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IMPLICATION-BASED T-FUZZY SUBGROUP OF A FINITE GROUP AND ITS PROPERTIES

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Abstract: Based on the definition of *implication-based fuzzy subgroup* of a finite group given by Yuan, *implication-based T- fuzzy subgroup* and *implication-based T- normal fuzzy subgroup* of a finite group is defined. Some properties of them are proved in this paper. Also $T-$ product, $\dot{T}-$ product and $T^{'}-$ product of these *implication-based T-fuzzy subgroups* of a finite group are defined and its properties are discussed.

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Key Words: implication-based fuzzy subgroup, implication-based T-fuzzy subgroup, implication-based T-fuzzy normal subgroup

1. Introduction

In 1965 the concept of *fuzzy* set was first introduced by Zadeh [1]. Rosenfeld [2] and many others [3], [4], [5] have studied about the *fuzzy normal subgroup.* Anthony and Sherwood [6] redefined *fuzzy* subgroups using t-norm. Many including Sessa [7] studied about these T-*fuzzy* subgroups and proved many prop-

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erties. Yuan [8] defined *implication-based fuzzy subgroup* in 2003. In this paper we define the concept of *implication-based T-fuzzy subgroup* of a finite group and its properties. We also define the $T-$ product, $\dot{T}-$ product and $T^{'}-$ product of these *implication-based T-fuzzy subgroups* of a finite group and proved some properties.

2. Preliminaries

Let X be an universe of discourse and $(G, \cdot)$ be a group. In *fuzzy* logic, truth value of *fuzzy* proposition α is denoted by $[\alpha]$. The *fuzzy* logical and the corresponding set theoretical notations used in this paper are:

1. $(x \in A) = A(x)$;
2. $(\alpha \wedge \beta) = min\{[\alpha], [\beta]\}$;
3. $(\alpha \rightarrow \beta) = min\{1, 1 - [\alpha] + [\beta]\}$;
4. $(\forall x \; \alpha(x)) = inf_{x \in X}[\alpha(x)]$;
5. $(\exists x \; \alpha(x)) = sup_{x \in X}[\alpha(x)]$;

$\models \alpha \Leftrightarrow [\alpha] = 1$ for all valuations. The truth valuation rules given above are those in Lukasiewicz system of continuous-valued logic.

Definition 1. [7] A triangular norm is a real continuous function $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ fulfilling the following properties, for every $a, b, c, d \in [0, 1]$

(i) $t(0, a) = 0, \; t(a, 1) = a$ [boundary conditions],

(ii) $t(a, b) \leq t(c, d)$ if $a \leq c$ and $b \leq d$ [monotonicity],

(iii) $t(a, b) = t(b, a)$ [commutativity],

(iv) $t\,(t(a, b), c) = t\,(a, t(b, c))$ [associativity]

Definition 2. [8] If a *fuzzy* subset A of a group G satisfies for any $x, y \in G$

(i) $\models (x \in A) \wedge (y \in A) \rightarrow (xy \in A)$

(ii) $\models (x \in A) \rightarrow (x^{-1} \in A)$

Then A is called a fuzzifying subgroup.

Definition 3. [8] Let A be a *fuzzy* subset of a finite group G and $\lambda \in (0, 1]$ is a fixed number. If for any $x, y \in G$

(i) $\models_\lambda (x \in A) \wedge (y \in A) \rightarrow (xy \in A)$

(ii) $\models_\lambda (x \in A) \rightarrow (x^{-1} \in A)$

Then A is called an *implication-based fuzzy subgroup* of G.

Definition 4. [9] Let A be an *implication-based fuzzy subgroup* of G and $f : G \rightarrow G$ be a function defined on G. Then the *implication-based fuzzy subgroup* B of $f(G)$ is defined by

$$\models_\lambda \left(\exists x\{(x \in A)\}; x \in f^{-1}(y)\right) \rightarrow (y \in B),$$

for all $y \in f(G)$.

Similarly if B is an *implication-based fuzzy subgroup* of $f(G)$ then the *implication-based fuzzy subgroup* $A = f \circ B$ in G is defined as $\models_\lambda (f(x) \in B) \rightarrow (x \in A)$ for all $x \in G$ and is called the pre-image of B under f.

