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FUNCTIONALS ON BV SPACE WITH CARATHÉODORY INTEGRANDS USING CONVEX DUALITY

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Abstract: We define nonlinear functionals $\int_\Omega \varphi(x, Du)$ for $u \in BV(\Omega)$, by using the convex dual $\varphi^*(x, q)$ of Carathéodory functions $\varphi(x, p)$, $\varphi : \Omega \times \mathbb{R}^N \to \mathbb{R}$, that for a.e. $x \in \Omega$ are convex and of a linear growth condition in p. Without the usual assumption of continuity in the x variable, we state conditions on φ in which $\int_\Omega \varphi(x, Du)$ is lower semicontinuous and compact in L^1, when $\int_\Omega \varphi(x, Du) = \int_\Omega \varphi(x, \nabla u)\, dx$ for all $u \in W^{1,1}(\Omega)$, and when the formula $\int_\Omega \varphi(x, Du) = \int_\Omega \varphi(x, \nabla u)\, dx + \int_\Omega \psi(x)|D^s u|$ holds for $u \in BV(\Omega)$. Additionally, we do not use the theory of convex functions of measures nor Reshetnyak continuity as is most commonly done in the literature. Such functionals have important applications to image restoration.

Key Words: bounded variation, image restoration, convex dual, Carathéodory function

1. Introduction

In this paper we present some new results for general functionals of the form $\int_\Omega \varphi(x, Du)$ that are defined, for $u \in BV(\Omega)$ on open bounded sets $\Omega \subset \mathbb{R}^N$, by using the convex dual φ^* of φ, for Carathéodory functions $\varphi(x, p)$, $\varphi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ that for a.e. x are convex and have a linear growth condition in the p variable. Functionals with this term have important applications in image restoration.

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Since Du is actually measure for $u \in BV(\Omega)$, the study of such functionals as is done in [2], [3] uses the theory of convex functions of measures, a recession function for φ, and Reshetnyak continuity for measures. The authors of [2], [3] require continuity in the x variable for some similar results presented here. The results we present here do not use the theory of convex functions of measures or Reshetnyak continuity, nor is continuity or lower semicontinuity in x assumed for φ. Instead we take a different approach and expand on the use of the convex dual φ^* as was used in [7] and [19].

As a motivation we consider an application for image restoration from [7]. The authors in [7] consider the function $\varphi_a : \Omega \times \mathbb{R}^N \to \mathbb{R}$ on bounded open $\Omega \subset \mathbb{R}^N$

$$\varphi_a(x,p) = \begin{cases} \frac{1}{r(x)}|p|^{r(x)} & \text{if } |p| \leq 1 \\ |p| - \frac{r(x)-1}{r(x)} & \text{if } |p| > 1 \end{cases}, \tag{1.1}$$

where $r \in L^\infty(\Omega)$ is given, $1 < \alpha \leq r(x) \leq 2$ a.e. x. They prove the existence of a unique minimizer for the following anisotropic functional F defined for $u \in BV(\Omega) \cap L^2(\Omega)$:

$$F(u) := \int_\Omega \varphi_a(x, Du) + \frac{\lambda}{2}\int_\Omega (u-I)^2\,dx$$

with chosen parameters $\alpha, \lambda > 0$, and given $I \in L^\infty(\Omega)$ representing the corrupted image. The restored image is then is taken to be the unique solution u of

$$\min_{u \in BV(\Omega)\cap L^2(\Omega)} F(u)$$

where in applications $\Omega \subset \mathbb{R}^2$ is typically a rectangle. Clearly φ_a in (1.1) is Carathéodory, and is in fact convex and C^1 in p. In [7] there is an additional trace term for u on $\partial\Omega$ included in the expression for $F(u)$, however trace terms are not considered here.

Minimization problems for image restoration in BV space, such as above, were first proposed in [17], where instead of the $\int_\Omega \varphi_a(x, Du)$ term from above, they used the total variation

$$\int_\Omega |Du| := \sup_{\phi \in C_0^\infty(\Omega,\mathbb{R}^N), |\phi(x)|\leq 1 \text{ all } x\in\Omega} \left\{ -\int_\Omega u\, div\phi \right\}$$

as a better way to retain edges of corrupted images and still remove unwanted noise, since BV space, unlike $W^{1,1}$, allows for functions with jump discontinuities . The $\int_\Omega (u-I)^2\,dx$ term is used to ensure that the restored image is not too far from the input image I. The above anisotropic model was later proposed as a better way to retain certain features of the input image over the original total variation model.

We recall that for open $\Omega \subset \mathbb{R}^N$, $BV(\Omega)$ is the space of all $u \in L^1(\Omega)$ such that $\int_\Omega |Du| < \infty$ with norm $\|u\|_{BV} = \int_\Omega |Du| + \|u\|_{L^1(\Omega)}$. The gradient Du is now a measure with $Du = \nabla u\, dx + D^s u$ decomposed into the mutually orthogonal measures $\nabla u\, dx$ and $D^s u$, with $\nabla u\, dx$ absolutely continuous with respect to N dimensional Lebesgue measure in $\mathbb{R}^N$ and $D^s u$ singular with respect to $\nabla u\, dx$. The term $\int_\Omega \varphi_a(x, Du)$ above is defined for $u \in BV(\Omega)$ by

$$\int_\Omega \varphi_a(x, Du) := \int_\Omega \varphi_a(x, \nabla u)\, dx + \int_\Omega |D^s u|, \tag{1.2}$$

as is done in [2], [3], or [9] for more general convex linear growth φ using convex functions of measures.

