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HYERS-ULAM STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS ON DIVISIBLE SQUARE-SYMMETRIC GROUPOID

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Abstract: Let $(X, \diamond)$ be a divisible square-symmetric groupoid, and $(Y, *, d)$ a complete metric divisible square-symmetric groupoid. In this paper, we obtain the Hyers-Ulam stability problem of functional inequality $d(f(x\diamond y)*f(x\diamond y^{-1}), \sigma_*(f(x)*f(y))) \leq \varepsilon(x, y)$ for approximate mapping $f : X \to Y$ of functional equation $h(x \diamond y) * h(x \diamond y^{-1}) = \sigma_*(h(x) * h(y))$ differing by $\varepsilon : X^2 \to [0, \infty)$.

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1. Introduction

In 1940, S.M. Ulam [19] proposed the Hyers-Ulam stability problem for functional equations concerning the stability of group homomorphisms : Under what condition does there exist a homomorphism near an approximately homomorphism?

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In 1941, D.H. Hyers [9] considered the case of approximate additive mappings with the Cauchy difference controlled by a positive constant in Banach spaces. Thereafter we usually say that the equation $E_1(h) = E_2(h)$ has the Hyers-Ulam stability if for an approximate solution f of this equation, i.e., for a function f with $|E_1(f) - E_2(f)| \leq \delta$, there exists a function g such that $E_1(g) = E_2(g)$ and $|f(x) - g(x)| \leq \varepsilon$.

D.G. Bourgin [3] and T. Aoki [2] treated this problem for approximate additive mappings controlled by unbounded function. In [17], Th. M. Rassias provided a generalization of Hyers' theorem for linear mappings which allows the Cauchy difference to be unbounded. In 1994, P. Găvruta [8] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the Hyers–Ulam stability and generalized Hyers–Ulam stability to a number of functional equations and mappings [1, 5, 7, 10, 16].

The functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is said to von-Nuemann functional equation. In particular, every solution of the von-Nuemann functional equation is said to be a quadratic function. The Hyers-Ulam stability of von-Nuemann functional equation was proved by Skof [18] for mapping $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. For more information of stability of von-Neumann functional equation, we refer [4, 6, 12, 15].

Let $(X, \diamond)$, $(Y, *)$ be groupoids with binary operations. If the binary operation $\diamond$ satisfies the following identity,

$$(x \diamond y) \diamond (x \diamond y) = (x \diamond x) \diamond (y \diamond y) \quad (x, y \in X),$$

the operation $\diamond$ will be called *square-symmetric*. Note that the square-symmetric of $\diamond$ implies that $\sigma_\diamond(x) := x \diamond x$ is an endomorphism. Thus we remark that

$$\sigma_\diamond^n(x) \diamond \sigma_\diamond^n(y) = \sigma_\diamond^n(x \diamond y)$$

for all $n \in \mathbb{N}$. It is an crucial relation when the mapping $\sigma_\diamond^n$ is an automorphism of $(X, \diamond)$, that is, when $\sigma_\diamond$ is bijective. Then, for each $y \in X$ the equation $\sigma_\diamond(x) = y$ can be solved uniquely with respect to $x \in X$. Such an element x is denoted by $x = \sigma_\diamond^{-1}(y)$. Then it is easy to see that $\sigma_\diamond^{-1} : X \to X$ is an endomorphism. A binary operation $\diamond$ such that $\sigma_\diamond$ is an automorphism of $(X, \diamond)$ is called *divisible*, the corresponding groupoid is said to be a divisible groupoid.

We remember that the Lipschitz modulus $Lip(\phi)$ for a function $\phi : Y \to Y$ on the space Y with metric d defined by

$$Lip(\phi) := \sup\Big\{ \frac{d(\phi(x).\phi(y))}{d(x,y)} \Big| x, y \in Y, x \neq y \Big\}.$$

Then it follows that $Lip(\phi \circ \psi) \leq Lip(\phi)Lip(\psi)$.

Over the setting of square-symmetric groupoid, G.H. Kim [11] obtain the Hyers-Ulam stability of the following functional inequality $d(f(x \diamond y), f(x) * f(y)) \leq \varepsilon(x,y)$ for approximate mapping $f : X \to Y$ of functional equation $h(x \diamond y) = h(x) * h(y)$ differing by $\varepsilon : X^2 \to [0,\infty)$.