Definition 5. [9] An *implication-based fuzzy subgroup* A of G is called an *implication-based fuzzy normal subgroup* if $\models_\lambda (xy \in A) \rightarrow (yx \in A) \quad \forall x, y \in G$

Lemma 1. *[9] An homomorphic image or pre-image of an implication-based fuzzy subgroup is an implication-based fuzzy subgroup provided in the former case the sup-property holds.*

Lemma 2. *[10] Generalised Associative Law Let* $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ *be a t-norm then* $T((T(a, b)), (T(c, d))) = T((T(a, c)), (T(b, d))) \quad \forall a, b, c, d \in [0, 1]$

Proof. Let $a, b, c, d \in [0, 1]$. Then

$$\begin{aligned} T((T(a, b)), (T(c, d))) &= T(a, T(b, T(c, d))) \text{ by (iii) and (iv) of definition of t-norm} \\ &= T(a, T(T(b, c), d)) \\ &= T(a, T(T(c, b), d)) \\ &= T(a, T(c, T(b, d))) \\ &= T(T(a, c), T(b, d)). \end{aligned}$$

□

Definition 6. [11] Given two t-norms T_1 and T_2, T_1 is said to be stronger than T_2, if $T_1(x, y) \geq T_2(x, y) \quad \forall x, y \in [0, 1]$ and is written as $T_1 \geq T_2$. And T_1 is said to dominate T_2, if $T_1((T_2(a, b)), (T_2(c, d))) \geq T_2((T_1(a, c)), (T_1(b, d)))$ $\forall a, b, c, d \in [0, 1]$ and is written as $T_1 \gg T_2$.

Let $(G, \cdot)$ be a finite group with the identity element $'e'$, $\lambda \in (0, 1]$ be a fixed number and let $T : [0, 1] \times [0, 1] \to [0, 1]$ be a $t - norm$.

3. Implication-Based T-Fuzzy Subgroup and its Properties

Definition 7. Let A be a *fuzzy* subset of G. For any $x, y \in G$ if

$$\models_\lambda (T((x \in A), (y \in A))) \to (xy \in A)$$

Then A is called an *implication-based T - fuzzy subgroupoid* of G.

Definition 8. Let A be a *fuzzy* subset of G. For any $x, y \in G$ if
(i) $\models_\lambda (T((x \in A), (y \in A))) \to (xy \in A)$
(ii) $\models_\lambda (x \in A) \to (x^{-1} \in A)$
Then A is called an *implication-based T-fuzzy subgroup* of G.

Example for *implication-based T-fuzzy subgroup* of a finite group.
Consider the group $G = \{e, a, b, c\}$ along with the binary operation $'*'$ whose closure table is as follows.

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

For the *fuzzy set* $A : G \to [0, 1]$ defined by $A(e) = 1, A(a) = .25, A(b) = .5, A(c) = .75$ with $\lambda = .2$ and the *implication* operator is that of Lukasiewicz, with the t-norm defined by $T(a, b) = ab$ we have

T	$[e]$	$[a]$	$[b]$	$[c]$
$[e]$	1	.25	.5	.75
$[a]$	.25	.0625	.125	.1875
$[b]$	.5	.125	.25	.375
$[c]$	.75	.1875	.375	.5625

Then A is an *implication-based T-fuzzy subgroup* of G.

Theorem 3. *Let f be a homomorphism of the group G and B be an implication-based T-fuzzy subgroup of $f(G)$ then $A = f \circ B$ is an implication-based T-fuzzy subgroup of G.*

Proof. Let $x, y \in G$

(i)

$\models_\lambda (T((x \in A), (y \in A)))$
$\to (T((f(x) \in B), (f(y) \in B)))$
$\to (f(x)f(y) \in B)$ $\because$ *B is an implication-based T-fuzzy subgroup of G*
$\to (f(xy) \in B)$ $\because f$ is an homomorphism
$\to (xy \in A)$

(ii) $\models_\lambda (x \in A) \to (x^{-1} \in A) \, \forall x \in G$ follows from lemma 1

$\therefore A = f \circ B$ is an *implication-based T-fuzzy subgroup of G.* □

Theorem 4. *Let A be an implication-based T-fuzzy subgroup of G and f be an homomorphism on G. Then B the image of A under f is also an implication-based T-fuzzy subgroup of f(G).*

