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ON THE STABILITY OF THE SOLUTIONS FOR A DELAY DIFFERENTIAL EQUATIONS WITH DISCONTINUITY

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Abstract: A numerical method of solving delay differential equations with fixed time delay and variable discontinuities of the solutions is considered. Runge-Kutta methods of higher order are used. The effectiveness of the method is shown by an example with proper initial data. The existence of a stable solution is discussed.

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1. Introduction

In this paper we consider functional differential equations (FDE) with delay, and discontinuities. Given an initial problem with a delay differential equation (DDE), which has the more general form:

$$\begin{array}{ll} (a) & \dot{x}(t) = f(t, x(t), x(t-\sigma)) \;\; t \in I = [0,T], \quad t \neq \tau_i(x), \\ (b) & x(s) = \psi(s), \;\; s \in [-\sigma, 0] \;\; (\sigma > 0), \\ (c) & \Delta x|_{t=\tau_i(x(t))} = S_i(x) \quad (i = 1, 2 \ldots, \end{array} \tag{1}$$

where $f : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently smooth function, the initial function

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$\psi : [-\sigma, 0] \to \mathbb{R}^n$ is continuous in its domain, $\Delta x|_{t=\tau_i(x(t))} = x(\tau_i) - x(\tau_i - 0)$, and $\tau_i : \mathbb{R}^n \to \mathbb{R}$ and $S_i : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous *switching surfaces* and *jump functions*, respectively. Assume for convenience that $\tau_0 = 0$.

The discontinuous systems have many applications and we refer the reader to [17], where many practical examples are constructed.

Note that in the applications there are mathematical models, where it is taken into account not only the given moments of the time, but also the pre-history. The general theory of FDE as well as many concrete examples are comprehensively studied in [14]. Solving such kind of problems numerically it turns out that although the solutions exist their derivatives are discontinuous at certain points which requires new form of numerical solving. Numerical solutions to these equations via Runge-Kutta (RK) methods are comprehensively studied in [6]. Numerical methods for ODEs including RK can be extended to DDEs, that is discussed in [7].

The problems including nonlinearity and FDEs are more complicated if the solutions contain jumps, i.e. under impulsive discontinuity. Note that the impulsive systems are applicable mainly in population dynamics, optimal control and economics. For the application of these equations we refer the reader to [17].

Note also that effective numerical solving of FDEs can be accomplished with aid of software as Maple, Mathematica, MatLab and so on, and systems of FDEs under impulsive effect could not be solved directly.

Further in the paper we study numerical approximations via Runge-Kutta (RK) methods of impulsive systems with fixed time of delay. The numerical methods concerning impulsive systems with no fixed jumps are studied mainly in [2, 3, 9, 10]. Also we refer the reader to [1], where numerical treatment of non smooth systems are studied.

In the next paragraph 2 "Preliminaries" we consider some definitions and known facts that use when we study FDEs.

In the paragraph 3 entitled "Runge-Kutta approximation of the solution" we study discrete approximation of a system with impulsive effect.

In the paragraph 4 entitled "Numerical examples" we construct an example in order to demonstrate existence of stable solution of a DDE under certain initial condition.

In the paragraph 5 we give some conclusive remarks.

2. Preliminaries

In this section we recall the main definitions and notations used in this paper. Assume that there exists a continuous extension of the solution obtained with such a precision which can be achieved by RK methods. Note that in this approach we use Hermite polynomials.

Recall that the piecewise continuously differentiable function $x(\cdot)$ is said to be a solution of (1) if:

(i) $x(\cdot)$ is right-side continuously differentiable function satisfying (1), and $\tau_i(x(t)) \neq t$ $(i = 1, \ldots, r)$, $t \in I$.

(ii) It possesses points of discontinuity (jumps) such that $\tau_i(x) = t$, $t \in I$, and with values satisfying $x(\tau_i) = x(\tau_i - 0) + S_i(x(\tau_i - 0))$.