In [7] the authors use the convex dual function

$$\varphi_a^*(x, q) := \sup_{p \in \mathbb{R}^n} \{p \cdot q - \varphi_a(x, p)\}$$

for φ_a given by (1.1) to also define $\int_\Omega \varphi_a(x, Du)$ as

$$\int_\Omega \varphi_a(x, Du) := \sup_{\phi \in C_0^\infty(\Omega, \mathbb{R}^N), |\phi(x)| \le 1 \text{ all } x \in \Omega} \left\{ -\int_\Omega u div\phi + \varphi_a^*(x, \phi(x))\, dx \right\}$$

where $\varphi^*(x, q)$ is computed explicitly. In fact it is easy to show by direct computation that

$$\varphi_a^*(x, q) = \begin{cases} \frac{r(x)-1}{r(x)} |q|^{\frac{r(x)}{r(x)-1}} & \text{if } |q| \le 1 \\ \infty & \text{if } |q| > 1 \end{cases} \tag{1.3}$$

giving

$$\int_\Omega \varphi_a(x, Du) = \tag{1.4}$$
$$\sup_{\phi \in C_0^\infty(\Omega, \mathbb{R}^N), |\phi(x)| \le 1 \text{ all } x \in \Omega} \left\{ -\int_\Omega u div\phi + \frac{r(x)-1}{r(x)} |\phi(x)|^{\frac{r(x)}{r(x)-1}} dx \right\}.$$

It is then shown in [7] that the above two expressions (1.2) and (1.4) of

$$\int_\Omega \varphi_a(x, Du)$$

are equal, that is

$$\sup_{\phi\in C_0^\infty(\Omega,\mathbb{R}^N),|\phi|\le 1 \text{ all } x}\left\{-\int_\Omega u div\phi + \frac{r(x)-1}{r(x)}|\phi(x)|^{\frac{r(x)}{r(x)-1}}\,dx\right\}$$
$$= \int_\Omega \varphi_a(x,\nabla u)\,dx + \int_\Omega |D^s u| = \int_\Omega \varphi_a(x,Du).$$

The advantage of the form (1.4) is that lower semicontinuity in $L^1(\Omega)$ follows immediately as in the proof of the lower semicontinuity theorem for $\int_\Omega |Du|$ from [13]. We also note earlier applications to plasticity of the convex dual to nonlinear functionals $\int_\Omega \varphi(Du)$ with C^1, convex, but x independent $\varphi:\mathbb{R}^N\to\mathbb{R}$ ([14][15]).

To summarize the main results of this paper, we generalize the definition of $\int_\Omega \varphi(x,Du)$ for $u\in BV(\Omega)$ by directly using the convex dual $\varphi^*(x,q)$ for Carathéodory functions $\varphi(x,p)$ that for a.e. x are both convex and of a linear growth condition in the p variable, and with no continuity or lower semicontinuity assumption in x. In fact we will (1) define $\int_\Omega \varphi(x,Du)$ as

$$\int_\Omega \varphi(x,Du) = \sup_{\phi\in\mathcal{V}}\left\{-\int_\Omega u div\phi + \varphi^*(x,\phi(x))\,dx\right\}$$

where $\mathcal{V}$ is an appropriate family functions $\phi\in C_0^\infty(\Omega)$ and

$$\varphi^*(x,q) := \sup_{p\in\mathbb{R}^n}\left\{p\cdot q-\varphi(x,p)\right\}.$$

The definition we use here is in contrast to that which is used in works such as [2], [3], and [8], where continuity or lower semicontinuity of φ in both x and p are assumed for some of the same results obtained here, notably the existence of minimizers as proved in the next section. As in the anisotropic case above, lower semicontinuity of $\int_\Omega \varphi(x,Du)$ in L^1 will easily follow in our case. We then show (2) compactness in L^1 of $\int_\Omega \varphi(x,Du)$ if the linear growth part of φ (ψ as given in the next section) is bounded away from 0. Furthermore, under additional assumptions of φ^*, we then prove (3) the consistency formula $\int_\Omega \varphi(x,Du) = \int_\Omega \varphi(x,\nabla u)\,dx$ for functions $u\in W^{1,1}(\Omega)$ and (4) the formula

$$\int_\Omega \varphi(x,Du) = \int_\Omega \varphi(x,\nabla u)\,dx + \int_\Omega \psi(x)\,|D^s u|,$$

thus showing that $\int_\Omega \varphi(x,Du)$ as defined in this paper and as defined from [2], [3], and [8] are equal under certain conditions for φ and φ^*.

Both compactness and lower semicontinuity are essential for proving existence of solutions to minimization problems such as

$$\min_{u\in BV(\Omega)}\left\{\int_\Omega \varphi(x,Du)+\int_\Omega g(u)\,dx\right\}$$

where $\int_\Omega g(u)\,dx$ is some appropriate convex penalty term. As mentioned above these results are well known for certain $\varphi(x,p)$ of linear growth and φ either continuous or lower semicontinuous in both x and p using the theory of convex functions of measures and Reshetnyak continuity. We refer the reader to [1] for a collection of many of these results as well as an extensive study of corresponding time flow problems $\frac{\partial u}{\partial t}=\mathrm{div}\nabla_p\varphi(x,Du)$, with initial value and boundary conditions. We note that these time flow problems are frequently used in applications for solving for minimizers by finding the steady state of $u(x,t)$ as $t\to\infty$. We further remark that convex duality is also used in Lemma 1 of [19] for $\int_\Omega \varphi(x,Du)$ but with the requirement that $\varphi\in C^1(\Omega\times\mathbb{R}^N)$ so as to use the implicit function theorem, and in [5] but in a more general functional analytic and measure theoretic setting. More recently, convex duality is also used in [8] to prove relations between generalized minimizers over $BV(\Omega)$ of $\int_\Omega\varphi(x,Du)$ for linear growth convex φ in p and its dual problem with divergence free vector fields. The authors of [8] also, however, incorporate the theory as used in [2], [3].

Finally, we mention the more recent use of convex duality in a more general setting for primal dual methods in [6] and [12], where, for example, the solution to the primal problem

$$\min_{u\in BV(\Omega)}\int_\Omega\varphi(x,Du)$$

may be obtained by solving

$$\begin{aligned}&\min_{u\in BV(\Omega)}\sup_{\phi\in\mathcal{V}}\left\{-\int_\Omega u div\phi+\varphi^*(x,\phi(x))\,dx\right\}\\ =\;&\min_{u\in BV(\Omega)}\sup_{\phi\in\mathcal{V}}\left\{\int_\Omega Du\cdot\phi-\int_\Omega\varphi^*(x,\phi(x))\,dx\right\}\end{aligned}$$

rather than the time flow, although the supremum in general will not be achieved for $\phi\in\mathcal{V}$. In our case we shall see that

$$\mathcal{V}=\left\{\phi\in C_0^1(\Omega,\mathbb{R}^N):|\phi(x)|\le\psi(x)\text{ for all }x\in\Omega\right\}$$

as described in the next section. Thus the results in this paper may be well suited for this method since we directly use the dual to define $\int_\Omega\varphi(x,Du)$.