Proof. (i) Let $y_1, y_2 \in f(G)$.
Let $A_1 = f^{-1}(y_1)$,
$A_2 = f^{-1}(y_2)$,
$A_{12} = f^{-1}(y_1 y_2)$
$A_1 A_2 = \{x \in G / x = a_1 a_2$ for some $a_1 \in A_1, a_2 \in A_2\}$.
Then $x \in G$
$\models_\lambda (x \in A_1 A_2) \to (a_1 a_2 \in A_1 A_2)$
where $a_1 \in A_1, a_2 \in A_2$
$\to (f(a_1 a_2) \in f(A_1 A_2))$
$\to (f(a_1)f(a_2) = y_1 y_2 \in f(A_{12}))$
$\to (y_1 y_2 \in f(A_{12}))$
$\to (f(x) \in f(A_{12}))$
$\to (x \in A_{12})$
$\therefore A_1 A_2 \leq A_{12}$

Now
$\models_\lambda (\exists x_1 x_2 \{T((x_1 \in A), (x_2 \in A))\};$
$x_1 \in A_1, x_2 \in A_2)$
$\to (\exists x_1 x_2 \{(x_1 x_2 \in A)\}; x_1 \in A_1, x_2 \in A_2)$
$\because$ A is an *implication-based T-fuzzy subgroupoid* of G
$\to (\exists x \{(x \in A_1 A_2)\}; x \in A_1 A_2)$
$\to (\exists x \{(x \in A_{12})\}; x \in A_{12})$
$\to \big(\exists x \{(x \in A_{12})\}; x \in f^{-1}(y_1 y_2)\big)$
$\to (y_1 y_2 \in B)$

Since T is a continuous t-norm, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$\models_\lambda ((\exists x_1\{(x_1 \in A)\}; x_1 \in A_1) - \delta) \to x_1^*$

$\models_\lambda ((\exists x_2\{(x_2 \in A)\}; x_2 \in A_2) - \delta) \to x_2^*$

$$\therefore \models_\lambda (T\,((\exists x_1\{(x_1 \in A)\}; x_1 \in A_1), (\exists x_2\{(x_2 \in A)\}; x_2 \in A_2)) - \epsilon) \to T(x_1^*, x_2^*) \qquad (1)$$

Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that

$\models_\lambda ((\exists x_1\{(x_1 \in A)\}; x_1 \in A_1) - \delta) \to (a_1 \in A_1)$

$\models_\lambda ((\exists x_2\{(x_2 \in A)\}; x_2 \in A_2) - \delta) \to (a_2 \in A_2)$

(1) implies

$$\models_\lambda (T\,((\exists x_1\{(x_1 \in A)\}; x_1 \in A_1), (\exists x_2\{(x_2 \in A)\}; x_2 \in A_2)) - \epsilon) \to T((a_1 \in A_1), (a_2 \in A_2))$$

Now $\models_\lambda (T((y_1 \in B), (y_2 \in B)))$

$\to (T((\exists x_1\{(x_1 \in A)\}; x_1 \in A_1), (\exists x_2\{(x_2 \in A)\}; x_2 \in A_2)))$

$\to (\exists x_1x_2\{T((x_1 \in A), (x_2 \in A))\}; x_1 \in A_1, x_2 \in A_2)$

$\to (\exists x_1x_2\{(x_1x_2 \in A)\}; x_1 \in A_1, x_2 \in A_2)$

$\to (y_1y_2 \in B)$

(ii) $\models_\lambda (x \in A) \to (x^{-1} \in A)\, \forall x \in G$ follows from lemma 1.

$\therefore$ B is an *implication-based T-fuzzy subgroup*of $f(G)$. □

Definition 9. An *implication-based T-fuzzy subgroup* A of G is called an *implication-based T-fuzzy normal subgroup* if

$\models_\lambda (xy \in A) \to (yx \in A) \quad \forall x, y \in G$

Theorem 5. *Let A be an implication-based T-fuzzy normal subgroup of G such that $\models_\lambda (e \in A) \to 1$ then $B = \{x \in G/ \models_\lambda (x \in A) \to (e \in A)\}$ is either empty or a normal subgroup of G.*

Proof. Let $x, y \in B$. Then

$\models_\lambda (x \in A) \to (e \in A)$ and

$\models_\lambda (y \in A) \to (e \in A)$.

By the boundary conditions of the t-norm T, we have

$T(a, 1) = 1 \Rightarrow T(1, 1) = 1$.

