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2-METRIC DIMENSION OF CARTESIAN PRODUCT OF GRAPHS

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Abstract: Let $G(V, E)$ be a connected graph. A subset S of V is said to be *2-resolving set* of G, if for every pair of distinct vertices $u, v \notin S$, there exists a vertex $w \in S$ such that $|d(u, w) - d(v, w)| \geq 2$. Among all 2-resolving sets of G, the set having minimum cardinality is called a *2-metric basis* of G and its cardinality is called the *2-metric dimension* of G and is denoted by $\beta_k(G)$. In this paper, we determine the 2-metric dimension of cartesian product of complete graph with some standard graphs. Further, we have determined the 2-metric dimension of the graphs $P_m \square P_n$, $C_m \square P_n$ and $C_m \square C_n$.

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Key Words: cartesian product, 2-metric basis, 2-metric dimension

1. Introduction

All the graphs considered in this paper are simple, finite, undirected and connected. The metric dimension of general graphs were first defined by F. Harary and R.A. Melter [7] and P.J. Slater [12]. Since then, this concept is widely investigated by various authors of [3], [4], [5], [6], [11] and arise in many diverse

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areas like network discovery and verification, robot navigation, chemical structure etc., The metric dimension of cartesian product of graphs was first studied by Jose Caceres et al.[10]. The motivation to study the metric dimension of cartesian product of graphs was its application in strategies for the Mastermind game, coin weighing problems and hypercubes. We have extended the notion of metric dimension to k-metric dimension in [8], [9], [13] and have discussed a few characterizations of k-metric dimension. In particular, we have focused on 2-metric dimension, obtained some characterization, determined the two metric dimension of trees and obtained the bounds of unicyclic graphs.

2. Earlier Results on 2-Metric Dimension

In this section, we have some known results on 2- metric dimension obtained in [8], [9].

Theorem 2.1. *For any non-trivial graph G on $n \geq 2$ vertices, $\beta_2(G) = n - 1$ if and only if $diam(G) \leq 2$.*

Lemma 2.2. *For a graph G of order at least 2, if S is a 2-metric basis of G, then no two vertices $u, v \in V(G) - S$ are adjacent in G.*

Lemma 2.3. *Let G be a graph of order at least 2. If S is a 2-metric basis of G then for every two vertices $u, v \in V(G) - S$, $d(u, v) \geq 2$.*

Lemma 2.4. *Let G be a graph with $n \geq 3$ vertices with S as a 2-metric basis. Let $w \in V(G)$ be adjacent to p pendant vertices $v_1, v_2, \ldots v_p$, for $p \geq 2$. Then exactly p vertices from the set $\{w, v_1, v_2, \ldots v_p\}$ are in S.*

Lemma 2.5. *Let S be a 2-metric basis of a graph G of order n. Then $\Delta(G) \leq |S| + 1$.*

A tree $T(V, E)$ has one or two centers. Here, we consider all trees $T(V, E)$ as rooted trees with center c as root and denote it by (T, c). In case of trees with two centers any one of the center is considered as a root. Then for a vertex $v \in V(T, c)$, if the distance $d(c, v) = i$, $0 \leq i \leq e(c)$ then we say that v is in i^{th} level of (T, c). Let $L(i)$ be the set of all vertices that are at level i. Let $L(odd) = \cup_{i\ is\ odd} L(i)$ and $L(even) = \cup_{i\ is\ even} L(i) \cup L(0)$.
Now, we denote the set of all the terminal vertices adjacent to an exterior major vertex v by $M(v)$.

Lemma 2.6. *Let v be an exterior major vertex of a tree (T, c). Let m be an arbitrary element of $M(v)$. Then the sets*

$S_1 = L(odd) \bigcup [\bigcup_{v \in L(odd)} M(v) - \{m\}]$ *and*

$S_2 = L(even) \bigcup [\bigcup_{v\in L(even)} M(v) - \{m\}]$ *are 2-resolving sets of* T.

Theorem 2.7. *If* (T, c) *is a tree, then* $\beta_2(T, c) = min\{|S_1|, |S_2|\}$ *where* S_1 *and* S_2 *are the 2-resolving set as stated in Lemma 2.6.*

Corollary 2.8. *For a path* P_n, $n \geq 2$, $\beta_2(P_n) = \lfloor \frac{n}{2} \rfloor$.

Corollary 2.9. *For a cycle* $C_n, n \geq 3$, $\beta_2(C_n) = \lceil \frac{n}{2} \rceil$.

Theorem 2.10. *If* (T, c) *be a tree of order at least 3 and* e *is an edge of* $(\bar{T}, c)$, *then* $\beta_2(T, c) \leq \beta_2(T + e, c) \leq \beta_2(T, c) + 1$.

3. Cartesian Product of Graphs

We recall the definitions of cartesian product of graphs, horizontal projection and vertical projection from [10] which we use in the further sections.

The cartesian product of two graphs G_1 with $V(G_1) = \{u_1, u_2, \dots, u_m\}$ and G_2 with $V(G_2) = \{v_1, v_2, \dots, v_n\}$, denoted by $G_1 \Box G_2$, is a graph G with vertex set $V(G_1) \times V(G_2) = \{(u_i, v_j) : u_i \in V(G_1), v_j \in V(G_2), 1 \leq i \leq n, 1 \leq j \leq m\}$, where (u_i, v_j) is adjacent to (u_k, v_l) whenever $[u_i = u_k \; and \; v_j v_l \in E(G_2)] or \; [v_j = v_l \; and \; u_i u_k \in E(G_1)]$. The cartesian product of two graphs is commutative.

Remark 3.1. If two graphs G_1 and G_2 are connected , then $G_1 \Box G_2$ is connected and $d((u_i, v_j), (u_k, v_l)) = d_{G_1}(u_i, u_k) + d_{G_2}(v_j, v_l)$ for all vertices (u_i, v_j), (u_k, v_l) of $G_1 \Box G_2$.

Definition 3.2. [10] Let G_1 and G_2 be two graphs with

$$V(G_1) = \{u_1, u_2, \dots, u_m\} \text{ and } V(G_2) = \{v_1, v_2, \dots, v_n\}.$$

Let $G = G_1 \Box G_2$.

1) For $1 \leq i \leq |V(G_1)|$, the column C_i of G is defined as

$$C_i = \{(u_i, v_j) : v_j \in V(G_2), 1 \leq j \leq |V(G_2)|.$$

2) For $1 \leq j \leq |V(G_2)|$, the row R_j of G is defined as

$$R_j = \{(u_i, v_j) : u_i \in V(G_1), 1 \leq i \leq |V(G_1)|.$$

Remark 3.3. For $1 \leq j \leq |V(G_2)|$, each $< R_j > \cong G_1$ and for $1 \leq i \leq |V(G_1)|$, each $< C_i > \cong G_2$.

Definition 3.4. [10] Let $G = G_1 \square G_2$ and let $S \subset G$.

1) For each $1 \leq j \leq |V(G_2)|$, the horizontal projection H_j with respect to S on to G_1, is defined as $H_j = R_j \cap S$.