2. Main Results

We recall the definition of a Carathéodory function $\varphi : \Omega \times \mathbb{R}^N \to \mathbb{R}$: for each $p \in \mathbb{R}^N$, $\varphi(\cdot, p) : \Omega \to \mathbb{R}$ is a measurable function, $\Omega \subset \mathbb{R}^N$, and for a.e. $x \in \Omega$, $\varphi(x, \cdot) : \mathbb{R}^N \to \mathbb{R}$ is continuous. In the sequel we assume that Ω is open and bounded with $N \geq 1$, that φ is Carathéodory, and is for a.e. x convex in the p variable on $\mathbb{R}^N$ (and hence continuous in p), with the following conditions:

1) (linear growth) for some constants $c, \beta \geq 0$ and some $\psi \in C(\Omega) \cap L^\infty(\Omega)$, $\psi > 0$ on Ω, we have for a.e. $x \in \Omega$ $\varphi(x,p) \leq \psi(x)|p| + c$ for each $|p| \geq \beta$, and if $|q| \leq \psi(x)$, $\sup_{p \in \mathbb{R}^N}\{q \cdot p - \varphi(x,p)\} = \sup_{p \in K}\{q \cdot p - \varphi(x,p)\}$ where K is some compact subset of $\mathbb{R}^N$ independent of q and x. Without loss of generality we may replace the condition $p \in K$ with $|p| \leq \beta$

2) $\sup_{|p| \leq \beta} |\varphi(x,p)| \in L^1(\Omega)$ a.e for all p for some $f \in L^1(\Omega)$.

The anisotropic functional φ_a (1.1) satisfies conditions 1 and 2 with $\psi(x) \equiv 1$ and $\beta = 1$, and that since we only assume $r \in L^\infty(\Omega)$, φ_a need not be continuous in x for any p. We may also replace $\psi = 1$ in φ_a by any appropriate ψ by multiplying φ_a through by ψ and adding $h \geq 0 \in L^\infty(\Omega)$ to arrive at

$$\varphi_b(x,p) = \begin{cases} \frac{\psi(x)}{r(x)}|p|^{r(x)} + h(x) & \text{if } |p| \leq 1 \\ \psi(x)|p| - \psi(x)\frac{r(x)-1}{r(x)} + h(x) & \text{if } |p| > 1 \end{cases}$$

and conditions 1 and 2 still hold with no continuity assumption for φ_b in x. In fact in this case if $|q| \leq \psi(x)$ we have $\sup_{p \in \mathbb{R}^N}\{q \cdot p - \varphi_b(x,p)\} = \sup_{|p| \leq 1}\{q \cdot p - \varphi_b(x,p)\}$ and if $|q| > \psi(x)$ we see that $\sup_{p \in \mathbb{R}^N}\{q \cdot p - \varphi(x,p)\} = \infty$ so condition 1 holds. Also since $\sup_{|p| \leq \beta} |\varphi_b(x,p)| = \psi(x)/r(x) + h(x)$, condition 2 holds as well. Clearly, many integrands of the form

$$\varphi_g(x,p) = \begin{cases} g(x,p) & \text{if } |p| \leq \beta \\ \psi(x)|p| + h(x) & \text{if } |p| > \beta \end{cases}$$

satisfy conditions 1 and 2, for example, with no continuity assumption in x.

We now recall the convex dual $g^* : \mathbb{R}^N \to (-\infty, \infty]$ defined for convex $g : \mathbb{R}^N \to \mathbb{R}$ (see e.g. [10]):

$$g^*(q) := \sup_{p \in \mathbb{R}^N} \{q \cdot p - g(p)\}.$$

Definition 1. *For Carathéodory function φ on $\Omega \times \mathbb{R}^N$ satisfying conditions 1 and 2 we define*

$$\int_\Omega \varphi(x, Du) = \sup_{\phi \in \mathcal{V}} \left\{ -\int_\Omega u \, div \phi + \varphi^*(x, \phi(x)) \, dx \right\}$$

where for a.e. fixed $x \in \Omega$, $\varphi^*(x,q) = \sup_{\{p\in\mathbb{R}^N, |p|\le\beta\}}\{q\cdot p - \varphi(x,p)\}$ *for each* $q \in \mathbb{R}^N$ *and*

$$\mathcal{V} := \left\{\phi \in C_0^1(\Omega, \mathbb{R}^N) : |\phi(x)| \le \psi(x) \text{ for all } x \in \Omega\right\}.$$

We show below that φ^* is in fact a Carathéodory function on the set

$$S = \{(x,q) : x \in \Omega \text{ and } |q| \le \psi(x)\}$$

and that $\varphi^*(x,q) = \sup_{p\in\mathbb{R}^N}\{q\cdot p - \varphi(x,p)\} = \max_{|p|\le\beta}\{p\cdot q - \varphi(x,p)\}$ on S.

Lemma 1. *Let* φ *satisfy conditions 1, 2 and let*

$$S = \{(x,q) : x \in \Omega \text{ and } |q| \le \psi(x)\}.$$

Then $\varphi^* : \Omega \times \mathbb{R}^N \to (-\infty, \infty]$ *is Carathéodory on the set* S *and for a.e.* x,

$$\varphi^*(x,q) = \begin{cases} \max_{|p|\le\beta}\{p\cdot q - \varphi(x,p)\} & \text{if } (x,q) \in S \\ \infty & \text{if } (x,q) \notin S. \end{cases}$$

Furthermore, if $B \subset \Omega$ *is open and* $q \in \mathbb{R}^N$ *with* $|q| \le \psi(x)$ *for all* $x \in B$ *we have* $\varphi^*(\cdot, q) \in L^1(B)$.