This paper aims to investigate the Hyers-Ulam stability for the following functional equation of $f : X \to Y$:

$$f(x \diamond y) * f(x \diamond y^{-1}) = \sigma_*(f(x) * f(y)) \quad (x, y \in X). \tag{1}$$

A particular case of (1) is the von-Nuemann functional equation when X, Y are semigroups with the operation $+$. Thus we will have the generalized stability results of von-Neumann functional equation.

2. The Hyers–Ulam Stability of Functional Equation (1) via Direct Method

Theorem 1. *Let $(X, \diamond)$ be a divisible square-symmetric groupoid and let $(Y, *, d)$ be a complete metric divisible square-symmetric groupoid. Assume that a mapping $f : X \to Y$ satisfies*

$$d(f(x \diamond y) * f(x \diamond y^{-1}), \sigma_*(f(x) * f(y))) \leq \varepsilon(x,y) \tag{2}$$

for all $x, y \in X$ with $f(id_X) = id_Y$ and $\varepsilon : X^2 \to [0,\infty)$ is a mapping such that

$$L_1(x,y) := \sum_{n=0}^{\infty} Lip(\sigma_*^{-2n})\varepsilon(\sigma_\diamond^n(x), \sigma_\diamond^n(y)) < \infty \tag{3}$$

for all $x, y \in X$. Then for every $x \in X$,

$$h(x) := \lim_{n\to\infty} \sigma_*^{-2n} \circ f \circ \sigma_\diamond^n(x)$$

converges and $h : X \to Y$ is the solution of (1) such that

$$d(f(x), h(x)) \leq Lip(\sigma_*^{-2})L_1(x,x) \tag{4}$$

for all $x \in X$.

Moreover, if $\{Lip(\sigma_*^{-2n})Lip(\sigma_*^{2n})\}_{n\geq 1}$ is bounded then h is the only solution of (1) such that the mapping $x \to d(f(x), h(x))$ is bounded with bound $Lip(\sigma_*^{-2})L(x,x)$.

Proof. Define $f_n : X \to Y$ by

$$f_0 := f, \qquad f_n := \sigma_*^{-2n} \circ f \circ \sigma_\diamond^n$$

for all positive integer $n \in \mathbb{N}$. First, we would like to prove that for each fixed $x \in X$ the sequence $\{f_n(x)\}$ is convergent. Replacing x and y by $\sigma_\diamond^n(x)$ in equality (2), we get

$$d(f(\sigma_\diamond^n(x) \diamond \sigma_\diamond^n(x)) * f(\sigma_\diamond^n(x) \diamond \sigma_\diamond^{-n}(x)), \sigma_*(f(\sigma_\diamond^n(x)) * f(\sigma_\diamond^n(x))) \\ \leq \varepsilon(\sigma_\diamond^n(x), \sigma_\diamond^n(x)),$$

which implies that

$$d(f \circ \sigma_\diamond^{n+1}(x), \sigma_*^2 \circ f \circ \sigma_\diamond^n(x)) \leq \varepsilon(\sigma_\diamond^n(x), \sigma_\diamond^n(x)). \tag{5}$$

Thus we obtain

$$\begin{aligned} d(f_{n+1}(x), f_n(x)) &= d(\sigma_*^{-2(n+1)} \circ f \circ \sigma_\diamond^{n+1}(x), \sigma_*^{-2(n+1)} \circ \sigma_*^2(x) \circ f \circ \sigma_\diamond^n(x)) \\ &\leq Lip(\sigma_*^{-2(n+1)}) d(f \circ \sigma_\diamond^{n+1}(x), \sigma_*^2 \circ f \circ \sigma_\diamond^n(x)) \\ &\leq Lip(\sigma_*^{-2(n+1)}) \varepsilon(\sigma_\diamond^n(x), \sigma_\diamond^n(x)). \end{aligned}$$