2) For each $1 \leq i \leq |V(G_1)|$, the vertical projection T_i with respect to S on to G_2, is defined as $T_i = C_i \cap S$.

Remark 3.5. Let $G = G_1 \square G_2$ and let $S \subset G$. For $1 \leq j \leq |V(G_2)|$, let H_j be the horizontal projection with respect to S and for $1 \leq i \leq |V(G_1)|$, T_i be the vertical projection with respect to S. Then $S = \bigcup_{j=1}^{|V(G_2)|} H_j = \bigcup_{i=1}^{|V(G_1)|} T_i$ and $|S| = \sum_{j=1}^{|V(G_2)|} |H_j| = \sum_{i=1}^{|V(G_1)|} |T_i|$

4. Results on 2-Metric Dimension of Cartesian Product of Graphs

In this section we characterize 2-metric dimension in terms of horizontal and vertical projections

Lemma 4.1. *Let G_1, G_2 be any two non-trivial connected graphs and $S \subset V(G_1 \square G_2)$. If S is a 2-resolving set of $G_1 \square G_2$, then the following holds:*

1) for every $1 \leq j \leq |V(G_2)|$, $H_j \neq \phi$;

2) for every $1 \leq i \leq |V(G_1)|$, $T_i \neq \phi$.

Proof. We prove the result by the method of contradiction. Let S be a 2-resolving set of $G_1 \square G_2$. Suppose, for some j, $1 \leq j \leq |V(G_2)|$, let $H_j = \phi$. Then R_j is totally disconnected (by Lemma 2.2) and hence G_2 is totally disconnected (since $R_j \cong G_2$)

The proof for 2 is analogous to proof of 1. □

Remark 4.2. For any two non-trivial graphs G_1, G_2 and $S \subset V(G_1 \square G_2)$, if S is a 2-resolving set and for some $u, v \in V(G_1 \square G_2) - S$, $w \in S$, $d(u, w) = d(v, w)$, then by the definition of 2-metric dimension the following holds:

1) No vertex in the column containing w in $G_1 \square G_2$ will 2-resolve u, v whenever u, v and w are in the same row of $G_1 \square G_2$.

2) No vertex in the row containing w in $G_1 \square G_2$ will 2-resolve u, v whenever u, v and w are in the same column of $G_1 \square G_2$.

Lemma 4.3. *Let $G_1 \square G_2$ be the cartesian product of any two non-trivial graphs G_1, G_2 such that $diam(G_1 \square G_2) > 2$ and $G_1 \ncong P_3$. Then any $S \subset V(G_1 \square G_2)$, S is a 2-resolving set of $G_1 \square G_2$ if and only if the following hold:*

1) for every j, $1 \leq j \leq |V(G_2)|$, H_j 2-resolves $< R_j >$;

2) for every $(u_i, v_j) \in V(G_1 \square G_2) - S$, $N((u_i, v_j)) \subset S$.

Proof. Let S be a 2-resolving set of G. For some j, $1 \leq j \leq |V(G_2)|$, if H_j is not a 2-resolving set of $< R_j >$ then there exists at least two vertices $u = (u_i, v_j), v = (u_k, v_j) \in R_j - S$ such that for every $w = (u_r, v_j) \in H_j$, we have $|d(u, v) - d(v, w)| \leq 1$. Then there are two possibilities: u, v are adjacent or $d(w, u) = d(w, v) = 1$.

Now, this $u, v \in R_j - S \subset V(G_1 \square G_2) - S$, a contradiction to first possibility since u, v are not adjacent in $V(G_1 \square G_2) - S$ as $V(G_1 \square G_2) - S$ is independent (by lemma 2.2). From the second possibility, we have $d(w, u) = d(w, v) = 1$ and this is possible when there is only one w such that $uw, vw \in E(G_1 \square G_2)$ which implies that G_1 is P_3. This is a contradiction to the hypothesis. Thus each j, $1 \leq j \leq |V(G_2)|$, H_j is a 2-resolving set.

Further, if for some $u = (u_i, v_j) \in V(G_1 \square G_2) - S$, $N((u_i, v_j)) \subsetneq S$, then there exists at least one vertex v or w not in S such that $v = (u_r, v_j)$ where u_i is adjacent to u_r in G_1 or $w = (u_i, v_m)$ where v_j is adjacent to v_m in G_2. In the first case, $u, v \in V(G_1 \square G_2) - S$ and are adjacent in $G_1 \square G_2$, which is a contradiction to Lemma 2.2. The other case follows similarly.

Conversely, let $S \subset V(G_1 \square G_2)$. For every j, $1 \leq j \leq |V(G_2)|$, by item (1) H_j is 2-resolving set of $< R_j >$ and by item (2), for every $(u_i, v_j) \in V(G_1 \square G_2) - S$, $N((u_i, v_j)) \subset S$.

Let $u = (u_i, v_j), v = (u_k, v_l) \in V(G_1 \square G_2) - S$. By item (2), u and v are not adjacent and hence $d(u, v) \geq 2$ for every $u, v \in V(G_1 \square G_2) - S$. Now we have two cases:

Case 1: u, v are in the same row R_j.

Since $N(u) \subset S$, there exists a $w \in S$ in $N(u)$ and $d(u, w) = 1$. Now, we have two subcases:

Subcase 1: $w \in N(v)$. If $w \in N(v)$ then $d(u, w) = d(v, w) = 1$ and $d(u, v) = 2$. By definition of 2-resolving set, w does not resolve u and v. Also, by item (2), there exists $s \in N(v) \subset S$ such that $vs \in E(G_1 \square G_2)$. Then $|d(u, s) - d(v, s)| = 2$. Hence s 2-resolves u and v.

Subcase 2: $w \notin N(v)$. If $w \notin N(v)$, then $d(u, w) \neq d(v, w)$ and $d(u, v) > 2$. We now have two possibilities:

1) w is in $u - v$ path.

From item (2), there exists a $s \in S$ such that $vs \in E(G_1 \square G_2)$. Then by triangular inequality $|d(u, s) - d(v, s)| = |(d(u, v) + d(v, s)) - d(v, s)| = d(u, v) > 2$. Hence, u and v are 2-revolved by s.

2) w is not in $u - v$ path. Then w may or may not be in the same row as u, v and from item (2), there exists a $s \in S$ such that $vs \in E(G_1 \Box G_2)$. Then by triangular inequality $|d(u, w) - d(v, w)| = |d(u, w) - (d(v, u) + d(u, w))| = d(u, v) > 2$. Hence w 2-resolves u, v.

Case 2: u, v are in different rows, say $u \in R_j$, $v \in R_k$ and $d(u, v) \geq 2$.

Since $N(u) \subset S$, there exists a $w \in S$ in $N(u)$ and $d(u, w) = 1$. Now, we have two subcases:

Subcase 1: $w \in N(v)$. If $w \in N(v)$ then $d(u, w) = d(v, w) = 1$, $d(u, v) = 2$ and w is in $u - v$ path. By definition of 2-resolving set, w does not 2-resolve u and v. Now, we have two possibilities $w \in R_j$ or $w \in R_k$. In either case there exists a s in $N(v)$ such that $s \in H_k$ (since R_k is 2-resolved by H_k), such that $|d(u, s) - d(v, s)| = |(d(u, v) + d(v, s)) - d(v, s)| = d(u, v) = 2$ by triangular inequality. Thus u and v are 2-resolved by s.