Proof. First, it is well known that convex functions are continuous on the interior of their effective domains, that is where $|q| < \psi(x)$ for φ^*. We thus prove continuity on all of $|q| \le \psi(x)$. By the linear growth condition $\varphi(x,p) \le \psi(x)|p| + c$, we have

$$\varphi^*(x,q) = \sup_{p\in\mathbb{R}^N}\{p\cdot q - \varphi(x,p)\} < \infty$$

if and only if $|q| \le \psi(x)$ and the supremum occurs for $|p| \le \beta$ by assumption. The fact that the supremum is a maximum follows by continuity in p. This proves that for a.e. $x \in \Omega$ $\varphi^*(x,q)$ is finite iff $(x,q) \in S$. Now we fix x where $\varphi(x,p)$ is continuous in p and conditions 1 and 2 hold, and prove that $\varphi^*(x,q)$ is continuous in q for all $|q| \le \psi(x)$. The case where $\beta = 0$ gives $\varphi^*(x,q) = \max_{\{p\in\mathbb{R}^N, |p|=0\}}\{q\cdot p - \varphi(x,p)\} = -\varphi(x,0)$ if $|q| \le \psi(x)$. Now let $\beta > 0$. First assume $\varphi(x,p)$ is strictly convex for $|p| \le \beta$. Then by strict convexity there is a unique $p^*(q)$ with $|p^*(q)| \le \beta$ so that $\varphi^*(x,q) = q\cdot p^*(q) - \varphi(x,p^*(q))$. To show that p^* is continuous we let $q_n \to q$ with $|q_n| \le \psi(x)$. Thus there is a subsequence q_{n_k} such that $p^*(q_{n_k}) \to p'$ for some $|p'| \le \beta$. Hence for each q_{n_k},

$$\varphi^*(x, q_{n_k}) = q_{n_k}\cdot p^*(q_{n_k}) - \varphi(x, p^*(q_{n_k})) \ge q_{n_k}\cdot p - \varphi(x,p)$$

for each $|p| \leq \beta$. Thus for each $|p| \leq \beta$

$$\begin{aligned} q \cdot p - \varphi(x,p) &\leq \lim_{k\to\infty} \varphi^*(x, q_{n_k}) \\ &= \lim_{k\to\infty} (q_{n_k} \cdot p^*(q_{n_k}) - \varphi(x, p^*(q_{n_k})) = q \cdot p^{'} - \varphi(x, p^{'}). \end{aligned}$$

Therefore $p^{'} = p^*(q)$. To show that the full sequence $p^*(q_n)$ converges to $p^*(q)$ we assume that there is another subsequence q_{n_i} and $\varepsilon > 0$ such that $q_{n_i} \to q$ but $|p^*(q_{n_i}) - p^*(q)| \geq \varepsilon$ for all n_i. We extract a further subsequence $q_{n_{i_j}}$ with $q_{n_{i_j}} \to q$ and $p^*(q_{n_{i_j}}) \to p''$. Repeating the above argument we have $p'' = p^*(q)$, but then $0 = |p'' - p^*(q)| \geq \varepsilon$, a contradiction. Clearly since $p^*(q)$ is continuous in q with $|q| \leq \psi(x)$, so is $\varphi^*(x,q)$. Without the strict convex assumption on $\varphi(x,p)$ we consider $\varphi_\varepsilon(x,p) := \varphi(x,p) + \varepsilon |p|^2$ for $|p| \leq \beta$. As $\varepsilon\beta^2 \geq \varphi^*(x,q) - \varphi^*_\varepsilon(x,q)$ and $\varphi^*(x,q) \geq \varphi^*_\varepsilon(x,q)$ we have $\varepsilon\beta^2 \geq |\varphi^*(x,q) - \varphi^*_\varepsilon(x,q)|$ and thus $\varphi^*_\varepsilon \to \varphi^*$ uniformly for all $|q| \leq \psi(x)$ as $\varepsilon \to 0$. Since $\varphi_\varepsilon(x,p)$ is strictly convex for $|p| \leq \beta$, $\varphi^*_\varepsilon(x,q)$ is continuous for all q with $|q| \leq \psi(x)$ by the previous argument and it follows that $\varphi^*(x,q)$ is continuous in q for all $|q| \leq \psi(x)$ from the uniform convergence of φ^*_ε to φ^*.

We now note that $\varphi^*(x,q) := \sup_{p\in\mathbb{R}^N}\{q\cdot p - \varphi(x,p)\}$ is measurable in x for each $q \in \mathbb{R}^N$, in fact from [16] $\varphi^*(x,\phi(x))$ is measurable for each measurable ϕ. We have shown that φ^* is Carathéodory on S.

To prove $\varphi^*(\cdot,q) \in L^1(B)$ we note that $|q| \leq \psi(x)$ for each $x \in B$ implies

$|\varphi^*(x,q)| < \infty$ for a.e. $x \in B$ with $\varphi^*(x,q) = p^* \cdot q - \varphi(x,p^*)$ for some $|p^*| \leq \beta$. Hence

$$|\varphi^*(x,q)| \leq \beta |q| + |\varphi(x,p^*)| \leq \beta |q| + \sup_{|p|\leq\beta} \varphi(x,p) \in L^1(B)$$

by condition 2. □

As in the proof in [13] we immediately have lower semicontinuity of $\int_\Omega \varphi(x,Du)$ in L^1:

Theorem 1. *If φ satisfy the conditions of Definition 1, then $\int_\Omega \varphi(x,Du)$ is lower semicontinuous in $L^1(\Omega)$. That is, if $u_n \to u$ in $L^1(\Omega)$ then $\int_\Omega \varphi(x,Du) \leq \liminf_{n\to\infty} \int_\Omega \varphi(x,Du_n)$.*

Proof. For fixed $\phi \in \mathcal{V}$ we see that $u_n \to u \in L^1(\Omega)$ implies

$$-\int_\Omega u div\phi + \varphi^*(x,\phi(x))\,dx = \lim_{n\to\infty}\left(-\int_\Omega u_n div\phi + \varphi^*(x,\phi(x))\,dx\right)$$

$$\leq \liminf_{n\to\infty} \int_\Omega \varphi(x, Du_n).$$

Taking the supremum over $\phi \in \mathcal{V}$ on the left side the theorem follows. □

We also have

Theorem 2. *If φ satisfy the conditions of Definition 1 and if in addition to $\psi \in L^\infty(\Omega) \cap C(\Omega)$ we have $\inf_\Omega \psi(x) > 0$ then $u \in BV(\Omega)$ if and only if $\int_\Omega \varphi(x, Du) < \infty$. Letting*

$$k_1 = \inf_\Omega \psi(x),\ \ k_2 = \sup_\Omega \psi(x),\ \ k_3 = \int_\Omega \sup_{|p|\leq\beta} \varphi(x,p)\,dx$$

we in fact have

$$\begin{aligned} k_1 \int_\Omega |Du| &\leq \int_\Omega \varphi(x, Du) + C(k_1, k_3, \beta, \Omega) \text{ and} \\ \int_\Omega \varphi(x, Du) &\leq k_2 \int_\Omega |Du| + C(k_2, k_3, \beta, \Omega) \end{aligned}$$

for constants $C(k_1, k_3, \beta, \Omega)$, $C(k_2, k_3, \beta, \Omega) \geq 0$.