Standing hypotheses (SH)

Suppose that $f(\cdot, \cdot)$ is sufficiently smooth w.r.t. its arguments (hence locally Lipschitz) with a growth condition, that is:
There exists a continuous function $v : I \times \mathbb{R}^+ \to \mathbb{R}^+$ with $|f(t, x)| \leq v(t, |x|)$ such that the maximal solution of $\dot{r} = v(t, r)$ exists on I for any initial condition $r(0) \geq 0$.

A1. $\tau_i(\cdot)$ and $S_i : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz with constants M and μ, respectively.

A2. $\tau_i(x + S_i(x)) \neq \tau_j(x)$, $\forall\ j \neq i$, and $\forall\ x \in \mathbb{R}^n$.

Denote by $\nabla\tau_i(\cdot)$ the gradient of $\tau_i(\cdot)$ and by $\nabla_C(\tau_j(x))$ the Clarke subdifferential (see e.g., [16]), having in $\mathbb{R}^n$ the form $\nabla_C(\tau_j(x)) = \overline{co} \bigcup \lim\limits_{x_i \to x} \{\nabla\tau_j(x)\}$, where $\overline{co}\ A$ is the closed convex hull of A.

We assume further that either **A3** or **A4** holds.

A3. $\tau_i(x) < \tau_{i+1}(x)$ for every $x \in \mathbb{R}^n$ and the following two conditions are satisfied:

1) There exists a constant $\alpha < 1$ such that $\langle \partial\tau_i(x), f(t, x, y) \rangle \leq \alpha$ $(i = 1, \ldots, r)$, $\forall\ (t, x, y) \in I \times \mathbb{R}^n \times \mathbb{R}^n$, where the derivatives exist;

2) $\tau_i(x) \geq \tau_i(x + S_i(x))$.

A4. $\tau_i(x) > \tau_{i+1}(x)$, $\forall\ x \in \mathbb{R}^n$, and the following two conditions are satisfied:

1) There exists a constant $\beta > 1$ such that $\langle \partial\tau_i(x), f(t, x, y) \rangle \geq \beta$ $(i = 1, \ldots, r)$, $\forall\ (t, x, y) \in I \times \mathbb{R}^n \times \mathbb{R}^n$, where the derivatives exist;

2) $\tau_i(x) \leq \tau_i(x + S_i(x))$.

A5.$f(t, \cdot)$ is Lipschitz with a constant K, $K \in \mathbb{R}^+$, (K ia a positive constant).

The multiple hitting of one switching surface is called **beating phenomenon**. By the following statements it can be demonstrated that if the hypotheses **SH** are satisfied, then the beating phenomenon is impossible.

Lemma 2.1. *Let* **A1, A2** *and* ***A3*** *or* ***A4*** *hold, and let* $x(\cdot)$ *be a solution of (1a)–(1c). Then every equation* $\tau_i(x(t)) = t$ $(i = 1, 2, \ldots)$ *admits no more than one solution.*

The proof can be seen in [3].

Theorem 2.2. *Under (SH) the system (1a)–(1c) admits a unique solution defined on* $[0, T]$*, and there exists a constant* K *such that the exact solution along with the approximate solutions are* K *- Lipschitzean on the intervals of continuity. Moreover, there exists a constant* $\lambda > 0$ *such that* $\tau_{i+1}(x(t)) - \tau_i(x(t)) \geq \lambda$ $(i = 1, 2, \ldots, r-1)$ *for every approximate solution* $x(\cdot)$*,* $t \in [0, T]$*.*

Existence and uniqueness for the solution of (1) is proved in [5], and the existence of λ is proved in [3].

3. Runge-Kutta Approximation of the Solution

In this section we study discrete approximation of the discontinuous system (1) with the RK scheme stated below. We refer the reader to [8] for the general theory of ODEs, and to [6] for RK methods used to the numerical solving of DDEs.