$\therefore \models_\lambda (T(1, 1))$

$\to (T((x \in A), (y \in A)))$

$\to (T((x \in A), (y^{-1} \in A)))$

$\to (xy^{-1} \in A)$

$\therefore \models_\lambda (xy^{-1} \in A) \to (e \in A)$

$\Rightarrow xy^{-1} \in B$.

Thus B is a subgroup of G.
Now let $g \in G, h \in B$

$\models_\lambda (ghg^{-1} \in A) \to (hgg^{-1} \in A)$

$\because$ A is an implication-based T-fuzzy normal subgroup of G

$\to (h \in A)$
$\to (e \in A)$
$\therefore ghg^{-1} \in B$
$\Rightarrow B$ is a normal subgroup of G. □

Theorem 6. *Let A be an implication-based T-fuzzy subgroup of G and if there is a sequence $\{x_n\}$ in G such that*
$\lim_{n\to\infty} \models_\lambda (T((x_n \in A),(x_n \in A))) \to 1$
then $\models_\lambda (e \in A) \to 1$

Proof. Let $x \in G$
$\models_\lambda (T((x \in A),(x \in A)))$
$\to (T((x \in A),(x^{-1} \in A)))$
$\to (xx^{-1} \in A)$
$\to (e \in A)$
Therefore for each n,
$\models_\lambda (T((x_n \in A),(x_n \in A))) \to (e \in A)$
$\lim_{n\to\infty} \models_\lambda (T((x_n \in A),(x_n \in A)))$
$\to (e \in A)$
By assumption,
$\lim_{n\to\infty} \models_\lambda (T((x_n \in A),(x_n \in A))) \to 1$
$\therefore \models_\lambda (e \in A) \to 1$ □

Theorem 7. *Let A be an implication-based T-fuzzy subgroup of G. If* $\models_\lambda (xy^{-1} \in A) \to 1$ *then*
(i) $\models_\lambda (x \in A) \to (y \in A)$
(ii) $\models_\lambda (y \in A) \to (x \in A) \quad \forall x, y \in G$.

Proof. Let $x, y \in G$

$$\begin{aligned}\models_\lambda (x \in A) &\to (x^{-1} \in A)\\ &\to (T((x^{-1} \in A), 1))\\ &\to (T((x^{-1} \in A),(xy^{-1} \in A)))\\ &\to (x^{-1}xy^{-1} \in A)\\ &\to (y^{-1} \in A)\\ &\to (y \in A)\end{aligned}$$

$$\begin{aligned}\vDash_\lambda (y \in A) &\to (T(1, (y \in A))) \\ &\to (T((xy^{-1} \in A), (y \in A))) \\ &\to (xy^{-1}y \in A) \\ &\to (x \in A)\end{aligned}$$

□

Theorem 8. *Let A be a fuzzy subset of G and T be a t-norm. If $\vDash_\lambda (e \in A) \to 1$ and $\vDash_\lambda (T((x \in A), (y \in A))) \to (xy^{-1} \in A) \quad \forall x, y \in G$ then A is an implication-based T-fuzzy subgroup of G.*

Proof. Let $x, y \in G$

$$\begin{aligned}\vDash_\lambda (y \in A) &\to (T(1, (y \in A))) \\ &\to (T((e \in A), (y \in A))) \\ &\to (ey^{-1} \in A) \\ &\to (y^{-1} \in A)\end{aligned}$$

$$\begin{aligned}\vDash_\lambda (y^{-1} \in A) &\to (T(1, y^{-1} \in A)) \\ &\to (T((e \in A), (y^{-1} \in A))) \\ &\to (e(y^{-1})^{-1} \in A) \\ &\to (ey \in A) \\ &\to (y \in A)\end{aligned}$$

$$\begin{aligned}&\vDash_\lambda (T((x \in A), (y \in A))) \\ &\qquad \to (T((x \in A), (y^{-1} \in A))) \\ &\qquad \to (x(y^{-1})^{-1} \in A) \\ &\qquad \to (xy \in A)\end{aligned}$$

Therefore A is an *implication-based T-fuzzy subgroup* of G. □

This serves as the necessary and sufficient condition for a *fuzzy* subset A of G to be an *implication-based T-fuzzy subgroup* of G.

