IMPLICATION-BASED T-FUZZY SUBGROUP OF
A FINITE GROUP AND ITS PROPERTIES

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Abstract: Based on the definition of implication-based fuzzy subgroup of a finite group given by Yuan, implication-based $T$-fuzzy subgroup and implication-based $T$-normal fuzzy subgroup of a finite group is defined. Some properties of them are proved in this paper. Also $T$-product, $T^\prime$-product and $T^\prime$-product of these implication-based $T$-fuzzy subgroups of a finite group are defined and its properties are discussed.

AMS Subject Classification: 03E72, 08A72, 20N25

Key Words: implication-based fuzzy subgroup, implication-based T-fuzzy subgroup, implication-based T-fuzzy normal subgroup

1. Introduction

In 1965 the concept of fuzzy set was first introduced by Zadeh [1]. Rosenfeld [2] and many others [3], [4], [5] have studied about the fuzzy normal subgroup. Anthony and Sherwood [6] redefined fuzzy subgroups using t-norm. Many including Sessa [7] studied about these $T$-fuzzy subgroups and proved many prop...
properties. Yuan [8] defined implication-based fuzzy subgroup in 2003. In this paper we define the concept of implication-based $T$-fuzzy subgroup of a finite group and its properties. We also define the $T$- product, $\hat{T}$- product and $T'$- product of these implication-based $T$-fuzzy subgroups of a finite group and proved some properties.

2. Preliminaries

Let $X$ be an universe of discourse and $(G, \cdot)$ be a group. In fuzzy logic, truth value of fuzzy proposition $\alpha$ is denoted by $[\alpha]$. The fuzzy logical and the corresponding set theoretical notations used in this paper are:

1. $(x \in A) = A(x)$;
2. $(\alpha \land \beta) = \min\{[\alpha], [\beta]\}$;
3. $(\alpha \rightarrow \beta) = \min\{1, 1 - [\alpha] + [\beta]\}$;
4. $(\forall x \alpha(x)) = \inf_{x \in X}[\alpha(x)]$;
5. $(\exists x \alpha(x)) = \sup_{x \in X}[\alpha(x)]$;

$\vdash \alpha \iff [\alpha] = 1$ for all valuations. The truth valuation rules given above are those in Lukasiewicz system of continuous-valued logic.

**Definition 1.** [7] A triangular norm is a real continuous function $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ fulfilling the following properties, for every $a, b, c, d \in [0, 1]$

(i) $t(0, a) = 0$, $t(a, 1) = a$ [boundary conditions],
(ii) $t(a, b) \leq t(c, d)$ if $a \leq c$ and $b \leq d$ [monotonicity],
(iii) $t(a, b) = t(b, a)$ [commutativity],
(iv) $t(t(a, b), c) = t(a, t(b, c))$ [associativity]

**Definition 2.** [8] If a fuzzy subset $A$ of a group $G$ satisfies for any $x, y \in G$

(i) $\vdash (x \in A) \land (y \in A) \rightarrow (xy \in A)$
(ii) $\vdash (x \in A) \rightarrow (x^{-1} \in A)$

Then $A$ is called a fuzzifying subgroup.
Definition 3. [8] Let $A$ be a fuzzy subset of a finite group $G$ and $\lambda \in (0, 1]$ is a fixed number. If for any $x, y \in G$

(i) $\models_{\lambda} (x \in A) \land (y \in A) \rightarrow (xy \in A)$

(ii) $\models_{\lambda} (x \in A) \rightarrow (x^{-1} \in A)$

Then $A$ is called an implication-based fuzzy subgroup of $G$.

Definition 4. [9] Let $A$ be an implication-based fuzzy subgroup of $G$ and $f : G \rightarrow G$ be a function defined on $G$. Then the implication-based fuzzy subgroup $B$ of $f(G)$ is defined by

$\models_{\lambda} (\exists x \{ (x \in A) \}; x \in f^{-1}(y)) \rightarrow (y \in B)$,

for all $y \in f(G)$.

Similarly if $B$ is an implication-based fuzzy subgroup of $f(G)$ then the implication-based fuzzy subgroup $A = f \circ B$ in $G$ is defined as $\models_{\lambda} (f(x) \in B) \rightarrow (x \in A)$ for all $x \in G$ and is called the pre-image of $B$ under $f$.