Let $0 = t_0 < t_1 < \ldots < t_N < t_{N+1} = T$ be a subdivision of $[0, T]$ for some natural number N. Note that an s-stage RK method computes iteratively the solution for the system (1) without jumps using the following relations:

$$\eta_h(t_{j+1}) = \eta_h(t_j) + h_j \sum_{m=1}^{p} b_m k_m, \tag{2}$$

$$k_j = f\left(t_j + c_m h_j, \eta_h(t_j) + h_j \textstyle\sum_{l=1}^{m} a_{ml} k_l, \eta_h(t_j + c_l h - \sigma)\right) \quad (j = 1, \ldots, p). \tag{3}$$

The RK method is *accurate up to order* p, if it provides the exact approximation of a polynomial solution $x(\cdot)$ up to degree p. It is known that the grid function $\eta_h(\cdot)$ of the RK approximation satisfies the estimate

$$\max_{j=0,\ldots,N} \|\eta_h(t_j) - x(t_j)\| \leq Ch^p \tag{4}$$

under appropriate smoothness conditions on the right-hand side of the FDE with delay and with suitable choice of the coefficients b_ν, c_ν, $a_{\nu,l}$ in (2), (3). Here $x(\cdot)$ is the solution of (1) without jumps.

Notice, however, that due to the delay terms there are discontinuities of the derivatives of the solution $x(\cdot)$ (see e.g., [6]). The discontinuity points must be included in the set of the grid points. We first include in the grid the points $\sigma, 2\sigma, \ldots, k\sigma$, where either $k = p$ or $k\sigma > T$. If we find a jump point τ, then we include in the grid $\tau, \tau + \sigma, \tau + 2\sigma$, etc.

To get an approximation of the term $x(t - \sigma)$ the continuous extension of the solution must be available. We must know the value of the solution not only on the grid point, but for every t. There are different methods to solve delay differential equations and we refer the reader to [6] for the theory.

Next, extend the solutions with their Hermite polynomials. In general, if p denotes the order of RK method used here, then the interpolation order of Hermite polynomials must be greater or equal to p. Let l_p denotes the number of support points for Hermite interpolation, then $2l_p > p$. Here in particular we use Hermite approximation polynomials of degree 3 in the cases when RK methods are of order 2 or of degree 5, and RK is of order 4. Further, we will not make difference between the approximate solution and its Hermite polynomials extension [2], [15].

The third order Hermite polynomial is defined for every coordinate x^k of the approximate solution x on (t_j, t_{j+1}) w.r.t. the values

$$H^k(t_j) = x^k(t_j), \quad \dot{H}^k(t_j) = f_k(t_j, x(t_j), x(t_j - \sigma)), \quad H^k(t_{j+1}) = x^k(t_{j+1}),$$
$$\text{and} \quad \dot{H}^k(t_{j+1}) = f_k(t_{j+1}, x(t_{j+1}), x(t_{j+1} - \sigma)).$$

Here $x(t_j - \sigma)$ is the value of Hermite extension of the approximate solution $x(\cdot)$.

Analogously Hermite polynomial of degree 5 is defined by using two successive intervals (t_{2k}, t_{2k+1}) and (t_{2k+1}, t_{2k+2}). The reader can consult [15] for details.

We will now apply the RK method to discontinuous systems and set

$$\begin{aligned} \varphi_i(t) &\equiv \tau_i(x(t)) - t, \quad t \in [0, T], \\ \varphi_{i,h}(t_j) &\equiv \tau_i(\eta_h(t_j)) - t_j, \quad j = 0, 1, \ldots, N. \end{aligned}$$