Subcase 2: $w \notin N(v)$. If $w \notin N(v)$, then $d(u, w) \neq d(v, w)$ and $d(u, v) > 2$. We now have two possibilities:

1) w is in $u - v$ path.

From item (2), there exists a $s \in S$ such that $vs \in E(G_1 \Box G_2)$. Then by triangular inequality $|d(u, s) - d(v, s)| = |(d(u, v) + d(v, s)) - d(v, s)| = d(u, v) \geq 2$. Hence, u and v are 2-revolved by s.

2) w in not in $u - v$ path. Then by triangular inequality, $|d(u, w) - d(v, w)| = |d(u, w) - (d(u, v) + d(u, w))| = d(u, v) \geq 2$. Hence w 2-resolves u, v.

Hence the proof. □

Lemma 4.4. *Let G_1, G_2 be any graphs, such that $diam(G_1 \Box G_2) > 2$ and $G_2 \ncong P_3$ then a set $S \subset V(G_1 \Box G_2)$ is a 2-metric basis of $G_1 \Box G_2$ if and only if the following hold:*

1) for every j, $j = 1, 2, \ldots, |V(G_2)|$, T_j 2-resolves $< C_j >$;

2) for every $(u_i, v_j) \in V(G_1 \Box G_2) - S$, $N((u_i, v_j)) \subset S$.

Proof. The proof for vertical projection is analogous to horizontal projection as in Lemma 4.3. □

Lemma 4.5. *Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be any two graphs with $|V_1| = m$ and $|V_2| = n$. Let S be a 2-metric basis of $G_1 \Box G_2$ with 2-metric dimension $|S| = \beta_2(G_1 \Box G_2)$, S_1 be a 2-metric basis of G_1 with 2-metric dimension $|S_1| = \beta_2(G_1)$ and S_2 be a 2-metric basis of G_2 with 2-metric dimension $|S_2| = \beta_2(G_2)$. Then, $\beta_2(G_1 \Box G_2) \geq max\{n\ \beta_2(G_1), m\ \beta_2(G_2)\}$.*

Proof. Let S and S_1 be metric bases of $G_1 \square G_2$ and G_1 respectively and hence are the resolving sets of $G_1 \square G_2$ and G_1 respectively. By Lemma 4.3, each $H_j = S_1$ resolves G_1. Since there are n copies of G_1 in $G_1 \square G_2$, $|S|$ must be at least n times $|S_1|$. Then, $|S| \geq n\ |S_1|$

$$\Rightarrow \beta_2(G_1 \square G_2) \quad \geq \quad n\ \beta_2(G_1) \tag{1}$$

The proof of the result

$$\beta_2(G_1 \square G_2) \quad \geq \quad m\ \beta_2(G_1) \tag{2}$$

is similar to proof of the result in equation 1. Now, equations 1 and 2 leads to required conclusion

$$\beta_2(G_1 \square G_2) \geq \max\{n\ \beta_2(G_1), m\ \beta_2(G_2)\}.$$ □

5. 2-Metric Dimension of the Cartesian Product of Some Standard Graphs with Complete Graphs

In this section, we obtain the 2-metric dimension of the cartesian product of a complete graph with various other standard graphs.

Remark 5.1. (from Theorem 2.1) For a complete graph K_n on n vertices,

$$\beta_2(K_n) = \begin{cases} n-1, & \text{if} \quad n \geq 2 \\ 1, & \text{if} \quad n = 1. \end{cases}$$

Remark 5.2. For any graph G, the disjoint union of G with K_n denoted by $G + K_n$ is a graph with diameter 2 and hence $\beta_2(G) = m + n - 1$, where $|V(G)| = m$. (from Theorem 2.1)

Theorem 5.3. *[8] For a connected graph G, $\beta_2(G) = 1$ if and only if $G \cong K_1$ or K_2.*

Theorem 5.4. *For all $m \geq n \geq 2$, $\beta_2(K_m \square K_n) = mn - 1$.*

Proof. The proof directly follows from Theorem 2.1 noting the fact that $K_m \square K_n$ is a diameter 2 graph. □

Theorem 5.5. *For a complete graph $K_n, n \geq 1$ and a path $P_m, m \geq 1$,*

$$\beta_2(K_n \square P_m) = \begin{cases} 1, & \text{if } m = 1, n = 1 \\ \left\lfloor \frac{m}{2} \right\rfloor, & \text{if } m \geq 2, n = 1 \\ n-1, & \text{if } m = 1, n \geq 2 \\ 5, & \text{if } m = 2, n = 3 \\ 2n-1, & \text{if } m = 2, n = 2 \text{and } n \geq 4 \\ (n-1)m, & \text{if } m \geq 3, n \geq 3 \end{cases}$$

Proof. The result is trivial when $n = m = 1$.

When $n = 1$ and $m \geq 2$, $K_n \square P_m \cong P_m$ and hence by Corollary 2.8, $\beta_2(K_1 \square P_m) = \beta_2(P_m) = \left\lfloor \frac{m}{2} \right\rfloor$. For $m = 1, n \geq 2$, the result follows by noting the fact that $K_n \square P_1 \cong K_n$.

If $m = 2, n = 3$ then $K_n \square P_2$ is a diameter 2 graph and the result follows by Theorem 2.1. If $m = 2, n \geq 2$ then $K_n \square P_2$ is a graph of diameter 2 and hence the result follows by Theorem 2.1.

Let $m \geq 3, n \geq 3$. For $1 \leq j \leq m$, each $< R_j > \cong K_n$ and also we know that $\beta_2(K_n) = n - 1$. By Lemma 4.5, $\beta_2(K_n \square P_m) \geq m(n-1)$. It can be easily verified that, the set S obtained by choosing $n-1$ vertices from each of the m rows R_j of $K_n \square P_m$, such that for every $(u_i, v_j) \in V(K_n \square P_m) - S$, $N((u_i, v_j)) \subset S$, S is a 2-metric basis of $K_n \square P_m$ and thus, $|S| = m(n-1) \Rightarrow \beta_2(K_n \square P_m) = m(n-1)$, for $m \geq 3, n \geq 3$. Hence the proof. $\square$

Theorem 5.6. *For a complete graph $K_n, n \geq 1$, $C_m, m \geq 3$,*

$$\beta_2(K_n \square C_m) = \begin{cases} \left\lceil \frac{m}{2} \right\rceil, & \text{if } n = 1, m \geq 3 \\ 5, & \text{if } n = 2, m = 3 \\ 2\left\lceil \frac{m}{2} \right\rceil, & \text{if } n = 2, m \geq 4 \\ (n-1)m, & \text{if } n \geq 3, m \geq 3 \end{cases}$$

Proof. If $n = 1, m = 3$, $K_1 \square C_m \cong C_m$, and hence by Theorem 2.9, $\beta_2(K_n \square C_m) = \left\lceil \frac{m}{2} \right\rceil$.