It produce the following inequality that for positive integers m, n with $m, n > 0$,

$$\begin{aligned} d(f_n(x), f_m(x)) &\leq \sum_{k=n}^{m-1} d(f_k(x), f_{k+1}(x)) \\ &\leq \sum_{k=n}^{m-1} Lip(\sigma_*^{-2(k+1)}) \varepsilon(\sigma_\diamond^k(x), \sigma_\diamond^k(x)) \\ &\to 0 \qquad as \quad n \to \infty, \end{aligned}$$

which shows that the sequence $\{f_n(x)\}$ is Cauchy and thus converges in Y. Since Y is complete, we can define a function $h : X \to Y$ as follows

$$h(x) := \lim_{n\to\infty} \sigma_*^{-2n} \circ f \circ \sigma_\diamond^n(x)$$

for all $x \in X$. Now, for $n \in \mathbb{N}$, we have

$$\begin{aligned} d(f_n(x), f(x)) &\leq \sum_{k=0}^{n-1} d(f_{k+1}(x), f_k(x)) \\ &\leq \sum_{k=0}^{n-1} Lip(\sigma_*^{-2(k+1)})\varepsilon(\sigma_\diamond^k(x), \sigma_\diamond^k(x)) \\ &\leq \sum_{k=0}^{\infty} Lip(\sigma_*^{-2(k+1)})\varepsilon(\sigma_\diamond^k(x), \sigma_\diamond^k(x)) \\ &\leq Lip(\sigma_*^{-2})L_1(x,x) \end{aligned}$$

for all $x \in X$. Taking $n \to \infty$, we get (4) for the limit function h of $\{f_n\}$.

Next, we wil prove that h satisfies (1). Replacing x and y by $\sigma_\diamond^n(x)$ and $\sigma_\diamond^n(y)$ in (2) and using the endomorphism of $\sigma_\diamond$, we get

$$\begin{aligned} d((f \circ \sigma_\diamond^n(x \diamond y)) * (f \circ \sigma_\diamond^n(x \diamond y^{-1})), \sigma_*((f \circ \sigma_\diamond^n(x)) * (f \circ \sigma_\diamond^n(x)))) \\ \leq \varepsilon(\sigma_\diamond^n(x), \sigma_\diamond^n(y)) \end{aligned}$$

for all $x, y \in X$. Since σ_* is an automorphism of $(Y, *)$, σ_*^{-2n} is also automorphism for all n. Therefore,

$$\begin{aligned} & d(f_n(x \diamond y) * f_n(x \diamond y^{-1}), \sigma_*(f_n(x) * f_n(y)) \\ & = d(\sigma_*^{-2n}((f \circ \sigma_\diamond^n)(x \diamond y) * (f \circ \sigma_\diamond^n)(x \diamond y^{-1})), \\ & \qquad \sigma_*^{-2n+1}((f \circ \sigma_\diamond^n(x)) * (f \circ \sigma_\diamond^n(y)))) \\ & \leq Lip(\sigma_*^{-2n}) d(f \circ \sigma_\diamond^n(x \diamond y) * f \circ \sigma_\diamond^n(x \diamond y^{-1}), \sigma_*((f \circ \sigma_\diamond^n(x)) * (f \circ \sigma_\diamond^n(y)))) \\ & \leq Lip(\sigma_*^{-2n})\varepsilon(\sigma_\diamond^n(x), \sigma_\diamond^n(y)) \end{aligned}$$

for all $x, y \in X$. Letting $n \to \infty$ in the last inequality, we get that h satisfies (4).

Finally we complete the proof by showing the uniqueness. Assume that $h' : X \to Y$ is an arbitrary solution of (1) such that the mapping $x \mapsto d(f(x), h'(x))$ is bounded with bound in (4). Putting $x = y$ in (1), we have that

$$h \circ \sigma_\diamond = \sigma_*^2 \circ h, \quad h' \circ \sigma_\diamond = \sigma_*^2 \circ h'.$$

Using induction, we get

$$\sigma_*^{-2n} \circ h \circ \sigma_\diamond^n = h, \quad \sigma_*^{-2n} \circ h' \circ \sigma_\diamond^n = h'.$$