Calculate for this purpose some approximations by RK method to the differential system in (1) and for subsequent grid points t_j, $j = 1, \ldots, N$. On each interval $[t_j, t_{j+1}]$ we check, whether one of the functions $\varphi_{i,h}(\cdot)$ changes its sign. If it does (for some i), then the discrete trajectory $\eta_h(\cdot)$ needs to jump within the interval (t_j, t_{j+1}) which is close to the i-th jump of the exact solution $x(\cdot)$. Afterward we use some strategies to determine the jump points. Using the Hermite extension $H(t)$ of $\eta_k(\cdot)$ we solve the equation $\tau_k(H(t))-t = 0$. The solution τ_k is then the first approximation of the jump point $\tau_k(x)$. We find then the RK solution at the point $\bar{\tau}_i$ and calculate the value of $\varphi_{i,h}(\eta_k(\bar{\tau}_i))-\tau_i$. If the value is less then h^{p+1} that we set $t_i = \bar{\tau}_i$. Otherwise we continue by the same method, i.e. verify the interval where $\varphi_i(\cdot)$ change it sign and then use Hermite interpolation of the solution in that subinterval. Again solve $\tau_i(H(t)) - t = 0$. The second approximation is almost enough. The approximate $\dot{\tau}_i$ is then included in the grid point and continue by using the standard delayed RK methods.

Although RK method provides the values of approximate solutions only on the grid points, we will consider the approximate solutions as it is defined on the whole interval $[0, T]$ with unknown values outside the grid.

The following theorem is proved in [3, 5].

Theorem 3.1. *Under (SH) there exists $\delta > 0$ such that for every $\varepsilon \in [0, \delta]$ the system* (1) *has a solution $x^\varepsilon(\cdot)$. If (A5) also holds, then the system* (1) *admits an unique solution $y(\cdot)$ and moreover, there exists a constant $\tilde{C}$ such that the distance between $x^\varepsilon(\cdot)$ and the solution $y(\cdot)$ of* (1) *satisfies the inequality $\rho(x^\varepsilon, y) \le \tilde{C}\varepsilon$.*

Here one may conclude that the precision of the method follows from the above stated Theorem 3.1. Let Q_T is a sphere, and the solution $z(t)$ of the problem ((1)a), $z(t) \in Q_T$, $t_0 \le t < \infty$.

Definition 1. The solution $z(t)$ of the problem (1) is called stable (in the Liapunov sense) if for any $\varepsilon \in (0, T)$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that all solutions $x(t)$ of the problem (1) satisfying the condition $\rho(x_{t_0}, z_{t_0}) \le \delta$ are defined on $[t_0, \infty)$, and also $|x(t) - z(t)| \le \varepsilon$ for $t \ge t_0$.

We remind that the Liapunov stability means that an arbitrarily narrow ε neighborhood of the solution $x(t)$ contains all those solutions of the problem ((1)a) which are sufficiently close to z_{t_0} at initial moment t_0. Here x_{t_0} and z_{t_0} are solutions sufficiently close to initial moment t_0.

4. Numerical Examples

In this section we construct an example in order to demonstrate existence of stable solution of a DDE under certain initial condition.

The accuracy of the method we use is shown in the next theorem.

Theorem 4.1. *([13]) Under assumption **SH** the measure of distance between the exact solution $y(\cdot)$ and the approximate solution $\eta_h(\cdot)$ is $O(h^p)$ for N being big enough.*

In numerical calculations it is used the implicit RK methods of order 4 with step 0.1 for approximate solution (the calculations are accomplished by MatLab 7.14.0.739).

Example 1. Consider the following system:

$$\left| \begin{array}{l} \dot{x}(t) = x(t) + \cos^2 x(t-0.5) + \sin^2 y(t-0.5) - 1.1, \; t \in [0,1] \\ \dot{y}(t) = y(t) + \sin^2 x(t-0.5) + \cos^2 y(t-0.5) + 0.1, \; t \neq \tau_i(x,y), \end{array} \right. \tag{5}$$

where

$$x(t) \equiv y(t) \equiv 0.05, \; t \in [-0.5, 0], \; \textit{initial condition} \tag{6}$$