If $n = 2, m = 3$, then $K_2 \square C_3$ is a diameter 2 graph and by Theorem 2.1, $\beta_2(K_n \square C_m) = 5$.

Let $n = 2, m \geq 4$ and by Theorem 2.9, for every i, $1 \leq i \leq m, \beta_2(C_m) = \left\lceil \frac{m}{2} \right\rceil$. By Lemma 4.5, $\beta_2(K_2 \square C_m) \geq 2\left\lceil \frac{m}{2} \right\rceil$. It can be easily verified that, the set S obtained by choosing $\left\lceil \frac{m}{2} \right\rceil$ vertices from each of the 2 columns C_j

of $K_2 \square C_m$, such that for every $(u_i, v_j) \in V(G) - S$, $N((u_i, v_j)) \subset S$, S is a 2-metric basis of $K_2 \square C_m$ and thus, $|S| = 2\left\lceil \frac{m}{2} \right\rceil$.

Let $n \geq 3, m \geq 3$. For $1 \leq j \leq m$, each $< R_j > \cong K_n$ and also we know that $\beta_2(K_n) = n - 1$. By Lemma 4.5, $\beta_2(K_n \square C_m) \geq m(n-1)$. It can be easily verified that, the set S obtained by choosing $n-1$ vertices from each of the m rows R_j of $K_n \square C_m$, such that for every $(u_i, v_j) \in V(K_n \square C_m) - S$, $N((u_i, v_j)) \subset S$, S is a 2-metric basis of $K_n \square C_m$ and thus, $|S| = m(n-1) \Rightarrow \beta_2(K_n \square C_m) = m(n-1)$ for $m, n \geq 3$. Hence the proof. □

6. 2-Metric Dimension of the Graph $G = P_m \square P_n$

We consider a path P_m on m vertices with $V(P_m) = \{u_1, u_2, u_3, \ldots, u_m\}$ and edge set $E(P_m) = \{(u_i, u_{i+1}) : i = 1, 2, \ldots, m-1\}$ and a path on n vertices with $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and edge set $E(P_n) = \{(v_i, v_{i+1}) : i = 1, 2, \ldots, n-1\}$. Consider the grid graph $P_m \square P_n$ with $V(P_m \square P_n) = \{(u_i, v_j) : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$ and edge set $E(P_m \square P_n) = \{((u_i, v_j)(u_r, v_s)) : [u_i = u_r \text{ and } v_j v_s \in E(G_2)] \text{ or } [v_j = v_s \text{ and } u_i u_r \in E(G_1)]\}$.

Remark 6.1. Let P_n be a path on $n \geq 2$ vertices. Then for any two vertices $v_i, v_j \in V(P_n)$, $d_{P_n}(v_i, v_j) = d(v_i, v_j) = |i - j|$.

Lemma 6.2. *In a grid graph $P_m \square P_n$, $m \geq n \geq 2$, for any two vertices $(u_i, v_j), (u_k, v_l)$, $d((u_i, v_j), (u_k, v_l)) = |i - k| + |j - l|$.*

Proof. Let $(u_i, v_j), (u_k, v_l) \in V(P_m \square P_n)$. Then

$$\begin{aligned} d((u_i, v_j), (u_k, v_l)) &= d_{P_m}(u_i, u_k) + d_{P_n}(v_j, v_l) \quad \text{(by Remark 3.1)} \\ &= |i - k| + |j - l|. \quad \text{(by Remark 6.1)} \end{aligned}$$

□

Remark 6.3. Consider two vertices $(u_i, v_j), (u_i, v_l)$ that lie in same column of a grid graph $P_m \square P_n$. Then $d((u_i, v_j), (u_i, v_l)) = d_{P_n}(v_j, v_l)$.

Similarly, $d((u_i, v_j), (u_k, v_j)) = d_{P_m}(u_i, u_k)$ where $(u_i, v_j), (u_k, v_j)$ are vertices in same row.

Theorem 6.4. *For a grid graph, $P_m \square P_n$, $m, n \geq 2$,*

$$\beta_2(P_m \square P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } m \text{ and } n \text{ are odd} \\ \frac{mn}{2}, & \text{otherwise.} \end{cases}$$

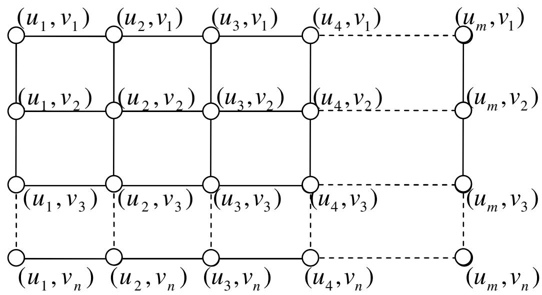

Figure 1: Grid $P_m \Box P_n$

Proof. Consider a grid graph, $P_m \Box P_n$, $m, n \geq 2$. As the cartesian product of any two graphs is commutative, without loss of generality we consider $m \geq n$.

For $1 \leq j \leq n$, each $< R_j >\cong P_m$ and from Corollary 2.8 we know that $\beta_2(P_m) = \left\lfloor \frac{m}{2} \right\rfloor$. By Lemma 4.5, $\beta_2(P_m \Box P_n) \geq n \left\lfloor \frac{m}{2} \right\rfloor$. We prove the result in four different cases.

Case 1: Both m and n are even

For $m = n = 2$, we know that $P_2 \Box P_2$ is a diameter 2 graph. Hence by theorem 2.1, $\beta(P_2 \Box P_2) = 3$.

We now discuss the proof for grid graphs for $m \geq 4, n \geq 2$ and show that $\beta_2(P_m \Box P_n) = n \left\lfloor \frac{m}{2} \right\rfloor$, as below;

Define a set S as

$S = \{(u_i, v_j) : i = 2, 4, \ldots, m, j = 1, 3, \ldots, n-1\} \bigcup \{(u_l, v_k) : l = 1, 3, \ldots, m-1, k = 2, 4, \ldots, n\}$.

We first claim that S is a 2 metric basis.

For $m + n \geq 6$, the horizontal projections of S on to P_m are $H_j = R_j \cap S, for\ j = 1, 3, \ldots, n-1$ and $H_k = R_k \cap S, for\ k = 2, 4, \ldots, n$. For $j = 1, 3, \ldots, n-1$, consider the horizontal projection

$$H_j = \{(u_2, v_j), (u_4, v_j), \ldots, (u_m, v_j)\}.$$

For any two vertices $(u_1, v_j), (u_3, v_j) \in V(P_m \Box P_n) - S$ and $(u_4, v_j) \in S$, we have $|d((u_1, v_j), (u_4, v_j)) - d((u_3, v_j), (u_4, v_j))| = |d_{P_m}(u_1, u_4) - d_{P_m}(u_4, u_3)| = 2$ (by Remark 6.3 and by Lemma 6.2)

Further, for every $i = 3, 5, 7, \ldots, p, \ldots, q \ldots, m-1$, with $(u_p, v_j), (u_q, v_j) \in V(P_m \Box P_n) - S$ and $(u_2, v_j) \in S$, $|d((u_p, v_j), (u_2, v_j)) - d((u_q, v_j), (u_2, v_j))| = |d_{P_m}(u_p, u_2) - d_{P_m}(u_q, u_2)| = 2$ (by Remark 6.3 and p, q are odd numbers ≥ 3)

Thus, $j = 1, 3, \ldots, n-1$ each H_j is a minimal 2-resolving set of each $< R_j >$.

