\textbf{τ-CURVATURE ON LORENTZIAN $\alpha$-SASAKIAN MANIFOLDS}

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\textbf{Abstract:} We study Lorentzian $\alpha$-Sasakian manifolds endowed with a $\tau$-curvature tensor which satisfies the conditions $\tau(\xi, X) \cdot R = 0$, $\tau(\xi, X) \cdot S = 0$, $\tau$-flat, $\xi$-$\tau$-flat. We also applied the results to several curvatures which are the particular cases of $\tau$-curvature tensor.

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\textbf{Key Words:} Lorentzian $\alpha$-Sasakian manifold, $\tau$-curvature tensor, $\tau$-flat, $\xi$-$\tau$-flat, $\eta$-Einstein manifold

\section{1. Introduction}

It is well known that, Weyl conformal curvature tensor [3] and conharmonic curvature tensor [5] are invariant under conformal and conharmonic transformations respectively. And, projective curvature tensor [3] vanishes if and only if semi-Riemannian manifold is locally projectively flat. The next important cur-
vature tensor in semi-Riemannian point of view is concircular curvature tensor [13], which is also invariant under concircular transformation. Yano and Sawaki [14] generalized the conformal curvature tensor and concircular curvature tensor to quasi-conformal curvature tensor.

With these ideas in mind, in 2011, Tripathi and Gupta [10] introduced a new curvature tensor called $T$-curvature tensor on the semi-Riemannian manifold and is given by

$$T(X, Y)Z = a_0 R(X, Y)Z + a_1 S(Y, Z)X + a_2 S(X, Z)Y$$

$$+ a_3 S(X, Y)Z + a_4 g(Y, Z)QX + a_5 g(X, Z)QY$$

$$+ a_6 g(X, Y)QZ + a_7 r[g(Y, Z)X - g(X, Z)Y],$$

where, $a_0, ..., a_7$ are some smooth functions on $M$; and $R, S, Q$ and $r$ are the curvature tensor, the Ricci tensor, the Ricci operator of type $(1, 1)$ and the scalar curvature respectively.

With this above generalized form they have shown that quasi-conformal, conformal, conharmonic, concircular, pseudo-projective, projective, $W_i$-curvature tensors ($i = 0, ..., 9$), $W_j^*$-curvature tensors ($j = 0, 1$) are the particular cases of the $T$-curvature tensor. Later, the authors Tripathi and Gupta [11] also studied $T$-curvature tensor in $k$-contact and Sasakian manifolds. In 2012 Nagaraja [8] and Somashekhara studied the $T$-curvature tensor in $(k, \mu)$ manifold. In 2013, Shah [9] studied the same curvature on LP-Sasakian manifold. Later in 2014, Ingalahalli and Bagewadi [4] studied the curvature on $N(k)$-contact manifold and arrived at some interesting results.

The present paper is organized as follows: In Section 2, some definitions and preliminary results are presented which will be needed thereafter. In Section 3 we study Lorentzian $\alpha$-Sasakian manifold satisfying the condition $T(\xi, X) \cdot R = 0$ and shown that as $\eta$-Einstein manifold. And in section 4 we study Lorentzian $\alpha$-Sasakian manifold satisfying the condition $T(\xi, X) \cdot S = 0$ and arrived at some interesting results. Section 5 and Section 6 contains discussions about $T$-flat and $\xi$-$T$-flat conditions respectively. We have also discussed the results of each sections to several curvatures which are particular cases of $T$-curvature tensor.

2. Preliminaries

A $(2n + 1)$-dimensional smooth manifold $M$ is said to be Lorentzian $\alpha$-Sasakian manifold if it admits a $(1, 1)$-tensor field $\phi$, a contravariant vector field field $\xi$, global
Differentiable 1-form \( \eta \) and a Lorentzian metric \( g \) which satisfy [15]

\[
\begin{align*}
\phi^2 &= I + \eta \otimes \xi, \quad \eta(\xi) = -1, \\
\phi \xi &= 0, \quad \eta \circ \phi = 0, \\
g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \\
g(X, \xi) &= \eta(X),
\end{align*}
\]

for all \( X, Y \in TM \).