Hence, for every $x \in X$, we get by (4),

$$\begin{aligned} d(h'(x), h(x)) &= d(\sigma_*^{-n} \circ h' \circ \sigma_\diamond^n(x), \sigma_*^{-n} \circ h \circ \sigma_*^n(x)) \\ &\leq Lip(\sigma_*^{-2n}) d(h' \circ \sigma_\diamond^n(x), h \circ \sigma_*^n(x)) \\ &\leq Lip(\sigma_*^{-2n})[d(h'(\sigma_\diamond^n(x)), f(\sigma_*^n(x))) + d(f(\sigma_\diamond^n(x)), h(\sigma_*^n(x)))] \\ &\leq 2Lip(\sigma_*^{-2n}) \sum_{k=0}^{\infty} Lip(\sigma_*^{-2k+2}) \varepsilon(\sigma_\diamond^k)(\sigma_\diamond^n(x)), \sigma_\diamond^k(\sigma_\diamond^n(x))) \\ &\leq 2Lip(\sigma_*^{-2n}) 2Lip(\sigma_*^{2n}) \sum_{k=n}^{\infty} Lip(\sigma_*^{-(2k+2)}) \varepsilon(\sigma_\diamond^k(x), \sigma_\diamond^k(x)). \end{aligned}$$

Since $\{Lip(\sigma_*^{-2n})Lip(\sigma_*^{2n})\}$ is bounded, taking the limit as $n \to \infty$ in the above inequality, we obtain $h'(x) = h(x)$ for all $x \in X$. This completes the proof. □

Theorem 2. *Let $(X, \diamond)$ be a divisible square-symmetric groupoid and let $(Y, *, d)$ be a complete metric divisible square-symmetric groupoid. Assume that a mapping $f : X \to Y$ satisfies*

$$d(f(x \diamond y) * f(x \diamond y^{-1}), \sigma_*(f(x) * f(y))) \leq \varepsilon(x, y) \tag{6}$$

for all $x, y \in X$ with $f(id_X) = id_Y$ and $\varepsilon : X^2 \to [0, \infty)$ is a mapping such that

$$L_2(x, y) := \sum_{n=1}^{\infty} Lip(\sigma_*^{2n}) \varepsilon(\sigma_\diamond^{-n}(x), \sigma_\diamond^{-n}(y)) < \infty \tag{7}$$

for all $x, y \in X$. Then for every $x \in X$,

$$h(x) := \lim_{n \to \infty} \sigma_*^{2n} \circ f \circ \sigma_\diamond^{-n}(x) \tag{8}$$

converges and $h : X \to Y$ is the solution of (1) such that

$$d(f(x), h(x)) \leq Lip(\sigma_*^2) L_2(x, x) \tag{9}$$

for all $x \in X$.

Moreover, if $\{Lip(\sigma_*^{-2n})Lip(\sigma_*^{2n})\}$ is bounded then h is the only solution of (1) such that the mapping $x \to d(h(x), f(x))$ is bounded with bound $L(x, y)$.

Proof. Define $f_n : X \to Y$ by

$$f_0 := f, \quad f_n := \sigma_*^{2n} \circ f \circ \sigma_\diamond^{-n}$$

for all positive integer $n \in \mathbb{N}$. Let $x \in X$ be fixed. Replacing x and y by $\sigma_\diamond^{-n}(x)$ in (7), we obtain

$$d(f \circ \sigma_\diamond^{-n+1}(x), \sigma_*^2 \circ f \circ \sigma_\diamond^{-n}(x)) \leq \varepsilon(\sigma_\diamond^{-n}(x), \sigma_\diamond^{-n}(x)). \tag{10}$$

Thus we figure out by (10)

$$\begin{aligned} d(f_{n-1}(x), f_n(x)) &= d(\sigma_*^{2n-2} \circ f \circ \sigma_\diamond^{-n+1}(x), \sigma_*^{2(n-1)} \circ \sigma_*^2 \circ f \circ \sigma_\diamond^{-n}(x)) \\ &\leq Lip(\sigma_*^{2n-2}) d(f \circ \sigma_\diamond^{-n+1}(x), \sigma_*^2 \circ f \circ \sigma_\diamond^{-n+1}(x)) \\ &\leq Lip(\sigma_*^{2n-2}) \varepsilon(\sigma_\diamond^{-n}(x), \sigma_\diamond^{-n}(x)) \end{aligned}$$

for all $x \in X$. Hence, it follows by the above convergence that for all $n, m \in \mathbb{N}$