Definition 5. [9] An implication-based fuzzy subgroup $A$ of $G$ is called an implication-based fuzzy normal subgroup if $\models_{\lambda} (xy \in A) \rightarrow (yx \in A) \ \forall x, y \in G$

Lemma 1. [9] An homomorphic image or pre-image of an implication-based fuzzy subgroup is an implication-based fuzzy subgroup provided in the former case the sup-property holds.

Lemma 2. [10] Generalised Associative Law

Let $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a t-norm then $T((T(a, b)), (T(c, d))) = T((T(a, c)), (T(b, d))) \ \forall a, b, c, d \in [0, 1]$

Proof. Let $a, b, c, d \in [0, 1]$. Then

$T((T(a, b)), (T(c, d))) = T(a, T(b, T(c, d)))$ by (iii) and (iv) of definition of t-norm

$= T(a, T(T(b, c), d))$

$= T(a, T(T(c, b), d))$

$= T(a, T(c, T(b, d)))$

$= T(T(a, c), T(b, d))$. $\square$

Definition 6. [11] Given two t-norms $T_1$ and $T_2$, $T_1$ is said to be stronger than $T_2$, if $T_1(x, y) \geq T_2(x, y) \ \forall x, y \in [0, 1]$ and is written as $T_1 \geq T_2$. And $T_1$ is said to dominate $T_2$, if $T_1((T_2(a, b)), (T_2(c, d))) \geq T_2((T_1(a, c)), (T_1(b, d))) \ \forall a, b, c, d \in [0, 1]$ and is written as $T_1 \gg T_2$. 
Let \((G, \cdot)\) be a finite group with the identity element \('e'\), \(\lambda \in (0, 1]\) be a fixed number and let \(T : [0, 1] \times [0, 1] \rightarrow [0, 1]\) be a \(t\) – norm.

3. Implication-Based T-Fuzzy Subgroup and its Properties

**Definition 7.** Let \(A\) be a fuzzy subset of \(G\). For any \(x, y \in G\) if

\[ \models_\lambda (T((x \in A), (y \in A))) \rightarrow (xy \in A) \]

Then \(A\) is called an *implication-based \(T\) - fuzzy subgroupoid* of \(G\).

**Definition 8.** Let \(A\) be a fuzzy subset of \(G\). For any \(x, y \in G\) if

(i) \[\models_\lambda (T((x \in A), (y \in A))) \rightarrow (xy \in A)\]

(ii) \[\models_\lambda (x \in A) \rightarrow (x^{-1} \in A)\]

Then \(A\) is called an *implication-based \(T\)-fuzzy subgroup* of \(G\).

Example for *implication-based \(T\)-fuzzy subgroup* of a finite group.

Consider the group \(G = \{e, a, b, c\}\) along with the binary operation \('\ast'\) whose closure table is as follows.

<table>
<thead>
<tr>
<th>*</th>
<th>e</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>e</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>e</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>e</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>e</td>
</tr>
</tbody>
</table>

For the fuzzy set \(A : G \rightarrow [0, 1]\) defined by \(A(e) = 1, A(a) = .25, A(b) = .5, A(c) = .75\) with \(\lambda = .2\) and the *implication* operator is that of Lukasiewicz, with the t-norm defined by \(T(a, b) = ab\) we have

<table>
<thead>
<tr>
<th>T</th>
<th>e</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>1</td>
<td>.25</td>
<td>.5</td>
<td>.75</td>
</tr>
<tr>
<td>a</td>
<td>.25</td>
<td>.0625</td>
<td>.125</td>
<td>.1875</td>
</tr>
<tr>
<td>b</td>
<td>.5</td>
<td>.125</td>
<td>.25</td>
<td>.375</td>
</tr>
<tr>
<td>c</td>
<td>.75</td>
<td>.1875</td>
<td>.375</td>
<td>.5625</td>
</tr>
</tbody>
</table>

Then \(A\) is an *implication-based \(T\)-fuzzy subgroup* of \(G\).