Also a Lorentzian \( \alpha \)-Sasakian manifold \( M \) satisfies the following [15]:

\[
\begin{align*}
\nabla_X \xi &= \alpha \phi X, \\
(\nabla_X \eta) Y &= \alpha g(X, \phi Y),
\end{align*}
\]

where \( \nabla \) denotes the operator of covariant differentiation with respect to the Lorentzian metric \( g \) and \( \alpha \) is constant.

Further, a Lorentzian \( \alpha \)-Sasakian manifold \( M \) holds the following relations:

\[
\begin{align*}
R(\xi, X)Y &= \alpha^2(g(X, Y)\xi - \eta(Y)X), \\
R(X, Y)\xi &= \alpha^2(\eta(Y)X - \eta(X)Y), \\
R(\xi, X)\xi &= \alpha^2(\eta(X)\xi + X), \\
S(X, \xi) &= 2n\alpha^2\eta(X), \\
Q\xi &= 2n\alpha^2\xi, \\
S(\xi, \xi) &= -2n\alpha^2,
\end{align*}
\]

for any vector fields \( X, Y, Z \), where \( S \) is the Ricci curvature and \( Q \) is Ricci operator given by \( S(X, Y) = g(QX, Y) \).

**Definition 1.** A Lorentzian \( \alpha \)-Sasakian manifold \( M \) is said to be \( \eta \)-Einstein if its Ricci tensor \( S \) is of the form

\[
S(X, Y) = \lambda_1 g(X, Y) + \lambda_2 \eta(X)\eta(Y),
\]

for any vector fields \( X, Y \), where \( \lambda_1, \lambda_2 \) are functions on \( M \). If \( \lambda_1 = 0 \) then \( M \) is a special \( \eta \)-Einstein manifold.

**3. Lorentzian \( \alpha \)-Sasakian Manifold Satisfying \( \mathcal{T}(\xi, X) \cdot R = 0 \)**

**Theorem 2.** A \((2n + 1)\) dimensional Lorentzian \( \alpha \)-Sasakian manifold \( M \) satisfying the condition \( \mathcal{T}(\xi, X) \cdot R = 0 \) is an \( \eta \)-Einstein manifold, provided \( a_1 \neq 0 \).
Proof. Suppose in a Lorentzian $\alpha$-Sasakian manifold

$$\mathcal{T}(\xi, X) \cdot R = 0. \quad (15)$$

Which implies,

$$\mathcal{T}(\xi, X)R(Y, Z)U - R(\mathcal{T}(\xi, X)Y, Z)U - R(Y, \mathcal{T}(\xi, X)Z)U - R(Y, Z)\mathcal{T}(\xi, X)U = 0. \quad (16)$$

By taking inner product with $\xi$ of the above equation, we have.

$$g(\mathcal{T}(\xi, X)R(Y, Z)U, \xi) - g(R(\mathcal{T}(\xi, X)Y, Z)U, \xi) - g(R(Y, \mathcal{T}(\xi, X)Z)U, \xi) - g(R(Y, Z)\mathcal{T}(\xi, X)U, \xi) = 0. \quad (17)$$

Putting $Y = U = \xi$ in (17) and using (1),(5),(8),(9),(10),(11), we obtain

$$\alpha^2 \{ -a_1 S(X, Z) - (a_0 \alpha^2 + a_1 2n \alpha^2 + a_4 2n \alpha^2 + a_7 r)\eta(X)\eta(Z) - (a_0 \alpha^2 + a_4 2n \alpha^2 + a_7 r)g(X, Z) - (a_0 \alpha^2 + a_2 2n \alpha^2)g(X, Z) - (a_0 \alpha^2 + a_2 2n \alpha^2)\eta(X)\eta(Z) \} = 0. \quad (18)$$

After simplifying (18), we get

$$-a_1 S(X, Z) = (\alpha^2 (2a_0 + 2na_1 + 2na_4) + a_7 r)g(X, Z) + (\alpha^2 (2a_0 + 2na_1 + 2na_2 + 2na_4) + a_7 r)\eta(X)\eta(Z), \quad (19)$$