Similarly, for $k = 2, 4, \ldots, n$, the horizontal projection H_k also forms a minimal 2-resolving set of each $< R_k >$.

Also, for every $u_r v_s \in V(P_m \Box P_n) - S$, we have the following possibilities:

1) For $r = 2, 4 \ldots, m-2, s = 2, 4, \ldots, n-2$,

$$N((u_r, v_s)) = \{(u_{r+1}, v_s), (u_{r-1}, v_s), (u_r, v_{s-1}), (u_r, v_{s+1})\} \subset S;$$

2) For $r = 3, 5, \ldots, m-1, s = 3, 5, \ldots, n-1$,

$$N((u_r, v_s)) = \{(u_{r-1}, v_s), (u_{r+1}, v_s), (u_r, v_{s+1}), (u_r, v_{s-1})\} \subset S;$$

3) For $r = 3, 5, \ldots, m-1$, $N((u_r, v_1)) = \{(u_r, v_2), (u_{r-1}, v_1), (u_{r+1}, v_1)\} \subset S$;

4) For $r = 2, 4, 6 \ldots, m-2$,

$$N((u_r, v_n)) = \{(u_r, v_{n-1}), (u_{r-1}, v_n), (u_{r+1}, v_n)\} \subset S;$$

5) For $s = 3, 5, \ldots, n-1$,

$$N((u_1, v_s)) = \{(u_2, v_s), (u_1, v_{s-1}), (u_1, v_{s+1})\} \subset S;$$

6) For $s = 2, 4, \ldots, n-2$,

$$N((u_m, v_s)) = \{(u_m, v_{s-1}), (u_m, v_{s+1}), (u_{m-1}, v_s)\} \subset S;$$

7) $N((u_1, v_1)) = \{(u_2, v_1), (u_1, v_2)\} \subset S$;

8) $N((u_m, v_n)) = \{(u_{m-1}, v_n), (u_m, v_{n-1})\} \subset S$.

Thus, from Lemma 4.3, S is a 2-metric basis of $P_m \Box P_n$ when m and n are both even.

Further in $H_j = \{(u_i, v_j) : i = 2, 4, \ldots, m, j = 1, 3, \ldots, n-1\}$, for $j = 1$, the horizontal projection $H_1 = \{(u_2, v_1), (u_4, v_1), \ldots, (u_m, v_1)\}$ and $|H_1| = \frac{m}{2}$. Since there are $\frac{n}{2}$ such horizontal projections, for $p = 1, 3, \ldots, n-1$, $\sum_{j \in p} |H_j| = \frac{m}{2}\frac{n}{2}$.

Similarly, if $H_k = \{(u_l, v_k) : l = 1, 3, \ldots, m-1, k = 2, 4, \ldots, n\}$, then for $q = 2, 4, \ldots, n$, $\sum_{k \in q} |H_k| = \frac{m}{2}\frac{n}{2}$.

Thus, for $p = 1, 3, \ldots, n-1$ and for $q = 2, 4, \ldots, n$, (since m and n are even) from Remark 3.5, $|S| = \sum_{j \in p} |H_j| + \sum_{k \in q} |H_k| = \frac{m}{2}\frac{n}{2} + \frac{m}{2}\frac{n}{2} = \frac{n}{2}[\frac{m}{2} + \frac{m}{2}] = \frac{mn}{2}$

$$\Rightarrow \qquad \beta_2(P_m \Box P_n) = \frac{mn}{2} \tag{3}$$

Case 2: Both m and n are odd.

For the grid graphs $P_3 \Box P_n$, when $n \geq 3$, let

$$S = \{(u_2, v_j), (u_1, v_k), (u_3, v_k), \ j = 1, 3, \ldots, n; \ k = 2, 4, \ldots, n-1\}.$$

It can be verified as in the above Case 1 that S is a 2-metric basis.

Let us prove the result for grid graphs with $m \geq 5, n \geq 3$. Consider two rows R_j, R_{j+1} of $P_m \Box P_n$. Then $< R_j > \cong < R_{j+1} > \cong P_m$. By Corollary 2.8,

$$S_j = \{(u_2, v_j), (u_4, v_j), \ldots, (u_{m-1}, v_j)\}$$

and

$$S_{j+1} = \{(u_2, v_{j+1}), (u_4, v_{j+1}), \ldots, (u_{m-1}, v_{j+1})\}.$$

In $P_m \Box P_n$, $(u_1, v_j) \in V(P_m \Box P_n) - S_j$, $(u_1, v_{j+1}) \in N(u_1, v_j)$, but $(u_1, v_{j+1}) \in V(P_m \Box P_n) - S_{j+1}$, a contradiction to item 2 of Lemma 4.3. Hence for $< V(R_j) \cup V(R_{j+1}) >$ to be 2-resolved either $(u_1, v_j) \in S_j$ or $(u_1, v_{j+1}) \in S_{j+1}$. This is true with every j, $j = 1, 3, 5 \ldots, n-2$, $R_j \in P_m \Box P_n$. Thus for every j, $j = 1, 3, 5 \ldots, n-2$, a vertex $(u_1, v_j) \in S_j$. Since there are $\left\lfloor \frac{n}{2} \right\rfloor$ rows, the metric dimension of $P_m \Box P_n$ has at least $\left\lfloor \frac{n}{2} \right\rfloor$ more vertices in addition to $n \left\lfloor \frac{m}{2} \right\rfloor$, that is

$$\begin{aligned}
\beta_2(P_m \Box P_n) &\geq n \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \left(\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil \right) \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \\
&= \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \\
&= \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{m}{2} \right\rfloor + 1 \right) + \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{m}{2} \right\rfloor \\
&= \left\lceil \frac{m}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{m}{2} \right\rfloor .
\end{aligned}$$

Thus 2-metric dimension of $P_m \Box P_n$ is at least $\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor$. To show that $\beta_2(P_m \Box P_n) = \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor$, for $m > 3,\ m + n \geq 7$ we define the set $S = \{(u_i, v_j) : i = 2, 4, \ldots, m-1; j = 1, 3, \ldots, n\} \bigcup \{(u_l, v_k) : l = 1, 3, \ldots, m; k = 2, 4, \ldots, n-1\}$. For $m > 3,\ m + n \geq 7$, the horizontal projections of S on to P_m are $H_j = R_j \cap S,$ for $j = 1, 3, \ldots, n$ and $H_k = R_k \cap S,$ for $k = 2, 4, \ldots, n-1$. The proof to show that S is a 2-metric basis is similar to case (1).