**Theorem 3.** Let \(f\) be a homomorphism of the group \(G\) and \(B\) be an implication-based \(T\)-fuzzy subgroup of \(f(G)\) then \(A = f \circ B\) is an implication-based \(T\)-fuzzy subgroup of \(G\).
Proof. Let $x, y \in G$

(i) 
\[ \begin{align*}
\models_\lambda (T((x \in A), (y \in A))) \\
\rightarrow (T((f(x) \in B), (f(y) \in B))) \\
\rightarrow (f(x)f(y) \in B) \therefore B \text{ is an implication-based T-fuzzy subgroup of } G \\
\rightarrow (f(xy) \in B) \therefore f \text{ is an homomorphism} \\
\rightarrow (xy \in A) \\
\end{align*} \]

(ii) \[ \begin{align*}
\models_\lambda (x \in A) \rightarrow (x^{-1} \in A) \forall x \in G \text{ follows from lemma 1} \\
\therefore A = f \circ B \text{ is an implication-based T-fuzzy subgroup of } G. \\
\end{align*} \]

Theorem 4. Let $A$ be an implication-based T-fuzzy subgroup of $G$ and $f$ be an homomorphism on $G$. Then $B$ the image of $A$ under $f$ is also an implication-based T-fuzzy subgroup of $f(G)$.

Proof. (i) Let $y_1, y_2 \in f(G)$.

Let $A_1 = f^{-1}(y_1)$,
$A_2 = f^{-1}(y_2)$,
$A_{12} = f^{-1}(y_1y_2)$

$A_1A_2 = \{ x \in G / x = a_1a_2 \text{ for some } a_1 \in A_1, a_2 \in A_2 \}$.

Then $x \in G$

\[ \begin{align*}
\models_\lambda (x \in A_1A_2) & \rightarrow (a_1a_2 \in A_1A_2) \\
& \text{ where } a_1 \in A_1, a_2 \in A_2 \\
& \rightarrow (f(a_1a_2) \in f(A_1A_2)) \\
& \rightarrow (f(a_1)f(a_2) = y_1y_2 \in f(A_{12})) \\
& \rightarrow (y_1y_2 \in f(A_{12})) \\
& \rightarrow (x \in A_{12}) \\
& \therefore A_1A_2 \leq A_{12} \\
\end{align*} \]

Now

\[ \begin{align*}
\models_\lambda (\exists x_1x_2\{T((x_1 \in A), (x_2 \in A))\}; \\
& x_1 \in A_1, x_2 \in A_2 \\
& \rightarrow (\exists x_1x_2\{(x_1x_2 \in A)\}; x_1 \in A_1, x_2 \in A_2) \\
& \therefore A \text{ is an implication-based T-fuzzy subgroupoid of } G \\
& \rightarrow (\exists x\{(x \in A_1A_2)\}; x \in A_1A_2) \\
& \rightarrow (\exists x\{(x \in A_{12})\}; x \in A_{12}) \\
& \rightarrow (\exists x\{(x \in A_1)\}; x \in A_1) \\
& \rightarrow (\exists x\{(x \in A_2)\}; x \in A_2) \\
& \rightarrow (y_1y_2 \in B) \\
\end{align*} \]
Since \( T \) is a continuous t-norm, for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that
\[
\models \lambda ((\exists x_1\{x_1 \in A\}; x_1 \in A_1) - \delta) \rightarrow x_1^*
\]
\[
\models \lambda ((\exists x_2\{x_2 \in A\}; x_2 \in A_2) - \delta) \rightarrow x_2^*
\]
\[
\therefore \models \lambda (T((\exists x_1\{x_1 \in A\}; x_1 \in A_1),
(\exists x_2\{x_2 \in A\}; x_2 \in A_2)) - \epsilon) \rightarrow T(x_1^*, x_2^*) \quad (1)
\]