If $a_1 \neq 0$, then

$$S(X, Z) = \lambda_1 g(X, Z) + \lambda_2 \eta(X)\eta(Z), \quad (20)$$

where,

$$\lambda_1 = \frac{-(2a_0 \alpha^2 + a_4 2n \alpha^2 + a_7 r)}{a_1} \quad (21)$$

and

$$\lambda_2 = \frac{-(2a_0 \alpha^2 + a_1 2n \alpha^2 + a_2 2n \alpha^2 + a_4 2n \alpha^2 + a_7 r)}{a_1}.$$ 

Hence $M$ is a $\eta$-Einstein manifold.
Further, from [10] and equation (20), we have also derived the result of the above theorem for particular cases of $\mathcal{T}$-curvature tensor as follows:
### Curvature Values of $\lambda_1$ and $\lambda_2$

<table>
<thead>
<tr>
<th>Curvature</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conformal curvature tensor $C$</td>
<td>$\lambda_1 = 2(n-1)\alpha^2 + \frac{r}{2n}$</td>
<td>$\lambda_2 = 2(n-1)\alpha^2 + \frac{r}{2n}$</td>
</tr>
<tr>
<td>Conharmonic curvature tensor $L$</td>
<td>$\lambda_1 = 2\alpha^2$</td>
<td>$\lambda_2 = 2(n-1)\alpha^2$</td>
</tr>
<tr>
<td>Projective curvature tensor $\mathcal{P}$</td>
<td>$\lambda_1 = 3\alpha^2$</td>
<td>$\lambda_2 = 4n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{M}$-Projective curvature tensor</td>
<td>$\lambda_1 = 8n\alpha^2$</td>
<td>$\lambda_2 = 6n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_0$-curvature tensor</td>
<td>$\lambda_1 = 4n\alpha^2$</td>
<td>$\lambda_2 = 2n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_0^*$-curvature tensor</td>
<td>$\lambda_1 = -4n\alpha^2$</td>
<td>$\lambda_2 = -6n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_1$-curvature tensor</td>
<td>$\lambda_1 = -2n\alpha^2$</td>
<td>$\lambda_2 = -2n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_1^*$-curvature tensor</td>
<td>$\lambda_1 = 4n\alpha^2$</td>
<td>$\lambda_2 = 4n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_6$-curvature tensor</td>
<td>$\lambda_1 = 4n\alpha^2$</td>
<td>$\lambda_2 = 4n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_7$-curvature tensor</td>
<td>$\lambda_1 = 6n\alpha^2$</td>
<td>$\lambda_2 = 4n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_8$-curvature tensor</td>
<td>$\lambda_1 = 2\alpha^2$</td>
<td>$\lambda_2 = 6n\alpha^2$</td>
</tr>
</tbody>
</table>

For Quasi conformal curvature tensor $\mathcal{C}^*$:

$$
\begin{align*}
\lambda_1 &= 2A\alpha^2 - \frac{r}{2n+1}\left(\frac{A}{2n} + 2B\right) \\
\lambda_2 &= -\frac{1}{B}\left[2A\alpha^2 + 2nB\alpha^2 - \frac{r}{2n+1}\left(\frac{A}{2n} + 2B\right)\right].
\end{align*}
$$

For Pseudo-projective curvature tensor $\mathcal{P}^*$:

$$
\begin{align*}
\lambda_1 &= 2A\alpha^2 + 2nB\alpha^2 - \frac{r}{2n+1}\left(\frac{A}{2n} + B\right) \\
\lambda_2 &= 2A\alpha^2 - \frac{r}{2n+1}\left(\frac{A}{2n} + B\right).
\end{align*}
$$

Here $A$ and $B$ are some smooth functions on $M$.

In all the above cases $M$ is a $\eta$-Einstein manifold. And since $a_1, a_7 = 0$ in $\mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_5, \mathcal{W}_9$-curvature tensors, the above result is not applicable for these curvature tensors.
4. Lorentzian $\alpha$-Sasakian Manifold Satisfying $\mathcal{T}(\xi, X) \cdot S = 0$

**Theorem 3.** A $(2n + 1)$ dimensional Lorentzian $\alpha$-Sasakian manifold $M$ satisfying the condition $\mathcal{T}(\xi, Y) \cdot S = 0$ with

$$a_0\alpha^2 + a_12n\alpha^2 + a_22n\alpha^2 + a_52n\alpha^2 - a_7r \neq 0,$$

is an $\eta$-Einstein manifold.