We have $H_j = \{(u_i, v_j) : i = 2, 4, \ldots, m-1; j = 1, 3, \ldots, n\}$. With respect to the first row, $H_1 = \{(u_2, v_1), (u_4, v_1), \ldots, (u_{m-1}, v_1)\}$ and $|H_1| = \left\lfloor \frac{m}{2} \right\rfloor$. Since there are $\left\lceil \frac{n}{2} \right\rceil$ such horizontal projections of P_m, for $p = 1, 3, \ldots, n$, $\sum_{j \in p} |H_j| = \left\lfloor \frac{m}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$. Similarly, for $H_k = \{(u_l, v_k) : l = 1, 3, \ldots, m, k = 2, 4, \ldots, n-1\}$, for $q = 2, 4, \ldots, n-1$, $\sum_{k \in q} |H_k| = \left\lceil \frac{m}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor$. Thus, for $p = 1, 3, \ldots, n$ and for $q = 2, 4, \ldots, n-1$, $|S| = \sum_{j \in p} |H_j| + \sum_{k \in q} |H_k|$

$$\Rightarrow \qquad \beta_2(P_m \Box P_n) = |S| = \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor \tag{4}$$

Case 3: m is even and n is odd.

We now prove the case $m \geq 4$, and $n \geq 3$ and show that $\beta_2(P_m \square P_n) = n\left\lfloor \frac{m}{2} \right\rfloor$. For this consider the set; $S = \{(u_i, v_j) : i = 2, 4, \ldots, m; j = 1, 3, \ldots, n\} \bigcup \{(u_l, v_k) : l = 1, 3, \ldots, m-1; k = 2, 4, \ldots, n-1\}$. The horizontal projections of S on to P_m are $H_j = R_j \cap S,$ for $j = 1, 3, \ldots, n$ and $H_k = R_k \cap S,$ for $k = 2, 4, \ldots, n-1$. The proof to show that S is a 2-metric basis is similar to case 1 and hence we omit the proof.

As

$$H_j = \{(u_i, v_j) : i = 2, 4, \ldots, m; j = 1, 3, \ldots, n\},$$

$$H_1 = \{(u_2, v_1), (u_4, v_1) \ldots, (u_m, v_1)\}$$

and $|H_1| = \left\lfloor \frac{m}{2} \right\rfloor$. Since there are $\left\lceil \frac{n}{2} \right\rceil$ rows of P_m, for $p = 1, 3, \ldots, n$, $\sum_{j \in p} |H_j| = \left\lfloor \frac{m}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$.

Similarly, for $H_k = \{(u_l, v_k) : l = 1, 3, \ldots, m-1, k = 2, 4, \ldots, n\}$, for $q = 2, 4, \ldots, n$, $\sum_{k \in q} |H_k| = \left\lceil \frac{m}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor$. Thus, $|S| = \sum_{j \in p} |H_j| + \sum_{k \in q} |H_k|$

$$\Rightarrow |S| = \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor = \frac{m}{2} \left[\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor \right] = \frac{mn}{2}$$

(since m is even and n is odd)

$$\Rightarrow \beta_2(P_m \square P_n) = \frac{mn}{2}. \tag{5}$$

Case 4: m is odd and n is even.

Let $G = P_3 \square P_n$. Then $S = \{(u_2, v_j), (u_1, v_k), (u_3, v_k), j = 1, 3, \ldots, n-1; k = 2, 4, \ldots, n\}$ is a 2-resolving set of G (follows similar to above Case 2).

For other graphs in this case, let $S = \{(u_i, v_j) : i = 2, 4, \ldots, m-1; j = 1, 3, \ldots, n-1\} \bigcup \{(u_l, v_k) : l = 1, 3, \ldots, m; k = 2, 4, \ldots, n\}$. Then as the Cartesian product is commutative, the proof follows from Case 3. □

7. 2-Metric Dimension of the Graph $G = C_m \square P_n$

In this section, we determine the 2-metric dimension of the cartesian product of cycle on m vertices with a path on n vertices. Consider cycle C_m on m vertices with $V(C_m) = \{u_1, u_2, \ldots, u_m\}$ and P_n be the path on n vertices with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Let $G = C_m \square P_n$.

Theorem 7.1. *For a graph, $G = C_m \square P_n$, $m \geq 3, n \geq 2$,*

$$\beta_2(C_m \square P_n) = \begin{cases} 5, & \text{if } m = 3, n = 2 \\ n\left\lceil \frac{m}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

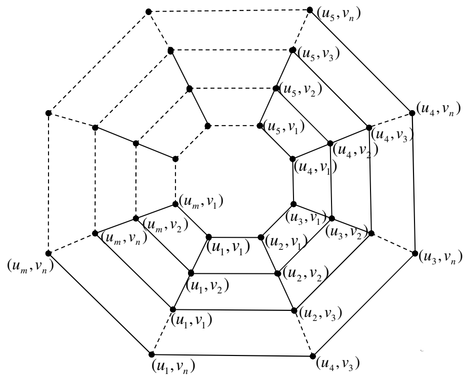

Figure 2: $C_m \square P_n$

Proof. Consider the graph $C_m \square P_n$ for $m \geq 3, n \geq 2$ as labeled in the figure 2.

As the cartesian product of any two graphs is commutative, without loss of generality we consider $m \geq n$.

We know that $C_3 \square P_2$ is a diameter 2 graph and hence by Theorem 2.1, $\beta_2(C_3 \square P_2) = 5$. Now, we prove the result for all $m \geq 3, n \geq 2$ (except for $m = 3$, $n = 2$). For each $1 \leq j \leq n$, $< R_j > \cong C_m$ and by Theorem 2.9, we know that $\beta_2(C_m) = \lceil \frac{m}{2} \rceil$. By Lemma 4.5, $\beta_2(C_m \square P_n) \geq n \lceil \frac{m}{2} \rceil$. To prove the equality $\beta_2(C_m \square P_n) = n \lceil \frac{m}{2} \rceil$, in the Figure 2, for every i, $1 \leq j \leq m$ we consider the cycle

$$C_j = (u_1, v_j), (u_2, v_j), \dots, (u_m, v_j), (u_1, v_j).$$

For each j, $1 \leq j \leq n$, let G_1 be the subgraph of $C_m \square P_n$ obtained by deleting the edges $\{(u_1, v_j), (u_m, v_j)\}$, from C_j. Then $G_1 \cong P_m \square P_n$.

Case 1: When m and n both are even in $C_m \square P_n$. Then m and n are both even in G_1 also. Thus by Theorem 6.4, $S = \{(u_i, v_j) : i = 2, 4, \dots, m; j = 1, 3, \dots, n-1\} \bigcup \{(u_l, v_k) := 1, 3, \dots, m-1; k = 2, 4, \dots, n\}$ and hence $\beta_2(G_1) = \frac{mn}{2}$

Now, by adding the edges $\{(u_1, v_j), (u_m, v_j)\}$ to G_1 we obtain the graph $C_m \square P_n$. The subset S of $V(G_1)$ is also a subset of $V(C_m \square P_n)$. Also for $j = 2, 4, \dots, n$, $N((u_1, v_j)) \subset S$, for $j = 1, 3, \dots, n-1$, $N((u_m, v_j)) \subset S$, for $j = 1, 3, \dots, n-1$, $(u_1, v_j) \in S$ and for $j = 2, 4, \dots, n$ $(u_m, v_j) \in S$. Thus by Lemma 4.3, S is a 2-resolving set of $C_m \square P_n$. Thus S is a 2-metric basis of G and $\beta_2(C_m \square P_n) = \frac{mn}{2}$ Since m is even, we have

$$\beta_2(C_m \square P_n) = \left\lceil \frac{m}{2} \right\rceil n \tag{6}$$

Case 2: When m and n both are odd in $C_m \Box P_n$.