$$\begin{aligned} d(f_n(x), f_{n+m}(x)) &\leq \sum_{k=0}^{m-1} d(f_{n+k}(x), f_{n+k+1}(x)) \\ &\leq \sum_{k=0}^{m-1} Lip(\sigma_*^{2(n+k)-2}) \varepsilon(\sigma_\diamond^{-n-k}(x), \sigma_\diamond^{-n-k}(x)) \\ &\leq \sum_{k=n}^{n+m-1} Lip(\sigma_*^{-2}) Lip(\sigma_*^{2(k+1)}) \varepsilon(\sigma_\diamond^{-k}(x), \sigma_\diamond^{-k}(x)) \\ &\to 0 \quad as \quad n \to \infty. \end{aligned}$$

which shows that the sequence $\{f_n(x)\}$ is Cauchy in Y. Therefore, a mapping $h : X \to Y$ given by (8) is well defined. For all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(f_{n+1}(x), f(x)) &\leq \sum_{k=0}^{n} d(f_{k+1}(x), f_k(x)) \\ &\leq \sum_{k=0}^{n} Lip(\sigma_*^{2k}) \varepsilon(\sigma_\diamond^{-k-1}(x), \sigma_\diamond^{-k-1}(x)) \\ &\leq \sum_{k=1}^{\infty} Lip(\sigma_*^{2k-2}) \varepsilon(\sigma_\diamond^{-k}(x), \sigma_\diamond^{-k}(x)). \end{aligned}$$

Letting $n \to \infty$, we obtain (8) for the limit function h of $\{f_n\}$.

Now we prove that h satisfies (1). Putting $\sigma_\diamond^{-n}(x)$ and $\sigma_\diamond^{-n}(y)$ into x and y in (7) and using the endomorphism of $\sigma_\diamond^n$, we obtain

$$\begin{aligned} &d(f_n(x \diamond y) * f_n(x \diamond y^{-1}), \sigma_*(f_n(x) * f_n(y))) \\ &= d((\sigma_*^{2n} \circ f \circ \sigma_\diamond^{-n})(x \diamond y) * (\sigma_*^{2n} \circ f \circ \sigma_\diamond^{-n})(x \diamond y^{-1}), \end{aligned}$$

$$
\begin{aligned}
&\sigma_*(\sigma_*^{2n}\circ f\circ\sigma_\diamond^{-n}(x))*(\sigma_*^{2n}\circ f\circ\sigma_\diamond^{-n}(y)))\\
&= d((\sigma_*^{2n}\circ f\circ\sigma_\diamond^{-n})(x\diamond y)*(\sigma_*^{2n}\circ f\circ\sigma_\diamond^{-n})(x\diamond y^{-1}),\\
&\quad \sigma_*^{2n+1}(f\circ\sigma_\diamond^{-n}(x))*(f\circ\sigma_\diamond^{-n}(y)))\\
&\le Lip(\sigma_*^{2n})d((f\circ\sigma_\diamond^{-n})(x\diamond y)*(f\circ\sigma_\diamond^{-n})(x\diamond y^{-1}),\\
&\quad \sigma_*(f\circ\sigma_\diamond^{-n}(x)*f\circ\sigma_\diamond^{-n}(y))\\
&\le Lip(\sigma_*^{2n})\varepsilon(\sigma_\diamond^{-n}(x),\sigma_\diamond^{-n}(y))
\end{aligned}
$$

Whence, taking the limit as $n\to\infty$ in the last inequality, we get that f is a homomorphism, that is, it satisfies (1). The proof of the uniqueness of h is now based on the identity

$$\sigma_*^{2n}\circ h\circ\sigma_\diamond^{-n}=h,\quad \sigma_*^{2n}\circ h'\circ\sigma_\diamond^{-n}=h'.$$