Choose \( a_1 \in A_1 \) and \( a_2 \in A_2 \) such that
\[
\models \lambda ((\exists x_1\{x_1 \in A\}; x_1 \in A_1) - \delta) \rightarrow (a_1 \in A_1)
\]
\[
\models \lambda ((\exists x_2\{x_2 \in A\}; x_2 \in A_2) - \delta) \rightarrow (a_2 \in A_2)
\]
(1) implies
\[
\models \lambda (T((\exists x_1\{x_1 \in A\}; x_1 \in A_1), (\exists x_2\{x_2 \in A\}; x_2 \in A_2)) - \epsilon)
\]
\[
\rightarrow T((a_1 \in A_1), (a_2 \in A_2))
\]
Now \( \models \lambda (T((y_1 \in B), (y_2 \in B)))
\]
\[
\rightarrow (T((\exists x_1\{x_1 \in A\}; x_1 \in A_1), (\exists x_2\{x_2 \in A\}; x_2 \in A_2)))
\]
\[
\rightarrow (\exists x_1 x_2\{T((x_1 \in A), (x_2 \in A))\}; x_1 \in A_1, x_2 \in A_2)
\]
\[
\rightarrow (\exists x_1 x_2\{(x_1x_2 \in A)\}; x_1 \in A_1, x_2 \in A_2)
\]
\[
\rightarrow (y_1y_2 \in B)
\]
(ii) \( \models \lambda (x \in A) \rightarrow (x^{-1} \in A) \forall x \in G \) follows from lemma 1.
\[\therefore B \text{ is an implication-based } T\text{-fuzzy subgroup of } f(G).\]

**Definition 9.** An implication-based \( T\)-fuzzy subgroup \( A \) of \( G \) is called an implication-based \( T\)-fuzzy normal subgroup if
\[
\models \lambda (xy \in A) \rightarrow (yx \in A) \quad \forall x, y \in G
\]

**Theorem 5.** Let \( A \) be an implication-based \( T\)-fuzzy normal subgroup of \( G \) such that \( \models \lambda (e \in A) \rightarrow 1 \) then \( B = \{ x \in G/ \models \lambda (x \in A) \rightarrow (e \in A) \} \) is either empty or a normal subgroup of \( G \).

**Proof.** Let \( x, y \in B \). Then
\[
\models \lambda (x \in A) \rightarrow (e \in A) \text{ and}
\]
\[
\models \lambda (y \in A) \rightarrow (e \in A).
\]
By the boundary conditions of the t-norm \( T \), we have
\[
T(a, 1) = 1 \Rightarrow T(1, 1) = 1.
\]
\[\therefore \models \lambda (T(1, 1))
\]
\[
\rightarrow (T((x \in A), (y \in A)))
\]
\[
\rightarrow (T((x \in A), (y^{-1} \in A)))
\]
\[
\rightarrow (xy^{-1} \in A)
\]
\[\therefore \models \lambda (xy^{-1} \in A) \rightarrow (e \in A)
\]
\[\Rightarrow xy^{-1} \in B.\]
Thus $B$ is a subgroup of $G$.

Now let $g \in G, h \in B$

\[ \vdash_\lambda (ghg^{-1} \in A) \rightarrow (hgg^{-1} \in A) \]

\[ \therefore A \text{ is an implication-based T-fuzzy normal subgroup of } G \]

$\rightarrow (h \in A)$

$\rightarrow (e \in A)$

\[ \vdash_\lambda ghg^{-1} \in B \]

$\Rightarrow B$ is a normal subgroup of $G$. 

\[ \square \]

**Theorem 6.** Let $A$ be an implication-based T-fuzzy subgroup of $G$ and if there is a sequence \( \{x_n\} \) in $G$ such that

\[ \lim_{n \to \infty} \vdash_\lambda (T((x_n \in A), (x_n \in A))) \to 1 \]

then \( \vdash_\lambda (e \in A) \to 1 \)

**Proof.** Let $x \in G$

\[ \vdash_\lambda (T((x \in A), (x \in A))) \]

$\rightarrow (T((x \in A), (x^{-1} \in A)))$

$\rightarrow (xx^{-1} \in A)$

$\rightarrow (e \in A)$

Therefore for each $n$,

\[ \vdash_\lambda (T((x_n \in A), (x_n \in A))) \to (e \in A) \]

\[ \lim_{n \to \infty} \vdash_\lambda (T((x_n \in A), (x_n \in A))) \to (e \in A) \]

By assumption,

\[ \lim_{n \to \infty} \vdash_\lambda (T((x_n \in A), (x_n \in A))) \to 1 \]

\[ \therefore \vdash_\lambda (e \in A) \to 1 \]

\[ \square \]

**Theorem 7.** Let $A$ be an implication-based T-fuzzy subgroup of $G$.

If \( \vdash_\lambda (xy^{-1} \in A) \to 1 \) then

(i) \( \vdash_\lambda (x \in A) \to (y \in A) \)

(ii) \( \vdash_\lambda (y \in A) \to (x \in A) \) \quad \forall x, y \in G.