Theorem 9. *Let A be an implication-based T-fuzzy subgroup of G such that $\vDash_\lambda (a \in A) \to 1$. $f_a : G \to G$ is called as right translation and is defined as $f_a(x) = xa$ and ${}_af : G \to G$ is called as left translation and is defined as ${}_af(x) = ax$. Then $f_a(A) =_a f(A) = A$.*

Proof. Let $x \in G$

$\vDash_\lambda (x \in A)$

$\rightarrow (T((x \in A), 1))$
$\rightarrow (T((x \in A), (a \in A)))$
$\rightarrow (T((x \in A), (a^{-1} \in A)))$
$\rightarrow (xa^{-1} \in A)$
$\rightarrow (\exists z\{(z \in A)\}; z \in {f_a}^{-1}(x))$
$\rightarrow (x \in f_a(A))$

$\models_\lambda (x \in f_a(A))$
$\rightarrow (xa^{-1} \in A)$
$\rightarrow (T((xa^{-1} \in A), 1))$
$\rightarrow (T((xa^{-1} \in A), (a \in A)))$
$\rightarrow (xa^{-1}a \in A)$
$\rightarrow (x \in A)$
$\therefore \quad f_a(A) = A.$
Similarly we can prove that ${}_af(A) = A$.
Let $x \in G$
$\models_\lambda (x \in A)$
$\rightarrow (T(1, (x \in A)))$
$\rightarrow (T((a \in A), (x \in A)))$
$\rightarrow (T((a^{-1} \in A), (x \in A)))$
$\rightarrow (a^{-1}x \in A)$
$\rightarrow (\exists z\{(z \in A)\}; z \in_a f^{-1}(x))$
$\rightarrow (x \in_a f(A))$

$\models_\lambda (x \in_a f(A))$
$\rightarrow (a^{-1}x \in A)$
$\rightarrow (T(1, (a^{-1}x \in A)))$
$\rightarrow (T((a \in A), (a^{-1}x \in A)))$
$\rightarrow (aa^{-1}x \in A)$
$\rightarrow (x \in A)$
$\therefore \quad {}_af(A) = A.$ □

Definition 10. Let A and B be two *implication-based T-fuzzy subgroups* of G. Then the *implication-based T-fuzzy product* of A and B denoted by $[A \cdot B]_T$ is defined as
$$\models_\lambda (\exists y, z\{T((y \in A), (z \in B))\}; yz = x; y, z \in G) \rightarrow (x \in A \cdot B)\ \forall x \in G.$$

Definition 11. Let A and B be two *implication-based T-fuzzy subgroups* of G. Then the *implication-based $\dot{T}$-fuzzy product* of A and B denoted by $[A \cdot B]_{\dot{T}}$

is defined as
$\vDash_\lambda (T\,((x \in A),(x \in B))) \to (x \in [A \cdot B]_{\dot{T}})\ \forall x \in G.$

Definition 12. Let G_1 and G_2 be two finite groups and $G = G_1 \times G_2$ be the direct product group of G_1 and G_2. Let A_1 and A_2 be two *implication-based T-fuzzy subgroups* of G_1 and G_2 respectively and T' be a t-norm. Then the *implication-based* T'*-fuzzy direct product* of A_1 and A_2 denoted by $[A_1 \times A_2]_{T'}$ is defined as

$$\vDash_\lambda \left(T'\,((a \in A_1),(b \in A_2))\right) \to \left((a,b) \in [A_1 \times A_2]_{T'}\right) \quad \forall a \in G_1, b \in G_2.$$

Theorem 10. *Let G_1 and G_2 be two finite groups and $G = G_1 \times G_2$ be the direct product of G_1 and G_2. Let A_1 and A_2 be two implication-based T-fuzzy subgroups of G_1 and G_2 respectively. Then $A = [A_1 \times A_2]_T$ is an implication-based T-fuzzy subgroup of G.*

Proof. Let $x = (a_1, b_1)$ and $y = (a_2, b_2)$ be any two elements of the group $G = G_1 \times G_2$.