The equations of the surfaces have the form:

$$\tau_1 : t = x + y - 0.20,$$

$$\tau_2 : t = x + y - 1,$$

The impulsive effect is formulated by the equations:

$$\Delta_1(x,y) = \left(\frac{\sin^2(x)}{20}; \quad \frac{\cos^2(x)}{10} \right)$$

$$\Delta_2(x,y) = \left(\frac{\cos^2(y)}{10}; \quad \frac{\sin^2(y)}{5} \right)$$

To solve the system (5) we set $z(t) = x(t) + y(t)$, then get $\dot{z}(t) = z(t) + 1$, with initial condition $z(0) = 0.10$. The solution of the differential equation $\dot{z}(t) = z(t) + 1$ is

$$z(t) = ce^t - 1, \quad t \in [0,1],$$

where $c = 1.10$, and hence $z(t) = 1.10e^t - 1$. Here we use implicit RK methods for order 4 with step 0.1 for approximate solution of the system (5) and Hermite polynomial of degree 5. The unique solution of the transcedental equation

$$1.10e^{\tau_1} - \tau_1 - 1.20 = 0$$

is the time of the first jump τ_1. The unique solution for the transcendental equation

$$1.1719011834e^{\tau_2} - \tau_2 - 2 = 0$$

is the time for the second jump τ_2. Note here that the impulsive times are $\tau_1 = 0.3298774626$, and $\tau_2 = 0.9092773408$. The approximate jump points are $\tau_{1ap} = 0.329547585$, and $\tau_{2ap} = 0.908368063$. In the table we put the exact values of $z(t)$, and approximate values of $x_{ap}(t)$ and $y_{ap}(t)$. The error is $r(t) = |x_{ap}(t) + y_{ap}(t) - z(t)|$.

The results for exact solutions and approximate solutions are given in the table below.

t	z_{ex}(t)	$x_{ap}(t)$	$y_{ap}(t)$	$r(t)$
0.00	0.1000000000	0.0900000000	0.0100000000	0.0000000000
0.10	0.2156880099	0.1077362697	0.1073608810	0.0005908602
0.20	0.3435430339	0.1715999170	0.1710020079	0.0009411090
0.30	0.4848446883	0.2421801640	0.2413363307	0.0013281937
0.33	0.6298774625	0.3146241071	0.3135278559	0.0017254995
0.40	0.7482711285	0.3737618024	0.3724594965	0.0020498295
0.50	0.9321384081	0.4656036004	0.4639812883	0.0025535194
0.60	1.1353431783	0.5671044847	0.5651285109	0.0031101828
0.70	1.3599191808	0.6792803101	0.6769134797	0.0037253910
0.80	1.6081140476	0.8032537700	0.8004549764	0.0044053012
0.91	2.1092773408	1.0535850853	1.0499140571	0.0057781983
0.95	2.2385088491	1.1181362883	1.1142403430	0.0061322178
1.00	2.4045507483	1.2010742999	1.1968893720	0.0065870765

Table 1: Exact values and the RK4 values of Example 1

Example 2. *Consider the same system* (5)*:*

$$\left|\begin{array}{l} \dot{x}(t) = x(t) + \cos^2 x(t-0.5) + \sin^2 y(t-0.5) - 1.1,\ t \in [0,1] \\ \dot{y}(t) = y(t) + \sin^2 x(t-0.5) + \cos^2 y(t-0.5) + 0.1,\ t \neq \tau_i(x,y), \end{array}\right. \tag{7}$$

with new initial condition

$$x(t) \equiv y(t) \equiv 0.03,\ t \in [-0.5, 0],\ \textit{initial condition} \tag{8}$$