**Proof.** Let $x, y \in G$

\[ \vdash_\lambda (x \in A) \to (x^{-1} \in A) \]

$\rightarrow (T((x^{-1} \in A), 1))$

$\rightarrow (T((x^{-1} \in A), (xy^{-1} \in A)))$

$\rightarrow (x^{-1}xy^{-1} \in A)$

$\rightarrow (y^{-1} \in A)$

$\rightarrow (y \in A)$
\[ \vdash_\lambda (y \in A) \rightarrow (T(1, (y \in A))) \]
\[ \rightarrow (T((xy^{-1} \in A), (y \in A))) \]
\[ \rightarrow (xy^{-1}y \in A) \]
\[ \rightarrow (x \in A) \]

**Theorem 8.** Let \( A \) be a fuzzy subset of \( G \) and \( T \) be a t-norm. If \( \vdash_\lambda (e \in A) \rightarrow 1 \) and \( \vdash_\lambda (T((x \in A), (y \in A))) \rightarrow (xy^{-1} \in A) \) \( \forall x, y \in G \) then \( A \) is an implication-based \( T \)-fuzzy subgroup of \( G \).

**Proof.** Let \( x, y \in G \)
\[ \vdash_\lambda (y \in A) \rightarrow (T(1, (y \in A))) \]
\[ \rightarrow (T((e \in A), (y \in A))) \]
\[ \rightarrow (ey^{-1} \in A) \]
\[ \rightarrow (y^{-1} \in A) \]
\[ \vdash_\lambda (y^{-1} \in A) \rightarrow (T(1, y^{-1} \in A)) \]
\[ \rightarrow (T((e \in A), (y^{-1} \in A))) \]
\[ \rightarrow (e(y^{-1})^{-1} \in A) \]
\[ \rightarrow (ey \in A) \]
\[ \rightarrow (y \in A) \]
\[ \vdash_\lambda (T((x \in A), (y \in A))) \]
\[ \rightarrow (T((x \in A), (y^{-1} \in A))) \]
\[ \rightarrow (x(y^{-1})^{-1} \in A) \]
\[ \rightarrow (xy \in A) \]
Therefore \( A \) is an implication-based \( T \)-fuzzy subgroup of \( G \).  

This serves as the necessary and sufficient condition for a fuzzy subset \( A \) of \( G \) to be an implication-based \( T \)-fuzzy subgroup of \( G \).

**Theorem 9.** Let \( A \) be an implication-based \( T \)-fuzzy subgroup of \( G \) such that \( \vdash_\lambda (a \in A) \rightarrow 1 \). \( f_a : G \rightarrow G \) is called as right translation and is defined as \( f_a(x) = xa \) and \( a \) is called as left translation and is defined as \( af(x) = ax \). Then \( f_a(A) = A \).

**Proof.** Let \( x \in G \)
\[ \vdash_\lambda (x \in A) \]
→ (T((x ∈ A), 1))
→ (T((x ∈ A), (a ∈ A)))
→ (T((x ∈ A), (a⁻¹ ∈ A)))
→ (xa⁻¹ ∈ A)
→ (∃z{ (z ∈ A)}; z ∈ fa⁻¹(x))
→ (x ∈ fa(A))

⊨₂₅₅₆ (x ∈ fa(A))
→ (xa⁻¹ ∈ A)
→ (T((xa⁻¹ ∈ A), 1))
→ (T((xa⁻¹ ∈ A), (a ∈ A)))
→ (xa⁻¹a ∈ A)
→ (x ∈ A)
∴ f_a(A) = A.