In fact, let $\{Lip(\sigma_*^{2n})Lip(\sigma_*^{-2n})\}$ be bounded and assume that f' is another solution of (1) such that the mapping $x\mapsto d(f(x).h'(x))$ is bounded with bound (9). Then for every $x\in X$ we get by (9) and (10)

$$
\begin{aligned}
d(h'(x),h(x)) &= d(\sigma_*^{2n}\circ h'\circ\sigma_\diamond^{-n}(x))\\
&\le Lip(\sigma_*^{2n})d(h'\circ\sigma_\diamond^{-n}),h\circ\sigma_\diamond^{-n})\\
&\le Lip(\sigma_*^{2n})[d(h'\circ\sigma_\diamond^{-n}),h\circ\sigma_\diamond^{-n})+d(f\circ\sigma_\diamond^{-n}),h\circ\sigma_\diamond^{-n})]\\
&\le 2Lip(\sigma_*^{2n})\sum_{k=1}^{\infty}Lip(\sigma_*^{2k})\varepsilon(\sigma_\diamond^{-k}(\sigma_\diamond^{-n}(x)),\sigma_\diamond^{-k}(\sigma_\diamond^{-n}(x)))\\
&\le 2Lip(\sigma_*^{2n})Lip(\sigma_*^{-2n})\sum_{k=1}^{\infty}Lip(\sigma_*^{2n+2k})\varepsilon(\sigma_\diamond^{-n-k}(x)),\sigma_\diamond^{-n-k}(x))\\
&\le 2Lip(\sigma_*^{2n})Lip(\sigma_*^{-2n})\sum_{k=n}^{\infty}Lip(\sigma_*^{2k})\varepsilon(\sigma_\diamond^{-k}(x)),\sigma_\diamond^{-k}(x))\\
&\to 0\quad as\quad n\to\infty,
\end{aligned}
$$

since $\{Lip(\sigma_*^{2n})Lip(\sigma_*^{-2n})\}$ is bounded. This proves the uniqueness of h. The proof of the theorem is complete. □

3. The Hyers–Ulam Stability of Functional Equation (1) via Fixed Point Theorem

In this section, we will mention about the stability of the functional equation (1) using another method. First, we give the definition of a generalized metric on a set X. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies the following properties

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in Y$.

In [13], Margolis and Diaz constructed the variable method using a fixed point theory, which is variously applied to the theory of functional equations.

Theorem 3. *(The Alternative Fixed Point [13]) Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) $d(J^n x, J^{n+1} x) < \infty$, $\forall n \ge n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) $d(y, y^*) \le \dfrac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

Using the alternative fixed point theorem, we investigate the stability of the functional equation (1).

Theorem 4. *Let $(X, \diamond)$ be a divisible square-symmetric groupoid and let $(Y, *, d)$ be a complete metric divisible square-symmetric groupoid and $L \in (0, 1)$. Assume that a mapping $f : X \to Y$ satisfies*

$$d(f(x \diamond y) * f(x \diamond y^{-1}), \sigma_*(f(x) * f(y))) \le \varepsilon(x, y) \tag{11}$$

for all $x, y \in X$ with $f(id_X) = id_Y$ and $\varepsilon : X^2 \to [0, \infty)$ is a mapping such that

$$\varepsilon(\sigma_\diamond(x), \sigma_\diamond(y)) \le \frac{L}{Lip(\sigma_*^{-2})} \varepsilon(x, y) \tag{12}$$

for all $x, y \in X$. Then for every $x \in X$,

$$H(x) := \lim_{n\to\infty} \sigma_*^{-2n} \circ f \circ \sigma_\diamond^n(x)$$

converges and $H : X \to Y$ *is the solution of (1) such that*

$$d(f(x), H(x)) \leq \frac{Lip(\sigma_*^{-2})}{1-L}\varepsilon(x,x) \tag{13}$$

for all $x \in X$.

Proof. Consider the set

$$S := \{g : X \to Y\}$$

and introduce a generalized metric on S :

$$d'(g,h) := \inf\{c > 0 : d(g(x), h(x)) \leq c \cdot \varepsilon(x,x)\}$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S, d') is complete (see the proof of Lemma 2.1 of [14])

Now we consider the linear mapping $T : S \to S$ such that

$$Tg := \sigma_*^{-2} \circ g \circ \sigma_\diamond$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d'(g,h) = c$. Then

$$\begin{aligned} d(Tg(x), Tk(x)) &= d(\sigma_*^{-2} \circ g \circ \sigma_\diamond(x), \sigma_*^{-2} \circ k \circ \sigma_\diamond(x)) \\ &\leq Lip(\sigma_*^{-2}) d(f \circ \sigma_\diamond(x), k \circ \sigma_\diamond(x)) \\ &\leq Lip(\sigma_*^{-2}) c\varepsilon(\sigma_\diamond(x), \sigma_\diamond(x)) \\ &\leq c \cdot L\varepsilon(x,x) \end{aligned}$$

for all $x \in X$. So $d'(g,h) = c$ implies that $d'(Tg, Th) \leq Lc$. This means that

$$d'(Tg, Th) \leq Ld'(g,h)$$

for all $g, h \in S$.