Proof. From the definition of φ^* we have

$$\varphi^*(x, \phi(x)) \leq |\phi(x)|\beta + \sup_{|p|\leq\beta} \varphi(x,p) \in L^1(\Omega) \tag{2.1}$$

and thus

$$\begin{aligned} k_1 \int_\Omega |Du| &= \sup_{\phi\in C_0^1(\Omega,\mathbb{R}^N),|\phi(x)|\leq k_1} \left\{ -\int_\Omega u div\phi\,dx \right\} \\ &\leq \sup_{\phi\in C_0^1(\Omega,\mathbb{R}^N),|\phi(x)|\leq k_1} \left\{ -\int_\Omega u div\phi + \varphi^*(x,\phi)\,dx \right\} \\ &\quad + \sup_{\phi\in C_0^1(\Omega,\mathbb{R}^N),|\phi(x)|\leq k_1} \left\{ \left| \int_\Omega \varphi^*(x,\phi)\,dx \right| \right\} \\ &\leq \sup_{\phi\in\mathcal{V}} \left\{ -\int_\Omega u div\phi + \varphi^*(x,\phi)\,dx \right\} \\ &\quad + \sup_{\phi\in C_0^1(\Omega,\mathbb{R}^N),|\phi(x)|\leq k_1} \left\{ \int_\Omega |\varphi^*(x,\phi)|\,dx \right\} \\ &\leq \int_\Omega \varphi(x, Du) + C(k_1, k_3, \beta, \Omega) \ \ \text{(from 2.1)} \end{aligned}$$

where $C(k_1, k_3, \beta, \Omega) \geq 0$; and also

$$\begin{aligned}\int_\Omega \varphi(x, Du) &= \sup_{\phi\in\mathcal{V}}\left\{-\int_\Omega u div\phi + \varphi^*(x,\phi)\,dx\right\}\\ &\leq \sup_{\phi\in C_c^1(\Omega,\mathbb{R}^N),|\phi(x)|\leq k_2}\left\{-\int_\Omega u div\phi\,dx\right\} + \sup_{\phi\in\mathcal{V}}\left\{\int_\Omega \varphi^*(x,\phi)\,dx\right\}\\ &\leq k_2\int_\Omega |Du| + C(k_2, k_3, \beta, \Omega).\end{aligned}$$

□

Compactness now easily follows from compactness for BV functions:

Corollary 1. *Let φ and ψ satisfy the conditions of Theorem 2. If $\{u_n\}$ is a sequence of functions from $L^1(\Omega)$ such that both $\int_\Omega \varphi(x, Du_n)$ and $\int_\Omega |u_n|\,dx$ are bounded, then there is a subsequence $\{u_{n_k}\}$ in $BV(\Omega)$ and a $u \in L^p(\Omega) \cap BV(\Omega)$ such that $u_{n_k} \to u$ strongly in $L^p(\Omega)$ for each $1 \leq p < \frac{N}{N-1}$ and $u_{n_k} \rightharpoonup u$ weakly in $L^p(\Omega)$ if $p = \frac{N}{N-1}$.*

Proof. We note that $\int_\Omega \varphi(x, Du_n)$ is bounded if and only if $\int_\Omega |Du_n|$ is bounded by Theorem 2, and thus from the compactness theorem of Chapter 1 in ([13]) the corollary follows. □

The existence theorem now follows from Theorem 1, the above corollary, and standard theory.

Theorem 3. *For φ and ψ satisfying the hypotheses of Theorem 2 and for $\Phi : L^p(\Omega) \to (-\infty, \infty]$ defined by*

$$\Phi(u) = \begin{cases} \int_\Omega \varphi(x, Du) + \int_\Omega (u - u_0)^p & u \in BV(\Omega) \\ \infty & u \notin BV(\Omega), \end{cases}$$

the problem $\min_{u\in L^p(\Omega)} \Phi(u)$ has a solution $u \in BV(\Omega) \cap L^p(\Omega)$ for each $1 \leq p \leq \frac{N}{N-1}$. If $p > 1$ the solution is unique due to strict convexity of the second term.

Proof. The proof follows from standard results of convex analysis ([10]) by noting that Φ is convex and lower semicontinuous on $L^p(\Omega)$ and hence weakly lower semicontinuous on $L^p(\Omega)$. From Corollary 1, for any minimizing sequence $\{u_n\}$ of $\Phi(u)$ there is a strongly convergent subsequence $u_{n_k} \to u$ in $L^p(\Omega)$ if $1 \leq p < \frac{N}{N-1}$ and a weakly convergent subsequence $u_{n_k} \rightharpoonup u$ in $L^p(\Omega)$ if

$p = \frac{N}{N-1}$. The term $\int_\Omega (u-u_0)^p$ guarantees the existence such of a minimizing sequence and hence we have

$$\Phi(u_*) \le \lim \inf_{n_k \to \infty} \Phi(u_{n_k}) = \inf_{u \in L^p(\Omega)} \Phi(u).$$

Uniqueness follows from strict convexity of Φ if $p > 1$. □

For the remaining results we also assume the following conditions on φ^*:

3) The set of Lebesgue points for the x variable of $\varphi^*(x,q)$ is independent of all q with $|q| \le \psi(x)$. That is, there is a set $\Omega' \subset \Omega$, $|\Omega'| = |\Omega|$, so that for each fixed $x \in \Omega'$, x is a Lebesgue point for $\varphi^*(x,q)$ for every q with $|q| < \psi(x)$.

4) For a.e. $x \in \Omega$, $\varphi^*(x,q)$ is strictly convex on the set $\left\{q \in \mathbb{R}^N | \ \ |q| \le \psi(x)\right\}$.

5) For a.e. $x \in \Omega$, $\varphi^*(x,q)$ is Lipschitz in all q such that $|q| \le \psi(x)$,with Lipschitz constant independent of x. That is for a.e. $x \in \Omega$ and $|q_1|, |q_2| \le \psi(x)$ we have $|\varphi^*(x,q_1) - \varphi^*(x,q_2)| \le C|q_1 - q_2|$ where C is independent of x.