**Proof.** Consider a Lorentzian $\alpha$-Sasakian manifold satisfying

$$\mathcal{T}(\xi, X) \cdot S = 0.$$  

(22)

Which implies,

$$S(\mathcal{T}(\xi, X)Y, Z) + S(Y, \mathcal{T}(\xi, X)Z) = 0.$$  

(23)

Taking inner product of (23) with $\xi$, we get

$$g(S(\mathcal{T}(\xi, X)Y, Z), \xi) + g(S(Y, \mathcal{T}(\xi, X)Z), \xi) = 0.$$  

(24)

And taking $Z = \xi$ in (24) and using (1),(5),(8),(9),(10),(11), we obtain

$$-a_12n\alpha^2S(X, Y) - 2n\alpha^2(a_0\alpha^2 + a_42n\alpha^2 + a_7r)g(X, Y)$$

$$+ 2n\alpha^2(a_0\alpha^2 + a_22n\alpha^2 + a_52n\alpha^2 - a_7r)\eta(X)\eta(Y)$$

$$+ 4n^2\alpha^4(a_3 + a_6 + a_1)\eta(X)\eta(Y)$$

$$+ 2n\alpha^2(a_0\alpha^2 + a_42n\alpha^2 + a_7r)\eta(X)\eta(Y)$$

$$- (a_0\alpha^2 + a_22n\alpha^2 + a_52n\alpha^2 - a_7r)S(X, Y)$$

$$+ 2n\alpha^2(a_32n\alpha^2 + a_62n\alpha^2)\eta(X)\eta(Y) = 0.$$  

(25)

Simplifying (25), we have

$$S(X, Y) = \lambda_1g(X, Y) + \lambda_2\eta(X)\eta(Y),$$  

(26)

where,

$$\lambda_1 = \frac{-2n\alpha^2(a_0\alpha^2 + a_42n\alpha^2 + a_7r)}{a_0\alpha^2 + a_12n\alpha^2 + a_22n\alpha^2 + a_52n\alpha^2 - a_7r}$$

and

$$\lambda_2 = \frac{2n\alpha^2(2a_0\alpha^2 + 2n\alpha^2[a_1 + a_2 + 2a_3 + a_4 + a_5 + 2a_6])}{a_0\alpha^2 + a_12n\alpha^2 + a_22n\alpha^2 + a_52n\alpha^2 - a_7r}.$$  

The particular cases of Theorem 3 for different curvature tensors is as follows:
$$\lambda_1 = 2n\alpha^2 \frac{2n(2n-1)\alpha^2 + 4n^2\alpha^2 - r}{2n(2n-1)\alpha^2 + 4n\alpha^2 - r}$$
$$\lambda_2 = \frac{8n^2\alpha^4(2n-1)}{2n(2n-1)\alpha^2 + 4n^2\alpha^2 - r}$$