Similarly we can prove that a f(A) = A.

Let x ∈ G
⊨₂₅₅₆ (x ∈ A)
→ (T(1, (x ∈ A)))
→ (T((a ∈ A), (x ∈ A)))
→ (T((a⁻¹ ∈ A), (x ∈ A)))
→ (a⁻¹x ∈ A)
→ (∃z{ (z ∈ A)}; z ∈ a f⁻¹(x))
→ (x ∈ a f(A))

⊨₂₅₅₆ (x ∈ a f(A))
→ (a⁻¹x ∈ A)
→ (T(1, (a⁻¹x ∈ A)))
→ (T((a ∈ A), (a⁻¹x ∈ A)))
→ (aa⁻¹x ∈ A)
→ (x ∈ A)
∴ a f(A) = A.

**Definition 10.** Let A and B be two implication-based T-fuzzy subgroups of G. Then the implication-based T-fuzzy product of A and B denoted by [A · B]_T is defined as

\[ \exists y, z \{ T((y ∈ A), (z ∈ B)); yz = x; y, z ∈ G \} \rightarrow (x ∈ A · B) \ \forall x ∈ G. \]

**Definition 11.** Let A and B be two implication-based T-fuzzy subgroups of G. Then the implication-based ˙T-fuzzy product of A and B denoted by [A · B] ˙T
is defined as
\[ \vDash_{\lambda} (T((x \in A), (x \in B))) \rightarrow (x \in [A \cdot B]) \forall x \in G. \]

**Definition 12.** Let \( G_1 \) and \( G_2 \) be two finite groups and \( G = G_1 \times G_2 \) be the direct product group of \( G_1 \) and \( G_2 \). Let \( A_1 \) and \( A_2 \) be two implication-based \( T \)-fuzzy subgroups of \( G_1 \) and \( G_2 \) respectively and \( T' \) be a \( t \)-norm. Then the implication-based \( T' \)-fuzzy direct product of \( A_1 \) and \( A_2 \) denoted by \([A_1 \times A_2]_{T'}\) is defined as
\[ \vDash_{\lambda} \left( T'((a \in A_1), (b \in A_2)) \right) \rightarrow ((a, b) \in [A_1 \times A_2]_{T'}) \forall a \in G_1, b \in G_2. \]

**Theorem 10.** Let \( G_1 \) and \( G_2 \) be two finite groups and \( G = G_1 \times G_2 \) be the direct product of \( G_1 \) and \( G_2 \). Let \( A_1 \) and \( A_2 \) be two implication-based \( T \)-fuzzy subgroups of \( G_1 \) and \( G_2 \) respectively. Then \( A = [A_1 \times A_2]_T \) is an implication-based \( T \)-fuzzy subgroup of \( G \).

**Proof.** Let \( x = (a_1, b_1) \) and \( y = (a_2, b_2) \) be any two elements of the group \( G = G_1 \times G_2 \).
\[ \vDash_{\lambda} (T((a \in A), (y \in A))) \]
\[ \rightarrow (T((a_1, b_1) \in A), ((a_2, b_2) \in A))) \]
\[ \rightarrow (T(T((a_1 \in A_1), (b_1 \in A_2)), \]
\[ T((a_2 \in A_1), (b_2 \in A_2)))) \]
\[ \rightarrow (T(T((a_1 \in A_1), (a_2 \in A_2)), T((b_1 \in A_1), (b_2 \in A_2)))) \]
by the generalised associative law
\[ \rightarrow (T((a_1 a_2^{-1} \in A_1), (b_1 b_2^{-1} \in A_2))) \]
\[ \rightarrow ((a_1 a_2^{-1}, b_1 b_2^{-1}) \in A) \]
\[ \rightarrow (xy^{-1} \in A) \]

Let \( e = (e_1, e_2) \in G \) where \( e_1, e_2 \) are the identity elements of the group \( G_1 \) and \( G_2 \) respectively.
\[ \vDash_{\lambda} (T((e_1 \in A_1), (e_2 \in A_2))) \]
\[ \rightarrow ((e_1, e_2) \in A) \]
\[ \rightarrow (e \in A) \]