The equations of the surfaces are:

$$\tau_1 : t = x + y - 0.20,$$

$$\tau_2 : t = x + y - 1,$$

The impulsive effect is the following:

$$\Delta_1(x,y) = \left(\frac{\sin^2(x)}{20}; \quad \frac{\cos^2(x)}{10} \right)$$

$$\Delta_2(x,y) = \left(\frac{\cos^2(y)}{10}; \quad \frac{\sin^2(y)}{5}\right)$$

To solve the system (5) we set $z(t) = x(t) + y(t)$, then the system takes the form $\dot{z}(t) = z(t) + 1$, with initial condition $z(0) = 0.06$. The solution of the differential equation $\dot{z}(t) = z(t) + 1$ is

$$z = ce^t - 1, \quad t \in [0, 1], \tag{9}$$

where $c = 1.06$, and thus $z(t) = 1.06e^t - 1$.

We use implicit Runge-Kutta methods for order 4 with step 0.1 for approximate solution of the system (5) and Hermite polynomial of degree 5.

The unique solution of the transcendental equation

$$1.06e^{\tau_1} - \tau_1 - 1.20 = 0$$

is the time of the first jump τ_1. The unique solution for the transcendental equation

$$1.1249985898e^{\tau_2} - \tau_2 - 2 = 0$$

is the time for the second jump τ_2. Exact impulsive times are $\tau_1 = 0.4308046117$, and $\tau_2 = 0.9711760782$. The approximate jump points are $\tau_{1ap} = 0.430373807$, and $\tau_{2ap} = 0.970204902$. In the table we put the exact values of $z(t)$, and approximate values of $x_{ap}(t)$ and $y_{ap}(t)$. The results for exact solutions and approximate solutions are given in the table below.

Γ1 is the graphics of the solution for the equation (5) with initial condition $x(t) \equiv y(t) \equiv 0.03$ corresponding to the results of Table 1.

Γ2 is the graphics of the solution for the equation (5) with initial condition $x(t) \equiv y(t) \equiv 0.05$ corresponding to the results of Table 2.

Here we discuss an example (5) with two initial conditions, $x(t) \equiv y(t) \equiv 0.03$, and $x(t) \equiv y(t) \equiv 0.05$. To solve the example with initial condition $x(t) \equiv y(t) \equiv 0.03$ first we set $z(t) = x(t) + y(t)$, then the system becomes $\dot{z}(t) = z(t) + 1$, $x(t) \equiv y(t) \equiv 0.03$. We use implicit RK method of order 4 with step 0.1 for approximate solution. Denote the components of the approximate solution by x_{ap} and y_{ap}. Exact jump times are $\tau_1 = 0.4308046117$ and $\tau_2 = 0.9711760782$. After passing the first jump we solve the initial problem with a new initial condition, and after the second jump solve the initial problem with the new initial condition.

Next, consider the same example with new jump surface.

t	z_{ex}(t)	$x_{ap}(t)$	$y_{ap}(t)$	$r(t)$
0.00	0.0600000000	0.0500000000	0.0100000000	0.0000000000
0.10	0.1714811732	0.0853140165	0.0857405866	0.0004265701
0.20	0.2946869236	0.1466104098	0.1473434618	0.0007330520
0.30	0.4308503360	0.2143534010	0.2154251680	0.0010717670
0.40	0.5813341795	0.2892209848	0.2906670898	0.0014461049
0.43	0.7308046117	0.3635843839	0.3654023059	0.0018179219
0.50	0.8465678232	0.4211780215	0.4232839116	0.0021058901
0.60	1.0407730564	0.5177975405	0.5203865282	0.0025889877
0.70	1.2554030324	0.6245786231	0.6277015162	0.0031228931
0.80	1.4926058399	0.7425899701	0.7463029200	0.0037129499
0.90	1.7547554845	0.8730126789	0.8773777423	0.0043650634
0.97	2.1579745591	1.0736191836	1.0789872796	0.0053680959
1.00	2.2619381942	1.1253423852	1.1309690971	0.0056267119

