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REMARKS ON δ-OPEN SETS INDUCED BY ENLARGEMENTS OF GENERALIZED TOPOLOGIES

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Abstract: The concepts of κ_μ-open set and enlargement κ of a generalized topology μ were introduced by Császár [5]. In this paper, we introduce the concept of δ_μ-open set induced by an enlargement κ of a generalized topology μ, and we study some basic properties for the set. Moreover, we establish the relationships among κ_μ-open sets, weak κ_μ-open [6] and δ_κ-open.

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1. Introduction

Let X be a nonempty set and $exp(X)$ be the power set of X. Then $\mu \subseteq exp(X)$ is called a *generalized topology* (briefly GT) [2] on X iff $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $G = \cup_{i\in I} G_i \in \mu$. We call the pair (X, μ) a *generalized topological space* (briefly GTS) on X. The elements of μ are called μ*-open* [1, 2] sets and the complements are called μ*-closed* sets. We call a GTS X is *strong* if $X \in \mu$ [4]. The generalized-closure of a subset S of X, denoted by $c_\mu(S)$, is

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the intersection of generalized closed sets including S. And the interior of S, denoted by $i_\mu(S)$, the union of generalized open sets included in S.

Let us define $\delta(\mu) = \delta \subseteq exp(X)$ by $A \in \delta$ iff $A \subseteq X$ and, if $x \in A$, then there is a μ-closed set Q such that $x \in i_\mu Q \subseteq A$ [4]. A is said to be *μr-open* (resp., *μr-closed*) [4] if $A = i_\mu(c_\mu(A))$ (resp., $A = c_\mu(i_\mu(A))$).

For a GT μ on X, a mapping $\kappa : \mu \to expX$ is called an *enlargement* [5] on X if $M \subseteq \kappa(M)$ whenever $M \in \mu$. Let us say that a subset $A \subseteq X$ is *κ_μ-open* [5] iff $x \in A$ implies the existence of a μ-open set M such that $x \in M$ and $\kappa(M) \subseteq A$. The collection κ_μ of all κ_μ-open sets is a GT on X and $\kappa_\mu \subseteq \mu$ [5]. Let us say that a subset $A \subseteq X$ is *κ_μ-closed* [5] iff $X - A$ is κ_μ-open.

2. Main Results

Let μ be a GT on X and $\kappa : \mu \to expX$ an enlargement of μ. For $M \in \mu$, M is said to be *κ-regular open* [6] if $i_\mu(\kappa(M)) = M$. The complement of κ-regular open set is called a *κ-regular closed* set.

Definition 2.1. Let (X, μ) be a GTS, and let κ be an enlargement of μ and $A \subseteq X$. Then A is said to be *δ_κ-open* if for each $x \in A$, there exists a κ-regular open set G such that $x \in G \subseteq A$. The complement of a δ_κ-open set is called a *δ_κ-closed set*. We will denote $\delta_\kappa = \{A \subseteq X : A \text{ is } \delta_\kappa\text{-open}\}$.

Remark 2.2. Let μ be a GT on X and $\kappa : \mu \to expX$ an enlargement of μ. Let us say that a subset $A \subseteq X$ is a *weak κ_μ-open* (briefly *$w\kappa_\mu$-open*) *set* [6] iff $x \in A$ implies the existence of a μ-open set M such that $x \in M$ and $i_\mu(\kappa(M)) \subseteq A$. Then the collection $w\kappa_\mu$ of all weak κ_μ-open sets is a GT on X and $\kappa_\mu \subseteq w\kappa_\mu \subseteq \mu$. Furthermore, we can show easily that $\delta_\kappa \subseteq w\kappa_\mu$ and the notions of κ_μ-open sets and δ_κ-open sets are independent. So, we have the following diagram:

$$\begin{array}{ccccc}
\kappa_\mu\text{-regular open} & \rightarrow & \delta_\kappa\text{-open} & & \\
 & & & \searrow & \\
 & & \not\downarrow\ \not\uparrow & & \text{weak } \kappa_\mu\text{-open} \\
 & & & \nearrow & \\
 & & \kappa_\mu\text{-open} & &
\end{array}$$