### Curvature Values of $\lambda_1$ and $\lambda_2$

<table>
<thead>
<tr>
<th>Curvature</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conformal curvature tensor $C$</td>
<td>$\lambda_1 = 2n\alpha^2 \frac{2n(2n-1)\alpha^2 + 4n^2\alpha^2 - r}{2n(2n-1)\alpha^2 + 4n\alpha^2 - r}$</td>
<td>$\lambda_2 = \frac{8n^2\alpha^4(2n-1)}{2n(2n-1)\alpha^2 + 4n^2\alpha^2 - r}$</td>
</tr>
<tr>
<td>Conharmonic curvature tensor $L$</td>
<td>$\lambda_1 = \frac{2n\alpha^2}{4n-1}$</td>
<td>$\lambda_2 = \frac{4n(2n-1)\alpha^2}{4n-1}$</td>
</tr>
<tr>
<td>Concircular curvature tensor $\mathcal{V}$</td>
<td>$\lambda_1 = \frac{-4n^2(2n+1)\alpha^2 + 2nr}{8n^2(2n+1)\alpha^2}$</td>
<td>$\lambda_2 = \frac{r}{8n^2(2n+1)\alpha^2}$</td>
</tr>
<tr>
<td>Projective curvature tensor $P$</td>
<td>$\lambda_1 = -2n\alpha^2$</td>
<td>$\lambda_2 = 4n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{M}$-Projective curvature tensor $\mathcal{P}$</td>
<td>$\lambda_1 = -2n\alpha^2$</td>
<td>$\lambda_2 = 8n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_0$-curvature tensor</td>
<td>$\lambda_1 = -2n\alpha^2$</td>
<td>$\lambda_2 = 4n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_0^*$-curvature tensor</td>
<td>$\lambda_1 = -2n\alpha^2$</td>
<td>$\lambda_2 = 4n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_1$-curvature tensor</td>
<td>$\lambda_1 = -2n\alpha^2$</td>
<td>$\lambda_2 = 4n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_1^*$-curvature tensor</td>
<td>$\lambda_1 = -2n\alpha^2$</td>
<td>$\lambda_2 = 4n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_2$-curvature tensor</td>
<td>$\lambda_1 = 0$</td>
<td>$\lambda_2 = 2n\alpha^2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Curvature</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{W}_4$-curvature tensor</td>
<td>$\lambda_1 = -n\alpha^2$</td>
<td>$\lambda_2 = n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_5$-curvature tensor</td>
<td>$\lambda_1 = -2n\alpha^2$</td>
<td>$\lambda_2 = 4n\alpha^2$</td>
</tr>
<tr>
<td>$\mathcal{W}_9$-curvature tensor</td>
<td>$\lambda_1 = 0$</td>
<td>$\lambda_2 = 6n\alpha^2$</td>
</tr>
</tbody>
</table>

For Quasi conformal curvature tensor $C_*$:

$$\lambda_1 = -2n\alpha^2 \left[ \frac{2n(2n+1)A\alpha^2 + 4n^2(2n+1)B\alpha^2 - r(A + 4nB)}{2n(2n+1)A\alpha^2 - 4n^2(2n+1)B\alpha^2 - r(A + 4nB)} \right]$$

$$\lambda_2 = 2n\alpha^2 \left[ \frac{4n(2n+1)A\alpha^2 - r(A + 4nB)}{2n(2n+1)A\alpha^2 - 2n(2n+1)B\alpha^2 - r(A + 4nB)} \right].$$
For Pseudo-projective curvature tensor $\mathcal{P}_*$:

$$
\lambda_1 = -2n\alpha^2 \left[ \frac{2n(2n+1)A\alpha^2 - r(A + 2nB)}{2n(2n+1)A\alpha^2 + r(A + 2nB)} \right]
$$

$$
\lambda_2 = 2n\alpha^2 \left[ \frac{4n(2n+1)A\alpha^2 - r(A + 2nB)}{2n(2n+1)A\alpha^2 + r(A + 2nB)} \right].
$$

Here $A$ and $B$ are some smooth functions on $M$.

From the above table one can observe that $M$ is special $\eta$-Einstein manifold for $W_2$ and $W_3$-curvature tensors and $\eta$-Einstein manifold for other curvature tensors. Since $a_0\alpha^2 + a_12n\alpha^2 + a_22n\alpha^2 + a_52n\alpha^2 - a_7r = 0$, in $W_3$, $W_6$, $W_7$, $W_8$-curvature tensors, the above result is not applicable for these curvature tensors.

\[\square\]

5. Lorentzian $\alpha$-Sasakian Manifold Satisfying $\mathcal{T}(X, Y)Z = 0$

**Theorem 4.** A $(2n + 1)$ dimensional Lorentzian $\alpha$-Sasakian manifold $M$ satisfying the condition $\mathcal{T}(X, Y)Z = 0$ with $a_3 \neq 0$ is an $\eta$-Einstein manifold.

**Proof.** Suppose Lorentzian $\alpha$-Sasakian manifold is $\mathcal{T}$-flat, i.e.,

$$
\mathcal{T}(X, Y)Z = 0. \tag{27}
$$

Then we have

$$
0 = a_0g(R(X, Y)Z, W) + a_1S(Y, Z)g(X, W)
+ a_2S(X, Z)g(Y, W) + a_3S(X, Y)g(Z, W)
+ a_4g(Y, Z)g(QX, W) + a_5g(X, Z)g(QY, W)
+ a_6g(X, Y)g(QZ, W) + a_7rg(Y, Z)g(X, W)
- a_7rg(X, Z)g(Y, W). \tag{28}
$$