Table 2: Exact values and the RK4 values of Example 2

Example 3. Consider the following system:

$$\left| \begin{array}{l} \dot{x}(t) = x(t) + \cos^2 x(t-0.5) + \sin^2 y(t-0.5) - 1.1, \; t \in [0,1] \\ \dot{y}(t) = y(t) + \sin^2 x(t-0.5) + \cos^2 y(t-0.5) + 0.1, \; t \neq \tau_i(x,y), \end{array} \right. \tag{10}$$

where

$$x(t) \equiv y(t) \equiv 0.05, \; t \in [-0.5, 0], \; \textit{initial condition} \tag{11}$$

The equations of the surfaces are:

$$\tau_1 : t = x + y - 0.21,$$

$$\tau_2 : t = x + y - 1,$$

The impulsive effect is demonstrated by the equations:

$$\Delta_1(x,y) = \left(\frac{\sin^2(x)}{20}; \quad \frac{\cos^2(x)}{10} \right)$$

$$\Delta_2(x,y) = \left(\frac{\cos^2(y)}{10}; \quad \frac{\sin^2(y)}{5} \right)$$

To solve the system (10) we set $z = x(t) + y(t)$, then the system becomes $\dot{z} = z + 1$, with initial condition $z(0) = 0.10$. The solution of the differential equation $\dot{z} = z + 1$ is

$$z = ce^t - 1, \quad t \in [0,1],$$

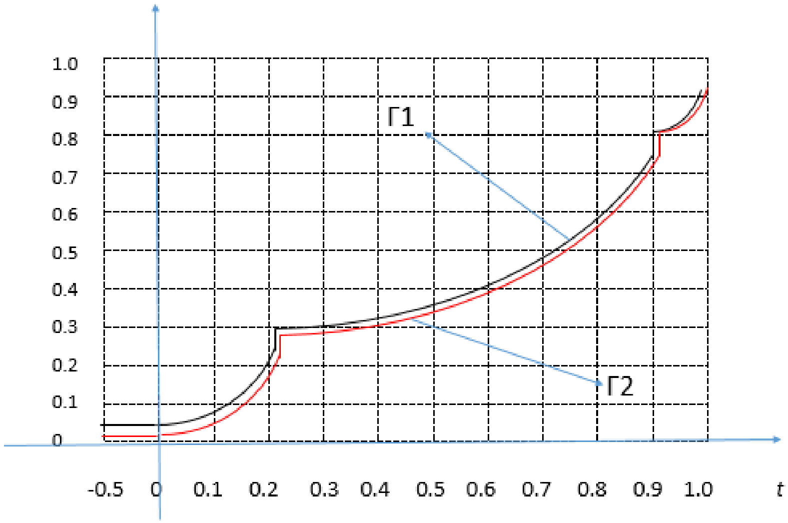

Figure 1: $\Gamma 1$ $x(t) \equiv y(t) = 0.10$, $\Gamma 2$ $x(t) \equiv y(t) = 0.03$.

where $c = 1.10$ and we get $z = 1.10e^t - 1$. Here use implicit RK methods for order 4 with step 0.1 for approximate solution of the system (5) and Hermite polynomial of degree 5. Then unique solution of the transcendental equation

$$1.10e^{\tau_1} - \tau_1 - 1.21 = 0$$

is the time of the firs jump τ_1. The unique solution for the transcendental equation

$$1.17e^{\tau_2} - \tau_2 - 2 = 0$$

is the time for the second jump τ_2.

Exact impulsive times are $\tau_1 = 0.3482589793$ and $\tau_2 = 0.9117508014$. The approximate jump points are $\tau_{1ap} = 0.34791072$ and $\tau_{2ap} = 0.910839051$. In the table we put the exact values of $z(t)$, and approximate values of $x(\cdot)$ and $y(\cdot)$. The error is $r(t) = |x_{ap}(t) + y_{ap}(t) - z(t)|$. The results for the exact solutions and approximate solutions are given in the table below.