Condition 3, for example, is satisfied for functions φ of the form $\varphi(x,p) = f(x)g(p)$, $f > 0$ as then $\varphi^*(x,q) = f(x)g^*(q/f(x))$ and hence the Lebesgue points only depend on f; and also for $\varphi^*(x,q)$ if $\varphi(x,p)$ if satisfies conditions 1 and 2 above and φ is continuous in x on a set $\Omega' \subset \Omega$ where Ω' is independent of p and $|\Omega'| = |\Omega|$, since then as in the proof of Lemma 1 below $\varphi^*(x,q)$ is continuous in x on Ω' for each q and hence the set of Lebesgue points of φ^* for each q is Ω'. We finally remark that condition 3 is also satisfied for Carathéodory functions $\varphi^* = g$ of the form $g(x,q) = f(r(x),q)$, $r \in L^\infty(\Omega)$, where $f(z,q)$, $f : I \times \mathbb{R}^N \to \mathbb{R}$, is a C^1 function of z on open interval I for each q and f_z bounded on $I \times \mathbb{R}^N$. In fact we have for each q

$$|g(y,q) - g(x,q)| = |f(r(y),q) - f(r(x),q)| \le |r(y) - r(x)| M.$$

Thus $\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)} |r(y) - r(x)|\,dy \to 0$ implies $\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)} |g(y,q) - g(x,q)|\,dy \to 0$ as $\rho \to 0$ and hence the Lebesgue set of g contains the Lebesgue set of r, independent of q (e.g. φ_a^* as given by (1.3)). We remark that φ_a^* (1.3), for example, satisfies all of conditions 1-5.

Lemma 2. *Let φ satisfy conditions 1 and 2 and φ^* satisfy conditions 4-5. Furthermore assume ψ is uniformly continuous on Ω with $\inf \psi > 0$ on Ω. Fix $p \in \mathbb{R}^N$. Let η be the standard mollifier on $\mathbb{R}^n$ with $\varphi_\varepsilon^*(x,q) := \int_\Omega \eta_\varepsilon(x-y)\varphi^*(y,q)\,dy$ for each fixed $x \in \Omega$, $q \in \mathbb{R}^N$ with $|q| \le \psi(x)$. Fix small $\sigma > 0$ so that $\inf_\Omega \psi - 2\sigma > 0$. Then for each sufficiently small $\varepsilon > 0$, $\varphi_\varepsilon^*(x,q)$ is finite if both $|q| \le \psi(x) - \sigma$ and $x \in \Omega_\varepsilon := \{x \in \Omega : dist(x, \partial\Omega) > \varepsilon\}$, and in fact the* *problem*

$$\max_{|q|\leq\psi(x)-2\sigma}\left(p\cdot q-\varphi_\varepsilon^*(x,q)\right)$$

has a unique solution $q_\sigma^\varepsilon(x)$ *for each* $x\in\Omega_\varepsilon$*. Furthermore* $q_\sigma^\varepsilon\in C\left(\Omega_\varepsilon,\mathbb{R}^N\right)$ *for each such* $\varepsilon>0$*.*

Proof. Let $p\in\mathbb{R}^N$. By uniform continuity of ψ, for fixed $\sigma>0$ we have $\psi(x)-\sigma<\psi(y)$ for all $|x-y|<\varepsilon$ with $\varepsilon>0$ sufficiently small. Note that we have for each $x\in\Omega_\varepsilon$

$$\begin{aligned}\varphi_\varepsilon^*(x,q) &= \int_\Omega \eta_\varepsilon(x-y)\varphi^*(y,q)\,dy\\ &= \int_{B(x,\varepsilon)}\eta_\varepsilon(x-y)\varphi^*(y,q)\,dy,\end{aligned}$$

where by Lemma 1 the integral, and thus $\varphi_\varepsilon^*(x,q)$, is finite if $|q|\leq\psi(x)-\sigma$ and all such sufficiently small ε since this implies $|q|\leq\psi(x)-\sigma<\psi(y)$ for all $y\in B(x,\varepsilon)$. For such $\varepsilon>0$ we also see that φ_ε^* satisfies condition 4 for all $x\in\Omega_\varepsilon$ and $|q|\leq\psi(x)-\sigma$. Therefore for all such $\varepsilon>0$ the problem

$$\max_{|q|\leq\psi(x)-2\sigma}\left(p\cdot q-\varphi_\varepsilon^*(x,q)\right)$$

has a unique solution $q_\sigma^\varepsilon(x)\in\mathbb{R}^N$ for each $x\in\Omega_\varepsilon$ giving

$$\max_{|q|\leq\psi(x)-2\sigma}\left(p\cdot q-\varphi_\varepsilon^*(x,q)\right)=p\cdot q_\sigma^\varepsilon(x)-\varphi_\varepsilon^*(x,q_\sigma^\varepsilon(x)).$$

To prove $q_\sigma^\varepsilon\in C\left(\Omega_\varepsilon,\mathbb{R}^N\right)$, fix $\varepsilon>0$ to be sufficiently small as chosen above. Let $x_n\to x$ and $q_\sigma^\varepsilon(x_n)\in\mathbb{R}^N$ be the unique solution to

$$\max_{|q|\leq\psi(x_n)-2\sigma}\left(p\cdot q-\varphi_\varepsilon^*(x_n,q)\right)$$

with $|q_\sigma^\varepsilon(x_n)|\leq\psi(x_n)-2\sigma$. Letting $F_\varepsilon(x,q):=p\cdot q-\varphi_\varepsilon^*(x,q)$ we then have for each $|q|\leq\psi(x)-2\sigma$

$$F_\varepsilon(x_n,q_\sigma^\varepsilon(x_n))\geq F_\varepsilon(x_n,q). \tag{2.2}$$