The converses are not true always as shown in the next examples:

Example 2.3. Let $X = \{a, b, c, d, e\}$ and a generalized topology $\mu = \{\emptyset, \{a\}\ \{b, c\}, \{a, b, c\}, \{a, b, c, d\}\}$ on X. Let us consider an enlargement $\kappa : \mu \to expX$ defined as the following:

$$\kappa(M) = \begin{cases} M \cup \{d\}, & \text{if } a \in M, \\ M \cup \{a\}, & \text{if } a \notin M. \end{cases}$$

Then κ is an enlargement. Note that:

$$\kappa(\{a\}) = \{a, d\}; \kappa(\{b, c\}) = \{a, b, c\}; \kappa(\{a, b, c\}) = \{a, b, c, d\};$$

$$i\kappa(\{a\}) = \{a\}; i\kappa(\{b, c\}) = \{a, b, c\}; i\kappa(\{a, b, c\}) = \{a, b, c, d\}.$$

For $A = \{a, b, c\}$, A is weak κ_μ-open but not δ_κ-open.

Example 2.4. Let $X = \{a, b, c, d\}$ and a generalized topology $\mu = \{\emptyset, \{a\}$ $\{b, c\}, \{a, b, c\}, X\}$ on X. Let us consider an enlargement $\kappa : \mu \to expX$ defined as the following:

$$\kappa(M) = \begin{cases} M \cup \{d\}, & \text{if } a, b \in M, \\ M \cup \{c\}, & \text{otherwise.} \end{cases}$$

Then κ is an enlargement. Note that:

$$\kappa(\{a\}) = \{a, c\}; \kappa(\{b, c\}) = \{b, c\}; \kappa(\{a, b, c\}) = X;$$

$$i\kappa(\{a\}) = \{a\}; i\kappa(\{b, c\}) = \{b, c\}; i\kappa(\{a, b, c\}) = X.$$

For $A = \{a\}$, A is δ_κ-open but not κ_μ-open. For $B = \{a, b, c\}$, B is δ_κ-open but not κ_μ-regular open.

Example 2.5. Let $X = \{a, b, c, d\}$ and a generalized topology $\mu = \{\emptyset, \{a, b\}$ $\{b, c\}, \{a, b, c\}, X\}$ on X. Let us consider an enlargement $\kappa : \mu \to expX$ defined as the following:

$$\kappa(M) = \begin{cases} M \cup \{d\}, & \text{if } a, c \in M, \\ M \cup \{c\}, & \text{otherwise.} \end{cases}$$

Then κ is an enlargement. Note that:

$$\kappa(\{a, b\}) = \{a, b, c\}; \kappa(\{b, c\}) = \{b, c\}; \kappa(\{a, b, c\}) = X;$$

$$i\kappa(\{a, b\}) = \{a, b, c\}; i\kappa(\{b, c\}) = \{b, c\}; i\kappa(\{a, b, c\}) = X.$$

For $A = \{a, b, c\}$, A is κ_μ-open but not δ_κ-open.

Let X be a nonempty set and $s \subseteq 2^X$. Then s is called a *σ-structure* [7] on X if for $i \in I \neq \emptyset$, $U_i \in s$ implies $\cup_{i \in I} U_i \in s$. The elements of s are called *σ-open* sets and the complements are called *σ-closed* sets.