Then m and n are both odd in G_1 also. Thus by Theorem 6.4, $S_1 = \{(u_i, v_j) : i = 2, 4, \ldots, m-1; j = 1, 3, \ldots, n\} \cup \{(u_l, v_k) : l = 1, 3, \ldots, m; k = 2, 4, \ldots, n-1\}$ and $\beta_2(G_1) = \lfloor \frac{m}{2} \rfloor \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor$.

Now, after adding the edges $\{(u_1, v_j), (u_m, v_j)\}$ to G_1 we get the graph $C_m \Box P_n$, the subset S of $V(G_1)$ fails to be a 2-metric basis of G since both the vertices $\{(u_1, v_j), (u_m, v_j)\} \in V(C_m \Box P_n) - S$ for $j = 1, 3, \ldots, n$. Thus for S_1 to be a 2-metric basis of $C_m \Box P_n$, either (u_1, v_j) or (u_m, v_j) must be in S_1. Let S be a 2-metric basis of $C_m \Box P_n$. Then

$$\begin{aligned} S &= S_1 \cup (u_1, v_j),\ for\ j = 1, 3, \ldots, n \\ \beta_2(G) &= \left\lfloor \frac{m}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil \left(\left\lfloor \frac{m}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{m}{2} \right\rceil \\ &= \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{m}{2} \right\rceil = \left\lceil \frac{m}{2} \right\rceil \left(\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor \right) = \left\lceil \frac{m}{2} \right\rceil n \end{aligned}$$

Hence

$$\beta_2(C_m \Box P_n) = \left\lceil \frac{m}{2} \right\rceil n. \tag{7}$$

Case 3: When m is even and n is odd in $C_m \Box P_n$.

Then m is even and n is odd in G_1 also. Thus by Theorem 6.4, $S = \{(u_i, v_j) : i = 2, 4, \ldots, m; j = 1, 3, \ldots, n\} \bigcup \{(u_l, v_k) : l = 1, 3, \ldots, m-1; k = 2, 4, \ldots, n-1\}$ and hence $\beta_2(G_1) = \lfloor \frac{mn}{2} \rfloor \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor$. Now, by adding the edges $\{(u_1, v_j), (u_m, v_j)\}$ to G_1 we obtain the graph G. The subset S of $V(G_1)$ is also a subset of $V(G)$ and it satisfies Lemma 4.3. Thus S is a 2-metric basis of G also and hence $\beta_2(C_m \Box P_n) = \frac{mn}{2}$. Since m is even, we have $\lceil \frac{m}{2} \rceil = \lfloor \frac{m}{2} \rfloor$, so

$$\beta_2(C_m \Box P_n) = \left\lceil \frac{m}{2} \right\rceil n. \tag{8}$$

Case 4: When m is odd and n is even in $C_m \Box P_n$.

Noting the fact that $P_m \Box P_n$ is commutative, the proof follows by Case 3. □

8. 2-Metric Dimension of the Graph $G = C_m \Box C_n$

In this section, we determine the 2-metric dimension of the cartesian product of cycle on m vertices on a cycle on n vertices. Consider cycle C_m on m vertices with $V(C_m) = \{u_1, u_2, \ldots, u_m\}$ and C_n be another cycle on n vertices with $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Let $G = C_m \Box C_n$.

Theorem 8.1. *For a graph,* $G = C_m \Box C_n$, $m, n \geq 3$,

$$\beta_2(C_m \Box C_n) = \begin{cases} 8, & \text{if } m = 3, n = 3 \\ n\left\lceil \frac{m}{2} \right\rceil, & \text{if } m \text{ and } n \text{ are even, } m \geq n \\ m\left\lceil \frac{n}{2} \right\rceil, & \text{if } m \text{ is even, } n \text{ is odd} \\ n\left\lceil \frac{m}{2} \right\rceil, & \text{if } m \text{ is odd, } n \text{ is even,} \\ \left\lceil \frac{m}{2} \right\rceil n + \left\lfloor \frac{m}{2} \right\rfloor - 1, & \text{if } m \text{ and } n \text{ are odd, } m \geq n. \end{cases}$$

Proof. We know that $C_3 \Box C_3$ is a diameter 2 graph and hence by theorem 2.1 $\beta_2(C_3 \Box C_3) = 8$.

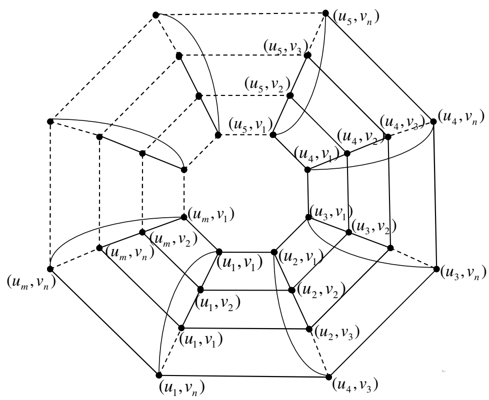

Figure 3: $C_m \Box C_n$

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Consider the graph $C_m \Box C_n$ for $m, n \geq 3, m \geq n$ as labeled in the figure 3.

As the cartesian product of any two graphs is commutative, without loss of generality we consider $m \geq n$.

For each $1 \leq j \leq n$, $< R_j > \cong C_m$ and by Theorem 2.9, we know that $\beta_2(C_m) = \left\lceil \frac{m}{2} \right\rceil$. By Lemma 4.5, $\beta_2(C_m \Box C_n) \geq n\left\lceil \frac{m}{2} \right\rceil$(since $m \geq n$). We now prove that $\beta_2(C_m \Box C_n) = n\left\lceil \frac{m}{2} \right\rceil$.

From the figure, for every i, $1 \leq i \leq m$ we consider the cycle $C_i = (u_i, v_1), (u_i, v_2), \ldots, (u_i, v_n), (u_i, v_1)$.

For each i, $1 \leq i \leq m$, let G_1 be the subgraph of $C_m \Box C_n$ obtained by deleting the edges $\{(u_i, v_1), (u_i, v_n)\}$, from C_i. Then $G_1 \cong C_m \Box P_n$.

Case 1: m and n both are even.

