T B_3 X] & Tr[(B_3 X)^T B_3 X] \end{pmatrix}, \\ \beta(x) &= \begin{pmatrix} Tr(X^T B_1^T B_1 X) & Tr(X^T B_2^T B_1 X) & Tr(X^T B_3^T B_1 X) \\ Tr(X^T B_1^T B_2 X) & Tr(X^T B_2^T B_2 X) & Tr(X^T B_3^T B_2 X) \\ Tr(X^T B_1^T B_3 X) & Tr(X^T B_2^T B_3 X) & Tr(X^T B_3^T B_3 X) \end{pmatrix}, \\ \beta(x) &= \begin{pmatrix} Tr(B_1^T B_1) & Tr(B_2^T B_1) & Tr(B_3^T B_1) \\ Tr(B_1^T B_2) & Tr(B_2^T B_2) & Tr(B_3^T B_2) \\ Tr(B_1^T B_3) & Tr(B_2^T B_3) & Tr(B_3^T B_3) \end{pmatrix}. \end{aligned}$$

Gradient Method can be applied for three-parameter matrix eigenvalue problem if $\beta^{-1}(x)$ exist i.e when $|\beta(x)| \neq 0$. Since all B_i of the matrix $\beta(x)$ above are known from equation (2) and hence $|\beta(x)|$ can be calculated directly to check validity.

Sl No	$(\lambda_3, \lambda_3, \lambda_3)$	Eigen Vectors
1	$(-0.0231, +0.0136, +0.2069)$	$(10000000000000000000000000000000)^T$
2	$(-0.0246, +0.0135, +0.2112)$	$(01000000000000000000000000000000)^T$
3	$(-0.0408, +0.0130, +0.2576)$	$(00100000000000000000000000000000)^T$
4	$(-0.0234, +0.0045, +0.2092)$	$(00010000000000000000000000000000)^T$
5	$(-0.0247, +0.0043, +0.2128)$	$(00001000000000000000000000000000)^T$
6	$(-0.0418, +0.0015, +0.2621)$	$(00000100000000000000000000000000)^T$
7	$(-0.0230, +0.0164, +0.2061)$	$(00000010000000000000000000000000)^T$
8	$(-0.0246, +0.0160, +0.2108)$	$(00000001000000000000000000000000)^T$
9	$(-0.0409, +0.0118, +0.2580)$	$(00000000100000000000000000000000)^T$
10	$(+0.1345, +0.0014, +0.1968)$	$(00000000010000000000000000000000)^T$
11	$(+0.1426, +0.0015, +0.1784)$	$(00000000001000000000000000000000)^T$
12	$(+0.01083, +0.0011, +0.2557)$	$(00000000000100000000000000000000)^T$
13	$(+0.1351, -0.0038, +0.1980)$	$(00000000000010000000000000000000)^T$
14	$(+0.1433, +0.0028, +0.1789)$	$(00000000000001000000000000000000)^T$
15	$(+0.1087, -0.0070, +0.2589)$	$(00000000000000100000000000000000)^T$
16	$(+0.1340, +0.0056, +0.1958)$	$(00000000000000010000000000000000)^T$
17	$(+0.1417, +0.0071, +0.1776)$	$(00000000000000001000000000000000)^T$
18	$(+0.1083, +0.0006, +0.2559)$	$(00000000000000000100000000000000)^T$
19	$(+0.2838, -0.0102, +0.1872)$	$(00000000000000000010000000000000)^T$
20	$(+0.2872, -0.0090, +0.1876)$	$(00000000000000000001000000000000)^T$
21	$(+0.2777, -0.0124, +0.2536)$	$(0000000000000000000010000000)^T$
22	$(+0.2841, -0.0116, +0.0358)$	$(0000000000000000000001000000)^T$
23	$(+0.2872, -0.0089, +0.1500)$	$(0000000000000000000000100000)^T$
24	$(+0.2786, -0.0165, +0.2552)$	$(0000000000000000000000001000)^T$
25	$(+0.2826, -0.0097, +0.1860)$	$(000000000000000000000000001000)^T$
26	$(+0.2856, -0.0007, +0.1490)$	$(000000000000000000000000000010)^T$
27	$(+0.2777, -0.0122, +0.2536)$	$(00000000000000000000000000000001)^T$

Table 1: Eigen value and eigen vector by Kronecker product method

Iteration	Initial Gauss	Approx Eigen 3-tuples	Approx Eigenvec-tors
3	$x = y = z = (111)^T$	$(0.1363, -0.0001, 0.1939)$	u
8	$x = (123)^T; y = (211)^T; z = (111)^T$	$(0.2838, -0.0089, 0.1631)$	v
6	$x = (223)^T; y = (256)^T; z = (345)^T$	$(0.2785, -0.0149, 0.2527)$	w

Table 2: Approximate eigen value and eigen vectors by gradient method

5. Numerical Illustrations

Let us consider the following three-parameter problems with real matrices.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \lambda_1 \begin{pmatrix} 20 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 20 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} + \lambda_3 \begin{pmatrix} 7 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

u	(0.0097 0.0064 -0.0020 0.0091 0.0060 -0.0018 0.0096 0.0063 -0.0019 0.9234 0.6052 -0.1855 0.8688 0.5695 -0.1745 0.9129 0.5984 -0.1834 0.0092 0.0061 -0.0019 0.0087 0.0057 -0.0017 0.0091 0.0060 -0.0018) ^T
v	(-0.1068 -0.1418 -0.0066 -0.0530 -0.0704 -0.0033 -0.0528 -0.0702 -0.0033 0.2583 0.3430 0.0160 0.1283 0.1704 0.0080 0.1278 0.1698 0.0079 3.5234 4.6790 0.2184 1.7499 2.3238 0.1085 1.7437 2.3156 0.1081) ^T
w	(-0.0098 0.0081 -0.0261 -0.0248 0.0206 -0.0662 -0.0306 0.0254 -0.0816 -0.0020 0.0016 -0.0052 -0.0050 0.0041 -0.0133 -0.0061 0.0051 -0.0164 -0.3240 0.2688 -0.8638 -0.8219 0.6819 -2.1912 -1.0133 0.8407 -2.7015) ^T

Table 3: Approximate eigenvectors of Table 2

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 50 & 0 & 0 \\ 0 & 52 & 0 \\ 0 & 0 & 26 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \lambda_1 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} + \lambda_3 \begin{pmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 23 \end{pmatrix}.$$

The exact number of common 3-tuples of the above system will be $3^3 = 27$ as shown in Table 1 calculated by Kronecker Product Method. In table 2 approximate eigenvalues and the corresponding approximate eigenvectors with three different initial guesses has been calculated using Gradient Method.

6. Conclusion

The Gradient Method discussed above can be easily extended to n -parameter cases in a similar manner. This Method is more efficient for the matrices of higher order than Kronecker Product Method, where the computation cost as well as time is more due to significantly increase of the dimension of Δ_i of order $N = n^n$. Since this Method does not require the inverting matrices as that in Kronecker product Method and hence it can handle large sparse matrices also. The prime benefit of this Method is that no initial approximation is required for eigenvalues. The convergence of this Method is very fast if we choose the initial guess of the eigenvectors near to the exact value. In near future, we have a plan to find the actual order of convergence of the Method for MMEPs.

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