Letting $y = x$ in (11), we get

$$d(f(\sigma_\diamond(x)), \sigma_*^2(f(x))) \leq \varepsilon(x,x)$$

for all $x \in X$. So

$$d(f(x), \sigma_*^{-2} \circ f \circ \sigma_\diamond(x))) \leq Lip(\sigma_*^{-2})\varepsilon(x,x)$$

for all $x \in X$. So $d'(f, Tf) \leq Lip(\sigma_*^{-2})$.

By Theorem 3, there exists a mapping $H : X \to Y$ satisfying the following :

(1) H is a fixed point of T, i.e.,

$$H(\sigma_\diamond(x)) = \sigma_*^2(H(x)) \tag{14}$$

for all $x \in X$. The mapping H is a unique fixed point of T in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This means that H is a unique fixed point satisfying (14).

(2) $d'(T^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \sigma_*^{-2n} \circ f \circ \sigma_\diamond^n(x) = H(x)$$

for all $x \in X$.

(3) $d'(f,H) \le \frac{1}{1-L} d'(f,Tf) \le \frac{Lip(\sigma_*^{-2})}{1-L}$, which implies the inequality

$$d(f(x), H(x)) \le \frac{Lip(\sigma_*^{-2})}{1-L}\varepsilon(x,x)$$

for all $x \in X$. This implies that the inequality (14) holds.

By (12),

$$\begin{aligned} &d((\sigma_*^{-2n} \circ f \circ \sigma_\diamond^n(x \diamond y)) * (\sigma_*^{-2n} \circ f \circ \sigma_\diamond^n(x \diamond y^{-1})), \\ &\qquad \sigma_*((\sigma_*^{-2n} \circ f \circ \sigma_\diamond^n(x)) * (\sigma_*^{-2n} \circ f \circ \sigma_\diamond^n(y)))) \\ &\le Lip(\sigma_*^{-2n})\varepsilon(\sigma_\diamond^n(x), \sigma_\diamond^n(y)) \\ &\le L^n \varepsilon(x,y) \end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Taking $n \to \infty$, we get H satisfies (1). This completes the proof. □

Theorem 5. *Let $(X, \diamond)$ be a divisible square-symmetric groupoid and let $(Y, *, d)$ be a complete metric divisible square-symmetric groupoid and $L \in (0,1)$. Assume that a mapping $f : X \to Y$ satisfies*

$$d(f(x \diamond y) * f(x \diamond y^{-1}), \sigma_*(f(x) * f(y))) \le \varepsilon(x,y) \tag{15}$$

for all $x, y \in X$ with $f(id_X) = id_Y$ and $\varepsilon : X^2 \to [0,\infty)$ is a mapping such that

$$\varepsilon(\sigma_\diamond^{-1}(x), \sigma_\diamond^{-1}(y)) \le \frac{L}{Lip(\sigma_*^2)}\varepsilon(x,y) \tag{16}$$

for all $x, y \in X$*. Then for every* $x \in X$*,*

$$H(x) := \lim_{n\to\infty} \sigma_*^{-2n} \circ f \circ \sigma_\diamond^n(x)$$

converges and $H : X \to Y$ *is the solution of (1) such that*

$$d(f(x), H(x)) \leq \frac{L}{Lip(\sigma_*^2)(1-L)}\varepsilon(x,x) \tag{17}$$

for all $x \in X$*.*

Proof. Putting $y = x$ in (15), we get

$$d(f(\sigma_\diamond(x)), \sigma_*^2 f(x)) \leq \varepsilon(x,x)$$

for all $x \in X$. If we set (S, d') which is introduced in Theorem 4 and define the operator T such that

$$Tg(x) := \sigma_*^2 \circ f \circ \sigma_\diamond^{-1}(x)$$

for all $g \in S$ and all $x \in X$. The rest proof is similar to that of Theorem 4. □

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