Theorem 2.6. *Let (X, μ) be a GTS, and let κ be an enlargement of μ. Then δ_κ is a σ-structure.*

Proof. Let $Q \in \delta_\kappa$ and $x \in Q$. Then there exists a κ-regular open set G_x such that $x \in G_x \subseteq Q$ and so $\cup_{x\in Q} G_x = Q$. □

In general, δ_κ is not a generalized topology in the sense of Császár as shown in the next example:

Example 2.7. Let $X = \{a, b, c, d\}$. Consider generalized topologies $\mu = \{\emptyset, \{a\}\ \{b, c\}, \{a, b, c\}\}$ on X. Let us consider an enlargements $\kappa : \mu \to expX$ defined as the following: $\kappa G = G \cup \{a\}$ for $G \in \mu$. Then for $G = \emptyset$,

$$\kappa(G) = \{a\}, \quad i\kappa(G) = i\{a\} = \{a\} \neq \emptyset.$$

Hence, the empty set $\emptyset$ is not κ-regular open.

Let μ be a GT on X and κ_μ an enlargement of μ. The enlargement κ is said to be *ordinary* [6] if $\kappa \subseteq c_\mu$ for $M \in \mu$.

Theorem 2.8. *Let μ be a GT on X and κ_μ an enlargement of μ. If κ is ordinary, then δ_κ is a generalized topology.*

Proof. From $\emptyset = i_\mu(c_\mu(\emptyset))$, it follows $\emptyset \subseteq i_\mu(\kappa(\emptyset)) \subseteq i_\mu(c_\mu(\emptyset))$. Hence $\emptyset$ is κ-regular open. Hence from Theorem 2.6, δ_κ is a generalized topology. □

Theorem 2.9. *Let (X, μ) be a GTS, and let κ be an enlargement of μ. If κ is quasi-idempotent and monotonic, then for every μ-open set M, $i_\mu(\kappa(M))$ is δ_κ-open.*

Proof. From hypothesis and $M \subseteq i_\mu(\kappa(M))$, it follows

$$i_\mu(\kappa(M)) \subseteq i_\mu(\kappa(i_\mu(\kappa(M)))) \subseteq i_\mu(\kappa(M)).$$

So $i_\mu(\kappa(i_\mu(\kappa(M)))) = i_\mu(\kappa(M))$. So, $i_\mu(\kappa(M))$ is κ-regular open for every μ-open set M, and $i_\mu(\kappa(M))$ is δ_κ-open. □

Theorem 2.10. *Let (X, μ) be a GTS, and let κ be a monotonic enlargement of μ. If μ is strong, then δ_κ is strong.*

Proof. Since μ is strong and κ is a monotonic enlargement of μ, $\kappa(X) = X$ and X is μ-open. So, X is κ-regular open. Hence, δ_κ is strong. □

A GT μ is called a *quasi-topology* (briefly QT) [3] on X if it satisfies the following: For $U_1, U_2 \in \mu$, $U_1 \cap U_2 \in \mu$.

Theorem 2.11. *Let (X, μ) be a GTS, and let κ be an enlargement of μ. If μ is QT and κ is monotonic and ordinary, then δ_κ is QT.*

Proof. First, since κ is ordinary, δ_κ is GT. In order to complete the proof, it is sufficient to show that $U_1 \cap U_2$ is κ-regular open, for κ-regular open sets U_1 and U_2. For κ-regular open sets U_1 and U_2, since μ is QT, $U_1 \cap U_2 \in \mu$. Since κ is monotonic, $U_1 \cap U_2 = i_\mu(U_1 \cap U_2) \subseteq i_\mu(\kappa(U_1 \cap U_2)) \subseteq i_\mu(\kappa(U_1)) \cap i_\mu(\kappa(U_2)) = U_1 \cap U_2$. Hence, $U_1 \cap U_2$ is κ-regular open. □

Corollary 2.12. *Let (X, μ) be a GTS, and let κ be an enlargement of μ. If μ is a topology and if κ is monotonic and ordinary, then δ_κ is a topology.*

Proof. Since a topology is a strong GT and a QT, it follows from Theorem 2.10 and Theorem 2.11. □

Definition 2.13. The δ_κ-closure of a subset A of X, denoted by $c_{\delta_\kappa}(A)$, is the intersection of δ_κ-closed sets including A. And the δ_κ-interior of A, denoted by $i_{\delta_\kappa}(A)$, the union of δ_κ-open sets included in A.