$$\begin{aligned}
&\vDash_\lambda (T((x \in A),(y \in A)))\\
&\to (T(((a_1,b_1) \in A),((a_2,b_2) \in A)))\\
&\to (T(T((a_1 \in A_1),(b_1 \in A_2)),\\
&\qquad T((a_2 \in A_1),(b_2 \in A_2)))\\
&\to (T(T((a_1 \in A_1),(a_2 \in A_2)),T((b_1 \in A_1),(b_2 \in A_2)))\\
&\qquad \text{by the generalised associative law}\\
&\to (T((a_1{a_2}^{-1} \in A_1),(b_1{b_2}^{-1} \in A_2)))\\
&\to ((a_1{a_2}^{-1}, b_1{b_2}^{-1}) \in A)\\
&\to (xy^{-1} \in A)
\end{aligned}$$

Let $e = (e_1, e_2) \in G$ where e_1, e_2 are the identity elements of the group G_1 and G_2 respectively.

$$\begin{aligned}
\vDash_\lambda (T((e_1 \in A_1),(e_2 \in A_2)))&\\
&\to ((e_1,e_2) \in A)\\
&\to (e \in A)
\end{aligned}$$

$$\begin{aligned}
\text{But } \vDash_\lambda (T((e_1 \in A_1),(e_2 \in A_2)))&\\
&\to (T(1,1))\\
&\to 1
\end{aligned}$$

$\therefore\ \vDash_\lambda (e \in A) \to 1$

By theorem 8 $A = [A_1 \times A_2]_T$ is an implication-based T-fuzzy subgroup of G. ☐

Theorem 11. *Let G_1 and G_2 be two finite groups and $G = G_1 \times G_2$ be the direct product of G_1 and G_2. Let A and B be two implication-based T-fuzzy subgroups of G_1 and G_2 respectively. Then $[A \times B]_{T'}$ is an implication-based T-fuzzy subgroup of G provided $T^{'}$ dominates T. Moreover if A and B are implication-based T-fuzzy normal subgroup of G then $[A \times B]_{T'}$ is an implication-based T-fuzzy normal subgroup of G.*

Proof. Let $x, y \in G$ where $x = (a_1, b_1)$ and $y = (a_2, b_2)$; $a_1, a_2 \in G_1$ and $b_1, b_2 \in G_2$.
$\models_\lambda (T((x \in [A \times B]_{T'}), (y \in [A \times B]_{T'})))$
$\to (T(((a_1, b_1) \in [A \times B]_{T'}), ((a_2, b_2) \in [A \times B]_{T'})))$
$\to (T((T^{'}((a_1 \in A), (b_1 \in B))), (T^{'}((a_2 \in A), (b_2 \in B)))))$
$\to (T^{'}((T((a_1 \in A), (a_2 \in A))), (T((b_1 \in B), (b_2 \in B)))))$
$\because T^{'}$ dominates T
$\to (T^{'}((a_1{a_2}^{-1} \in A), ((b_1{b_2}^{-1} \in B)))$
$\to ((a_1{a_2}^{-1}, b_1{b_2}^{-1}) \in [A \times B]_{T'})$
$\to (xy^{-1} \in [A \times B]_{T'})$
Let $e = (e_1, e_2) \in G$
$\models_\lambda (T^{'}((e_1 \in A), (e_2 \in B)))$
$\to ((e_1, e_2) \in [A \times B]_{T'})$
$\to (e \in [A \times B]_{T'})$
But $\models_\lambda (T^{'}((e_1 \in A), (e_2 \in B)))$
$\to (T^{'}(1, 1) \in [A \times B]_{T'})$
$\to 1$
$\therefore \models_\lambda (e \in [A \times B]_{T'}) \to 1$
By theorem 8
$[A \times B]_{T'}$ is an implication-based T-fuzzy subgroup of G.
$\models_\lambda (T^{'}((xy \in A), (xy \in B)))$
$\to (xy \in [A \times B]_{T'})$
But $\models_\lambda (T^{'}((xy \in A), (xy \in B)))$
$\to (T^{'}((yx \in A), (yx \in B)))$
$\because$ A and B are implication-basedT-fuzzy normal subgroups of G
$\to (yx \in [A \times B]_{T'})$
$[A \times B]_{T'}$ is an *implication-based T-fuzzy normal subgroup* of G. □

Theorem 12. *Let $i^2 : G \to G \times G$ defined by $i^2(x) = (x, x) \forall x \in G$. If A and B are two implication-based T-fuzzy subgroups of G then $[A \cdot B]_{\dot{T}}$ is the pre-image of $[A \times B]_T$ under i^2.*