Then m and n are both even in G_1 also. Thus by Theorem 7.1, $S = \{(u_i, v_j) : i = 2, 4, \ldots, m; j = 1, 3, \ldots, n-1\} \bigcup \{(u_l, v_k) := 1, 3, \ldots, m-1; k = 2, 4, \ldots, n\}$ and hence $\beta_2(G_1) = \lceil \frac{m}{2} \rceil n$ Now, by adding the edges $\{(u_i, v_1), (u_i, v_n)\}$ to G_1 we obtain the graph $C_m \Box C_n$. The subset S of $V(G_1)$ is also a subset of $V(C_m \Box C_n)$.

Further, for every $u_r v_s \in V(C_m \Box C_n) - S$, we have the following possibilities:

1) For $r = 2, 4 \ldots, m-2, s = 2, 4, \ldots, n-2$,
$N((u_r, v_s)) = \{(u_{r+1}, v_s), (u_{r-1}, v_s), (u_r, v_{s-1}), (u_r, v_{s+1})\} \subset S$,

2) For $r = 3, 5, \ldots, m-1, s = 3, 5, \ldots, n-1$,
$N((u_r, v_s)) = \{(u_{r-1}, v_s), (u_{r+1}, v_s), (u_r, v_{s+1}), (u_r, v_{s-1})\} \subset S$,

3) For $r = 3, 5, \ldots, m-1$,
$N((u_r, v_1)) = \{(u_r, v_2), (u_{r-1}, v_1), (u_{r+1}, v_1)\} \subset S$,,

4) For $r = 2, 4, 6 \ldots, m-2$,
$N((u_r, v_n)) = \{(u_r, v_{n-1}), (u_{r-1}, v_n), (u_{r+1}, v_n)\} \subset S$,

5) For $s = 3, 5, \ldots, n-1$,
$N((u_1, v_s)) = \{(u_2, v_s), (u_1, v_{s-1}), (u_1, v_{s+1})\} \subset S$,

6) For $s = 2, 4, \ldots, n-2$,
$N((u_m, v_s)) = \{(u_m, v_{s-1}), (u_m, v_{s+1}), (u_{m-1}, v_s)\} \subset S$,

7) $N((u_1, v_1)) = \{(u_2, v_1), (u_1, v_2)\} \subset S$,

8) $N((u_m, v_n)) = \{(u_{m-1}, v_n), (u_m, v_{n-1})\} \subset S$.

Thus by Lemma 4.3, S is a 2-resolving set of $C_m \Box C_n$. Thus S is a 2-metric basis of G.

$$\beta_2(C_m \Box C_n) = \left\lceil \frac{m}{2} \right\rceil n \tag{9}$$

Case 2: When m and n both are odd in $C_m \Box C_n$. Consider two rows R_j, R_{j+1} of $C_m \Box C_n$. Then $< R_j > \cong < R_{j+1} > \cong C_m$. By Theorem 2.9, $S_j = \{(u_2, v_j), (u_4, v_j), \ldots, (u_{m-1}, v_j)\}$ and

$$S_{j+1} = \{(u_2, v_{j+1}), (u_4, v_{j+1}), \ldots, (u_{m-1}, v_{j+1})\}.$$

In $P_m \Box P_n$, $(u_1, v_j) \in V(C_m \Box C_n) - S_j$, $(u_1, v_{j+1}) \in N(u_1, v_j)$ but $(u_1, v_{j+1}) \in V(C_m \Box C_n) - S_{j+1}$, a contradiction to item 2 of Lemma 4.3. Hence for $< V(R_j) \cup V(R_{j+1}) >$ to be 2-resolved either $(u_1, v_j) \in S_j$ or $(u_1, v_{j+1}) \in S_{j+1}$. This is true with every j, $j = 1, 3, 5 \ldots, n-1$, $R_j \in P_m \Box P_n$. Thus for every j, $j = 1, 3, 5 \ldots, n-1$, a vertex $(u_1, v_j) \in S_j$. Since there are $\frac{n}{2}$ rows, a metric basis of $C_m \Box C_n$ has at least $\lfloor \frac{m}{2} \rfloor - 1$ more vertices in addition to $m \lfloor \frac{n}{2} \rfloor$, that is

$$\beta_2(C_m \Box C_n) \geq m \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor - 1.$$

Now we show that $\beta_2(C_m\Box C_n) = m\left\lfloor \frac{n}{2}\right\rfloor + \left\lfloor \frac{m}{2}\right\rfloor - 1$. Consider the graph G_1. Then m and n are both odd in G_1 also. Thus by Theorem 6.4,

$$S_1 = \{(u_i, v_j) : i = 2, 4, \ldots, m-1, j = 1, 3, \ldots, n\}$$
$$\cup \{(u_l, v_k) : l = 1, 3, \ldots, m, k = 2, 4, \ldots, n-1\} \cup \{(u_1, v_j) : j = 1, 3, \ldots, n-2\}$$
$$\cup \{u_m, v_n\}$$

and $\beta_2(G_1) = \left\lceil \frac{m}{2}\right\rceil n$.

Now, after adding the edges $\{(u_i, v_1), (u_i, v_n)\}$ to G_1 we get the graph $C_m\Box C_n$, the subset S of $V(G_1)$ fails to be a 2-metric basis of $C_m\Box C_n$ since both the vertices $\{(u_i, v_1), (u_m, v_i)\} \in V(C_m\Box P_n) - S$ for $i = 3, 5 \ldots, n-2$. Thus for S_1 to be a 2-metric basis of $C_m\Box C_n$, for $i = 3, 5 \ldots, n-2$, (u_i, v_n) must be in S_1. Let S be a 2-metric basis of $C_m\Box C_n$. Then $S = S_1 \cup (u_i, v_n),\ for\ i = 3, 4 \ldots, n$ $\Rightarrow \beta_2(C_m\Box C_n) = \left\lceil \frac{m}{2}\right\rceil n + \left\lfloor \frac{m}{2}\right\rfloor - 1$.
Hence

$$\beta_2(C_m\Box C_n) = \left\lceil \frac{m}{2}\right\rceil n + \left\lfloor \frac{m}{2}\right\rfloor - 1. \tag{10}$$

Case 3: When m is even and n is odd in $C_m\Box C_n$, then m is even and n is odd in G_1 also. Thus by Theorem 6.4,

$S = \{(u_i, v_j) : i = 2, 4, \ldots, m; j = 1, 3, \ldots, n\} \bigcup \{(u_l, v_k) : l = 1, 3, \ldots, m-1; k = 2, 4, \ldots, n-1\}$ and hence $\beta_2(G_1) = \left\lceil \frac{m}{2}\right\rceil n = \left\lceil \frac{n}{2}\right\rceil m$ (since cartesian product is commutative).

Now, by adding the edges $\{(u_i, v_1), (u_i, v_n)\}$ to G_1, we obtain the graph $C_m\Box C_n$. The subset S of $V(G_1)$ is also a subset of $V(C_m\Box C_n)$ and it satisfies Lemma 4.3. Thus S is a 2-metric basis of $C_m\Box C_n$ also and hence

$$\beta_2(C_m\Box C_n) = \left\lceil \frac{n}{2}\right\rceil m. \tag{11}$$

Case 4: When m is odd and n is even in $C_m\Box C_n$.

Let us mark the fact that $C_m\Box C_n$ is commutative, the proof follows by Case 3. $\Box$

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