Theorem 2.14. *Let μ be a GT on X, $A \subseteq X$ and κ an enlargement of μ. Then*

(1) $x \in c_{\delta_\kappa}(A)$ if and only if for every κ-regular open set G containing x, $G \cap A \neq \emptyset$.

(2) $x \in i_{\delta_\kappa}(A)$ if and only if there exists a κ-regula open set M containing x such that $M \subseteq A$.

Proof. Obvious. □

Theorem 2.15. *Let μ be a GT on X and κ an ordinary enlargement of μ. Then the following hold.*

(1) If $A \subseteq B \subseteq X$, then $c_{\delta_\kappa}(A) \subseteq c_{\delta_\kappa}(B)$ and $i_{\delta_\kappa}(A) \subseteq i_{\delta_\kappa}(B)$.

(2) For $A \subseteq X$, $i_{\delta_\kappa}(A)$ is μ-open.

Proof. It is obvious since δ_κ is a GT on X. □

Theorem 2.16. *Let μ be a GT on X, $A \subseteq X$ and κ an enlargement of μ. If κ is quasi-idempotent and monotonic, then the following hold.*

(1) $x \in c_{\delta_\kappa}(A)$ if and only if $x \in M$ and $M \in \mu$ implies $i_\mu(\kappa(M)) \cap A \neq \emptyset$.

(2) $x \in i_{\delta_\kappa}(A)$ if and only if there exists a μ-open set M containing x such that $i_\mu(\kappa(M)) \subseteq A$.

Proof. (1) For each μ-open set M containing x, by Theorem 2.9, $i_\mu(\kappa(M))$ is a κ-regular open set containing x. Since $x \in c_{\delta_\kappa}(A)$, we have $i_\mu(\kappa(M)) \cap A \neq \emptyset$.

For the converse, let Q be any δ_κ-open set containing x. Then there is a κ-regular open set G such that $x \in G \subseteq Q$. Since G is a μ-open set of x, from hypothesis, it follows $\emptyset \neq i_\mu(\kappa(G)) \cap A = G \cap A \subseteq Q \cap A$. Hence $x \in c_{\delta_\kappa}(A)$.

(2) Let $x \in i_{\delta_\kappa}(A)$. Then there is a δ_κ-open set Q containing x such that $x \in Q \subseteq A$. Moreover, there is a κ-regular open set M such that $x \in M \subseteq Q$. Since M is a μ-open set of x and $i_\mu(\kappa(M)) = M$, $i_\mu(\kappa(M)) \subseteq A$.

For the converse, suppose that there exists a μ-open set M containing x such that $i_\mu(\kappa(M)) \subseteq A$. Then since $i_\mu(\kappa(M))$ is κ-regular open, $x \in i_{\delta_\kappa}(A)$. □

Theorem 2.17. *Let μ be a GT on X, $A \subseteq X$ and κ an enlargement of μ. If κ is quasi-idempotent, monotonic and ordinary, then for any $M \in \mu$, $x \in c_{\delta_\kappa}(M)$ if and only if $x \in G$ and $G \in \mu$ implies $\kappa(G) \cap M \neq \emptyset$.*

Proof. Suppose that $x \in G$ and $G \in \mu$ implies $\kappa(G) \cap M \neq \emptyset$. Then for any μ-open set G containing x, since κ is ordinary, $\emptyset \neq \kappa(G) \cap M \subseteq c_\mu(G) \cap M$. Since $c_\mu(G) \cap M \neq \emptyset$, we have $G \cap M \neq \emptyset$ and $\emptyset \neq G \cap M \subseteq i_\mu(\kappa(G)) \cap M$. Hence, $x \in c_{\delta_\kappa}(M)$.

Conversely, let $x \in c_{\delta_\kappa}(M)$ for $M \in \mu$. Then for any μ-open set G containing x, from Theorem 2.15, $\emptyset \neq i_\mu(\kappa(G)) \cap M \subseteq \kappa(G) \cap M$. So the statement is obtained. □

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