Since $|q_\sigma^\varepsilon(x_n)|\leq\psi(x_n)-2\sigma$, $q_\sigma^\varepsilon(x_n)$ is bounded and thus there is $q'\in\mathbb{R}^n$, $|q'|\leq\psi(x)-2\sigma$, such that $q_\sigma^\varepsilon(x_{n_k})\to q'$ for some subsequence $x_{n_k}\to x$. By condition 5, the second statement of Lemma 1, the dominated convergence theorem, and noting that $|q_\sigma^\varepsilon(x)|\leq\psi(x_{n_k})-\sigma$ for all sufficiently large n_k it is straightforward to show that $\varphi_\varepsilon^*(x_{n_k}, q_\sigma^\varepsilon(x_{n_k})) \to \varphi_\varepsilon^*(x, q')$. We then have, after letting $n_k \to \infty$ in (2.2),

$$F_\varepsilon(x, q') \geq F_\varepsilon(x, q)$$

for each $|q| \leq \psi(x) - 2\sigma$, implying $q' = q_\sigma^\varepsilon(x)$ by uniqueness of maximizers. Again by uniqueness we have $q_\sigma^\varepsilon(x_n) \to q'$ for the entire sequence and hence $q_\sigma^\varepsilon(x_n) \to q_\sigma^\varepsilon(x)$. Continuity of q_σ^ε is thus proved. □

Lemma 3. *Let φ satisfy conditions 1 and 2 and φ^* satisfy conditions 3-5. Assume ψ is uniformly continuous on Ω with $\inf_\Omega \psi > 0$. Then there exists a function $q(x,p)$, $q : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, so that for each fixed $p \in \mathbb{R}^N$, we have $q(\cdot, p) \in L^\infty\left(\Omega, \mathbb{R}^N\right)$ with $|q(x,p)| \leq \psi(x)$ for a.e. $x \in \Omega$, and for a.e. $x \in \Omega$ $q(x, \cdot)$ is continuous in the p variable on $\mathbb{R}^N$ and hence Carathéodory. Furthermore, for each x in the set Ω', $|\Omega'| = |\Omega|$, of condition 3, and each $p \in \mathbb{R}^N$ there holds*

$$\varphi(x,p) = p \cdot q(x,p) - \varphi^*(x, q(x,p)) = \max_{|q| \leq \psi(x)} \{p \cdot q - \varphi^*(x,q)\}.$$

Proof. First we fix σ from Lemma 2. By condition 3 there is $\Omega' \subset \Omega$ with $|\Omega'| = |\Omega|$ such that for each $x \in \Omega'$ and each q with $|q| \leq \psi(x) - 2\sigma$, x is a Lebesgue point of $\varphi^*(x,q)$. Hence from the properties of mollifiers (see e.g. [11]) we have

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon^*(x,q) = \varphi^*(x,q)$$

for each $x \in \Omega'$. Letting for fixed $p \in \mathbb{R}^N$,

$$\begin{aligned} F_\varepsilon(x,q) &= p \cdot q - \varphi_\varepsilon^*(x,q) \text{ and} \\ F(x,q) &= p \cdot q - \varphi^*(x,q) \end{aligned}$$

we then have for each $|q| \leq \psi(x) - 2\sigma$, $x \in \Omega'$

$$\lim_{\varepsilon \to 0} F_\varepsilon(x,q) = F(x,q). \tag{2.3}$$

By condition 5 we see that for each q_1, q_2 with $|q_i| \leq \psi(x) - 2\sigma$

$$\begin{aligned} |\varphi_\varepsilon^*(x,q_1) - \varphi_\varepsilon^*(x,q_2)| &= \left| \int_\Omega \eta_\varepsilon(x-y)(\varphi^*(y,q_1) - \varphi^*(y,q_2)\,dy \right| \\ &\leq C\,|q_1 - q_2| \end{aligned}$$

and thus for each $p \in \mathbb{R}^N$

$$|F_\varepsilon(x,q_1) - F_\varepsilon(x,q_2)| \leq |p|\,|q_1 - q_2| + C\,|q_1 - q_2|\,. \tag{2.4}$$

Fix $x \in \Omega'$. For all sufficiently small $\varepsilon > 0$ we have $x \in \Omega' \cap \Omega_\varepsilon$ and from Lemma 2 there exists unique $q_\sigma^\varepsilon(x)$, $|q_\sigma^\varepsilon(x)| \le \psi(x) - 2\sigma$, $q_\sigma^\varepsilon \in C(\Omega_\varepsilon, \mathbb{R}^N)$, that satisfies

$$F_\varepsilon(x, q_\sigma^\varepsilon(x)) \ge F_\varepsilon(x, q) \tag{2.5}$$

for each $|q| \le \psi(x) - 2\sigma$. Thus there exists $q_\sigma(x)$ with $|q_\sigma(x)| \le \psi(x) - 2\sigma$ and a subsequence $q_\sigma^{\varepsilon'}(x)$ with $q_\sigma^{\varepsilon'}(x) \to q_\sigma(x)$ as $\varepsilon' \to 0$. Since F is continuous in q for all $|q| \le \psi(x)$ by Lemma 1, and from (2.4), (2.3) we have by letting $\varepsilon' \to 0$ in (2.5)

$$F(x, q_\sigma(x)) \ge F(x, q)$$

for each q with $|q| \le \psi(x) - 2\sigma$. Hence $q_\sigma(x)$ is the unique solution of

$$\max_{|q| \le \psi(x) - 2\sigma} F(x, q)$$

and the uniqueness of $q_\sigma(x)$ implies the convergence of the full sequence $q_\sigma^\varepsilon(x) \to q_\sigma(x)$ as $\varepsilon \to 0$.

Now for each $x \in \Omega'$ $q_\sigma(x)$ satisfies $F(x, q_\sigma(x)) = \max_{|q| \le \psi(x) - 2\sigma} F(x, q)$. Thus we have a subsequence $q_{\sigma'}(x)$ with $q_{\sigma'}(x) \to q(x)$. As in the previous arguments we then have for the full sequence

$$\lim_{\sigma \to 0} q_\sigma(x) = q(x) \text{ for all } x \in \Omega'$$

where $q(x)$ solves

$$p \cdot q(x) - \varphi^*(x, q(x)) = \max_{|q| \le \psi(x)} \{p \cdot q - \varphi^*(x, q)\}.$$

Since for a.e. x $\varphi^*(x, q) = \infty$ iff $|q| > \psi(x)$ by Lemma 1 we see that

$$\max_{|q| \le \psi(x)} \{p \cdot q - \varphi^*(x, q)\} = \sup_{q \in \mathbb{R}^N} \{p \cdot q - \varphi^*(x, q)\} = \varphi^{**}(x, p).$$