But \( \vDash_{\lambda} (T((e_1 \in A_1), (e_2 \in A_2))) \)
\[ \rightarrow (T(1, 1)) \]
\[ \rightarrow 1 \]

\[ \therefore \vDash_{\lambda} (e \in A) \rightarrow 1 \]

By theorem 8 \( A = [A_1 \times A_2]_T \) is an implication-based \( T \)-fuzzy subgroup of \( G \). \( \square \)
Theorem 11. Let $G_1$ and $G_2$ be two finite groups and $G = G_1 \times G_2$ be the direct product of $G_1$ and $G_2$. Let $A$ and $B$ be two implication-based $T$-fuzzy subgroups of $G_1$ and $G_2$ respectively. Then $[A \times B]_{T'}$ is an implication-based $T$-fuzzy subgroup of $G$ provided $T'$ dominates $T$. Moreover if $A$ and $B$ are implication-based $T$-fuzzy normal subgroup of $G$ then $[A \times B]_{T'}$ is an implication-based $T$-fuzzy normal subgroup of $G$.

Proof. Let $x, y \in G$ where $x = (a_1, b_1)$ and $y = (a_2, b_2)$; $a_1, a_2 \in G_1$ and $b_1, b_2 \in G_2$.

\[
\Rightarrow \lambda \left( T((x \in [A \times B]_{T'}), (y \in [A \times B]_{T'})) \right)
\Rightarrow (T(((a_1, b_1) \in [A \times B]_{T'}), ((a_2, b_2) \in [A \times B]_{T'})))
\Rightarrow (T'(T'((a_1 \in A), (b_1 \in B))), (T'((a_2 \in A), (b_2 \in B)))))
\Rightarrow (T'(T((a_1 \in A), (a_2 \in A)), (T((b_1 \in B), (b_2 \in B))))))
\therefore T' \text{ dominates } T
\Rightarrow (T'(a_1a_2^{-1} \in A), (b_1b_2^{-1} \in B))
\Rightarrow ((a_1a_2^{-1}, b_1b_2^{-1}) \in [A \times B]_{T'}
\Rightarrow (xy^{-1} \in [A \times B]_{T'}))
\text{Let } e = (e_1, e_2) \in G
\Rightarrow \lambda \left( T'(e_1 \in A), (e_2 \in B)) \right)
\Rightarrow ((e_1, e_2) \in [A \times B]_{T'})
\Rightarrow (e \in [A \times B]_{T'})
\text{But } \Rightarrow \lambda \left( T'(e_1 \in A), (e_2 \in B)) \right)
\Rightarrow (T'(1, 1) \in [A \times B]_{T'})
\Rightarrow 1
\therefore \Rightarrow \lambda \left( e \in [A \times B]_{T'} \right) \Rightarrow 1
\text{By theorem 8}
\Rightarrow [A \times B]_{T'} \text{ is an implication-based $T$-fuzzy subgroup of } G.
\Rightarrow \lambda \left( T'(xy \in A), (xy \in B)) \right)
\Rightarrow (xy \in [A \times B]_{T'})
\text{But } \Rightarrow \lambda \left( T'(xy \in A), (xy \in B)) \right)
\Rightarrow (T'(yx \in A), (yx \in B))
\therefore A \text{ and } B \text{ are implication-based } T \text{-fuzzy normal subgroups of } G
\Rightarrow (yx \in [A \times B]_{T'})
\Rightarrow [A \times B]_{T'} \text{ is an implication-based } T \text{-fuzzy normal subgroup of } G.
\]

Theorem 12. Let $i^2 : G \rightarrow G \times G$ defined by $i^2(x) = (x, x) \forall x \in G$. If $A$ and $B$ are two implication-based $T$-fuzzy subgroups of $G$ then $[A \cdot B]_T$ is the pre-image of $[A \times B]_T$ under $i^2$.