Example 4. Consider the impulsive differential equation, which is constructed in such a way that in the moment t the solution decays. Here we shell use the RK methods. The example is constructed such that the solution $x(t)$ "decays", i.e. we have **beating phenomena** of the solution.

t	z_{ex}(t)	$x_{RK4}(t)$	$y_{RK4}(t)$	$r(t)$
0.00	0.1000000000	0.0500000000	0.0500000000	0.0000000000
0.10	0.2156880099	0.1077362697	0.1073608810	0.0005908602
0.20	0.3435430339	0.1715999170	0.1710020079	0.0009411090
0.30	0.4848446883	0.2421801640	0.2413363307	0.0013281937
0.35	0.6582589793	0.3274920295	0.3291294897	0.0016374601
0.80	1.6038828863	0.7979516847	0.8019414432	0.0039897584
0.90	1.8777356401	0.9341968359	0.9388678201	0.0046709842
0.91	2.1117508014	1.0506222893	1.0558754007	0.0052531114
0.95	2.2321370741	1.1105159573	1.1160685371	0.0055525798
1.00	2.3978522856	1.1929613361	1.1989261428	0.0059648067

Table 3: Exact values and the RK4 values of Example 3

$$\begin{aligned} &\dot{x}(t) = x, \quad x(t_0) = x_0 - \text{initial condition}, \\ &\tau : \quad x - 1 = 0 - \text{impulsive surface}. \end{aligned} \tag{12}$$

The solution of the equation (4) is

$$x(t) = x_0 e^{t-t_0}.$$

The solution takes value $= 1$, n-times).

$$\Delta_n = e^{-\frac{1}{5^n}} - 1.$$

Let $t_0 = 0,\ x_0 = e^{-\frac{1}{5}}$. Thus for the solution we have $x(t) = e^{-\frac{1}{5}} e^{(t)}$, and the time of the first jump is $t_1 = \frac{1}{5}$, second jump is $t_2 = \frac{1}{5} + \frac{1}{5^2}$, ...,$n$-th jump is $t_n = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^n}$. So it is easy to see that $t_n < t_{n+1}$. On the other hand we have

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{5^n} = \frac{1}{5} . \frac{1}{1 - \frac{1}{5}} = \frac{1}{4}.$$

Hence the solution $x(t)$ "decays" at the point $t = \frac{1}{4} = 0.25$. Here we use implicit RK methods for order 4 with step 0.01 for approximate solution of the system (5). Denote the approximate solution by $x_{ap}(t)$, and exact solution x_{ex}, $error = |x_{ex} - x_{ap}|$ In the table below we show results from the considered RK-method of second (fourth) order, and with step $h = 0.01$.

t	$x_{ex}(t)$	$x_{ap}(t)$	$error$
0.00	0.820000000	0.819918000	0.000082000
0.05	0.862069535	0.861983328	0.000086207
0.10	0.906297418	0.906206788	0.000090630
0.15	0.952794383	0.952699103	0.000095279
0.19	0.991703724	0.991604554	0.000099170
0.20	0.801676842	0.801596674	0.000080168
0.21	0.828241137	0.827412896	0.000828241
0.22	0.836565099	0.835728534	0.000836565
0.23	0.844972718	0.844127745	0.000844973
0.24	0.853464835	0.852611370	0.000853465
0.249	0.861180688	0.865398795	0.004218107

Table 4: Exact values and the RK4 values of Example 4

5. Conclusion

Note here that by the same method one could show stability for impulsive FDEs, also it is applicable for problems with inclusions, [2, 3] as well as fuzzy FDEs, [12]. Similar methods can be used for FDEs with maxima, and delay in the cases considered in [4]. Other applications of the methods used in the present paper are the cases of evolutionary DEs (for instance parabolic PDEs) with maxima and/or delay. The problem for stability and asymptotic stability can be resolved also by the aid of similar estimates. In parabolic case one may reduce the problem to an FDE, and the above stated estimates can be applied as well (see, e.g., [11]).

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