By the Fenchel-Moreau theorem ([4], [10]), for a.e. fixed x $\varphi^{**}(x, p) = \varphi(x, p)$ for each $p \in \mathbb{R}^N$, by convexity and continuity of φ in p. We therefore finally have for a.e. $x \in \Omega$

$$\varphi(x, p) = p \cdot q(x) - \varphi^*(x, q(x))$$

$$\text{where } q(x) = \lim_{\sigma \to 0} \lim_{\varepsilon \to 0} q_\sigma^\varepsilon(x).$$

By continuity of q_σ^ε for each fixed σ, q is thus a measurable function of x with $|q(x)| \le \psi(x)$ for a.e. $x \in \Omega$ and

$$\varphi(x, p) = p \cdot q(x) - \varphi^*(x, q(x)).$$

Now we note that in fact for a.e. x, q is also a function of p, so that

$$\varphi(x,p) = p \cdot q(x,p) - \varphi^*(x, q(x,p))$$

and continuity of q in p for each fixed $x \in \Omega'$ follows from the strict convexity assumption and the same method of proof as in Lemmas 1 or 2 and the theorem follows. □

Theorem 4. *Let φ satisfy conditions 1 and 2, φ^* satisfy conditions 3-5,and assume ψ is uniformly continuous on Ω with $\inf_\Omega \psi > 0$. For each $u \in BV(\Omega)$ we have $\int_\Omega \varphi(x, Du) = \int_\Omega \varphi(x, \nabla u)\, dx + \int_\Omega \psi(x) |D^s u|$, where $\int_\Omega \varphi(x, Du)$ is defined as in Definition 1.*

Proof. Since $u \in BV(\Omega)$, integration by parts for BV functions (e.g. [11]) gives

$$-\int_\Omega u \mathrm{div}\phi + \varphi^*(x, \phi(x))\, dx = \int_\Omega \phi \cdot \nabla u - \varphi^*(x, \phi(x))\, dx + \int_\Omega \phi \cdot D^s u$$

for each $\phi \in C_0^1(\Omega, \mathbb{R}^N)$. And from Lemma 3 there exists $g(x) = q(x, \nabla u(x))$ with $|g(x)| \leq \psi(x)$ a.e. so that for a.e. x

$$\varphi(x, \nabla u(x)) = \nabla u(x) \cdot g(x) - \varphi^*(x, g(x)) = \max_{|q| \leq \psi(x)} \{\nabla u(x) \cdot q - \varphi^*(x,q)\}$$

and hence

$$\varphi(x, \nabla u(x)) \geq \nabla u(x) \cdot \phi(x) - \varphi^*(x, \phi(x)) \text{ a.e. for each } \phi \in \mathcal{V}.$$

The proof now follows as in Lemma 2.3 from [7] or as in Lemma 1 in [19] since then

$$\begin{aligned} \int_\Omega \varphi(x, \nabla u)\, dx &= \int_\Omega \nabla u(x) \cdot g(x) - \varphi^*(x, g(x))\, dx \\ &\geq \sup_{\phi \in \mathcal{V}} \int_\Omega \nabla u(x) \cdot \phi(x) - \varphi^*(x, \phi(x))\, dx, \end{aligned}$$

and by choosing an appropriate sequence of $\phi_n \in \mathcal{V}$ with $\phi_n \to g$ in $L^1(\Omega, \mathbb{R}^N)$, noting condition 5 for φ^* we actually have

$$\int_\Omega \varphi(x, \nabla u)\, dx = \sup_{\phi \in \mathcal{V}} \int_\Omega \nabla u(x) \cdot \phi(x) - \varphi^*(x, \phi(x))\, dx. \tag{2.6}$$

Since the measures dx and $D^s u$ are mutually singular we have then

$$\begin{aligned}\int_\Omega \varphi(x, Du) &= \sup_{\phi\in\mathcal{V}} \int_\Omega \nabla u(x)\cdot\phi(x) - \varphi^*(x,\phi(x))\,dx + \int_\Omega \psi(x)\,|D^s u| \\ &= \int_\Omega \varphi(x,\nabla u)\,dx + \int_\Omega \psi(x)\,|D^s u| \text{ from (2.6).}\end{aligned}$$

We remark that in the proof of Lemma 2.3 from [7] the function g was explicitly given for that special case, whereas here we used Lemma 3. □

Theorem 5. *Let φ satisfy conditions 1 and 2, φ^* satisfy conditions 3-5,and assume ψ is uniformly continuous on Ω with $\inf_\Omega \psi > 0$. For each $u \in BV(\Omega)$ let $\int_\Omega \varphi(x, Du)$ be defined as in Definition 1. Then for each $u \in W^{1,1}(\Omega)$ we have*

$$\int_\Omega \varphi(x, Du) = \int_\Omega \varphi(x,\nabla u)\,dx.$$

Proof. This follows immediately from the above since for $u \in W^{1,1}(\Omega)$ we have $|D^s u| \equiv 0$. □

If φ, φ^*, ψ satisfy the conditions in Theorem 4, and if additionally for a.e. $x \in \Omega$ φ is C^1 in the p variable, as in the case (1.1) of φ_a, we can use Theorem 4 to obtain the Euler-Lagrange equation for the solution to

$$\min_{u\in BV(\Omega)} \left\{ \int_\Omega \varphi(x, Du) + \int_\Omega g(u)\,dx \right\}$$

for appropriate C^1 convex function g. In fact as in [14], [15] we have for any $v \in BV(\Omega) \cap H_0(\Omega)$ with $D^s v << |D^s u|$

$$\int_\Omega \nabla_{\mathbf{p}}\varphi(x,\nabla u)\cdot(\nabla v - \nabla u)\,dx +$$
$$\int_\Omega \psi(x)\frac{D^s u}{|D^s u|}\cdot(D^s v - D^s u) + \int_\Omega g'(u)(v-u)\,dx = 0$$

where $\frac{D^s u}{|D^s u|}$ denotes the Radon-Nikodym derivative of $D^s u$ with respect to $|D^s u|$. Note that $\left|\frac{D^s u}{|D^s u|}\right| = 1$, $|D^s u|$-a.e. Now letting $v = u + \phi$ for any $\phi \in C_0^\infty(\Omega)$

$$div\,(\nabla_{\mathbf{p}}\varphi(x,\nabla u)) = g'(u) \text{ in } \mathcal{D}'(\Omega)$$

as $D^s \phi = 0$.

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