Proof. Let $x \in G$. By definition we have

$\vDash_\lambda (T((x \in A), (x \in B))) \to ((x, x) \in [A \times B]_T)$
$\vDash_\lambda (T((x \in A), (x \in B))) \to (x \in [A \cdot B]_{\dot{T}})$
$\Rightarrow \quad \vDash_\lambda ((x, x) \in [A \times B]_T) \to (x \in [A \cdot B]_{\dot{T}})$
$\Rightarrow \quad \vDash_\lambda (i^2(x) \in [A \times B]_T) \to (x \in [A \cdot B]_{\dot{T}})$
$\therefore \quad [A \cdot B]_{\dot{T}}$ is the pre-image of $[A \times B]_T$ under i^2. □

Theorem 13. *Let A and B be two implication-based T-fuzzy normal subgroups of a group G. Let T' be a t-norm which dominates T. Then $[A \cdot B]_{\dot{T}'}$ is an implication-based T-fuzzy normal subgroup of G.*

Proof. $\vDash_\lambda (T'((e \in A), (e \in B))) \to (e \in [A \cdot B]_{\dot{T}'})$
But $\vDash_\lambda (T'((e \in A), (e \in B))) \to (T'(1, 1)) \to 1$
$\therefore \quad \vDash_\lambda (e \in [A \cdot B]_{\dot{T}'}) \to 1$
Let $x, y \in G$
$\vDash_\lambda (T((x \in [A.B]_{\dot{T}'}), (y \in [A.B]_{\dot{T}'})))$
$\to ((T(T'((x \in A), (x \in B)), T'((y \in A), (y \in B))))$
$\to ((T'(T((x \in A), (y \in A)), T((x \in B), (y \in B))))$
$\because T'$ dominates T
$\to (T'((xy^{-1} \in A), (xy^{-1} \in B)))$
$\to (xy^{-1} \in [A \cdot B]_{\dot{T}'})$
By theorem 8 $[A \cdot B]_{\dot{T}'}$ is an implication-based T-fuzzy subgroup of G.
$\vDash_\lambda (T'((xy \in A), (xy \in B))) \to (xy \in [A.B]_{\dot{T}'})$
But
$\vDash_\lambda (T'((xy \in A), (xy \in B))) \to (T'((yx \in A), (yx \in B)))$
$\because$ A and B are implication-based T-fuzzy normal subgroups of G
$\to (yx \in [A \cdot B]_{\dot{T}'})$
$[A \cdot B]_{\dot{T}'}$ is an *implication-based T-fuzzy normal subgroup* of G. □

We have if f is an homomorphism on a group G and A and B are *implication-based T-fuzzy subgroups* of $f(G)$. If the t-norm T' dominates T. Then $[A \cdot B]_{\dot{T}'}$ is an *implication-based T-fuzzy subgroup* of $f(G)$. By theorem 3 $f \circ A$, $f \circ B$ and $f \circ [A \cdot B]_{\dot{T}'}$ are *implication-based T-fuzzy subgroups of G.*

Theorem 14. *Let f be an homomorphism on a finite group G. Let B_1 and B_2 be two implication-based T-fuzzy subgroups of f(G). Let the t-norm T' dominate T. If $B = [B_1 \cdot B_2]_{\dot{T}'}$ is the implication-based $\dot{T}$-fuzzy product of B_1 and B_2 and $[A_1 \cdot A_2]_{\dot{T}'}$ is the implication-based $\dot{T}$-fuzzy product of $A_1 = f \circ B_1$ and $A_2 = f \circ B_2$ then*
$\vDash_\lambda (x \in [A_1 \cdot A_2]_{\dot{T}'}) \to (x \in A) \quad \forall x \in G$ *where $A = f \circ B$.*

Proof. Let $x \in G$

$$\models_\lambda (T^{'}((x \in A_1), (x \in A_2))) \to (x \in [A_1 \cdot A_2]_{\dot{T}'})$$

But

$$\begin{aligned}\models_\lambda (T^{'}((x \in A_1), (x \in A_2))) \\ &\to (T^{'}((f(x) \in B_1), (f(x) \in B_2))) \\ &\to (f(x) \in [B_1 \cdot B_2]_{\dot{T}'}) \\ &\to (f(x) \in B) \\ &\to (x \in A)\end{aligned}$$

□

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