Proof. Let $x \in G$. By definition we have
\[
\vdash_\lambda (T((x \in A), (x \in B))) \rightarrow ((x, x) \in [A \times B]_T)
\]
\[
\vdash_\lambda (T((x \in A), (x \in B))) \rightarrow (x \in [A \cdot B]_T)
\]
\[
\Rightarrow \quad \vdash_\lambda ((x, x) \in [A \times B]_T) \rightarrow (x \in [A \cdot B]_T)
\]
\[
\vdash_\lambda (\iota^2(x) \in [A \times B]_T) \rightarrow (x \in [A \cdot B]_T)
\]
\[
\therefore \quad [A \cdot B]_T is the pre-image of [A \times B]_T under \iota^2.
\]

**Theorem 13.** Let A and B be two implication-based T-fuzzy normal subgroups of a group G. Let \( T' \) be a t-norm which dominates T. Then \([A \cdot B]_{T'}\) is an implication-based T-fuzzy normal subgroup of G.

**Proof.** \( \vdash_\lambda (T'((e \in A), (e \in B))) \rightarrow (e \in [A \cdot B]_{T'}) \)
But \( \vdash_\lambda (T'((e \in A), (e \in B))) \rightarrow (T'(1,1)) \rightarrow 1 \)
\[
\therefore \quad \vdash_\lambda (e \in [A \cdot B]_{T'}) \rightarrow 1
\]
Let \( x, y \in G \)
\[
\vdash_\lambda (T((x \in [A \cdot B]_{T'}), (y \in [A \cdot B]_{T'})))
\]
\[
\rightarrow ((T(T'((x \in A), (x \in B))), T'((y \in A), (y \in B))))
\]
\[
\rightarrow ((T'(T((x \in A), (y \in A))), T((x \in B), (y \in B))))
\]
\[
\therefore \quad T' dominates T
\]
\[
\rightarrow (T'((xy^{-1} \in A), (xy^{-1} \in B)))
\]
\[
\rightarrow (xy^{-1} \in [A \cdot B]_{T'})
\]
By theorem 8 \([A \cdot B]_{T'}\) is an implication-based T-fuzzy subgroup of G.
\[
\vdash_\lambda (T'((xy \in A), (xy \in B))) \rightarrow (xy \in [A \cdot B]_{T'})
\]
But
\[
\vdash_\lambda (T'(\{(xy \in A), (xy \in B)\}) \rightarrow (T'(yx \in A), (yx \in B)))
\]
\[
\therefore \quad A and B are implication-based T-fuzzy normal subgroups of G
\]
\[
\rightarrow (yx \in [A \cdot B]_{T'})
\]
\([A \cdot B]_{T'}\) is an implication-based T-fuzzy normal subgroup of G.  

We have if \( f \) is an homomorphism on a group G and A and B are implication-based T-fuzzy subgroups of \( f(G) \). If the t-norm \( T' \) dominates T. Then \([A \cdot B]_{T'}\) is an implication-based T-fuzzy subgroup of \( f(G) \). By theorem 3 \( f \circ A, f \circ B \) and \( f \circ [A \cdot B]_{T'} \) are implication-based T-fuzzy subgroups of G.

**Theorem 14.** Let \( f \) be an homomorphism on a finite group G. Let \( B_1 \) and \( B_2 \) be two implication-based T-fuzzy subgroups of \( f(G) \). Let the t-norm \( T' \) dominate T. If \( B = [B_1 \cdot B_2]_{T'} \) is the implication-based T-fuzzy product of \( B_1 \) and \( B_2 \) and \([A_1 \cdot A_2]_{T'}\) is the implication-based T-fuzzy product of \( A_1 = f \circ B_1 \) and \( A_2 = f \circ B_2 \) then
\[
\vdash_\lambda (x \in [A_1 \cdot A_2]_{T'}) \rightarrow (x \in A) \quad \forall x \in G \text{ where } A = f \circ B.\]
**Proof.** Let $x \in G$

\[
\models_\lambda (T'((x \in A_1), (x \in A_2))) \rightarrow (x \in [A_1 \cdot A_2]_{T'})
\]

But

\[
\models_\lambda (T'((x \in A_1), (x \in A_2))) \\
\rightarrow (T'((f(x) \in B_1), (f(x) \in B_2))) \\
\rightarrow (f(x) \in [B_1 \cdot B_2]_{T'}) \\
\rightarrow (f(x) \in B) \\
\rightarrow (x \in A)
\]

\[
\square
\]

**References**