Taking $Z = W = \xi$ in (28), we get

$$
S(X, Y) = \lambda_1g(X, Y) + \lambda_2\eta(X)\eta(Y), \tag{29}
$$

where,

$$
\lambda_1 = -\frac{a_62n\alpha^2}{a_3},
$$

and

$$
\lambda_2 = 2n\alpha^2.
$$
Further, from [10] and equation (29), we have also derived the result of the above theorem for particular cases of \( T \)-curvature tensor as follows:

Since \( a_3 = 0 \) in conformal, conharmonic, concircular, pseudo-projective, projective, \( M \)-projective, \( \mathcal{W}_0, \mathcal{W}_0^*, \mathcal{W}_1, \mathcal{W}_1^*, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_5, \mathcal{W}_6, \) and \( \mathcal{W}_7 \)-curvature tensors, the above result is not applicable for these curvature tensors.

So, curvatures which are applicable and corresponding results with the values of \( \lambda_1 \) and \( \lambda_2 \) are given below:

<table>
<thead>
<tr>
<th>Curvature</th>
<th>Values of ( \lambda_1 ) and ( \lambda_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{W}_8 )-curvature tensor</td>
<td>( \lambda_1 = 0, \lambda_2 = -2n\alpha^2 )</td>
</tr>
<tr>
<td>( \mathcal{W}_9 )-curvature tensor</td>
<td>( \lambda_1 = 0, \lambda_2 = -2n\alpha^2 )</td>
</tr>
</tbody>
</table>

From the above table we can observe that \( M \) is Special \( \eta \)-Einstein manifold in both curvature tensors. \( \square \)

### 6. Lorentzian \( \alpha \)-Sasakian Manifold Satisfying \( T(X, Y)\xi = 0 \)

**Theorem 5.** A \((2n + 1)\) dimensional Lorentzian \( \alpha \)-Sasakian manifold \( M \) satisfying the condition \( T(X, Y)\xi = 0 \) with \( a_3 \neq -2n\alpha^2a_0 \) is a special \( \eta \)-Einstein manifold with scalar curvature given by

\[
r = -\frac{2n\alpha^2(a_1 + a_4 + a_5)}{a_02n\alpha^2 + a_3}.
\]

**Proof.** Suppose Lorentzian \( \alpha \)-Sasakian manifold is \( \xi \)-\( T \)-flat, i.e.,

\[
T(X, Y)\xi = 0.
\]

Then we have

\[
(a_3 + a_62n\alpha^2)S(X, Y)\xi + (a_0\alpha^2 + a_12n\alpha^2 + a_42n\alpha^2 + a_7r)\eta(Y)X
+ (-a_0\alpha^2 + a_52n\alpha^2 - a_7r)\eta(X)Y = 0.
\]

By taking inner product of (32) with \( \xi \), we have

\[
S(X, Y) = \lambda_1 g(X, Y) + \lambda_2 \eta(X)\eta(Y),
\]
where,

\[ \lambda_1 = 0 \quad \text{and} \quad \lambda_2 = \frac{2na^2(a_1 + a_4 + a_5)}{a_62na^2 + a_3}. \]

On contracting above equation one can get the scalar curvature \( r \) as

\[ r = -\frac{2na^2(a_1 + a_4 + a_5)}{a_62na^2 + a_3}. \]

Further, from [10] the result of the Theorem 5 for different curvature tensors is as follows:

<table>
<thead>
<tr>
<th>Curvature</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conformal curvature tensor ( C )</td>
<td>( \frac{1}{2n-1} )</td>
</tr>
<tr>
<td>Conharmonic curvature tensor ( \mathcal{C} )</td>
<td>( \frac{1}{2n-1} )</td>
</tr>
<tr>
<td>Concircular curvature tensor ( \mathcal{V} )</td>
<td>0</td>
</tr>
<tr>
<td>Projective curvature tensor ( \mathcal{P} )</td>
<td>( \frac{1}{2n} )</td>
</tr>
<tr>
<td>( \mathcal{M} )-Projective curvature tensor</td>
<td>0</td>
</tr>
<tr>
<td>( \mathcal{W}_0 )-curvature tensor</td>
<td>0</td>
</tr>
<tr>
<td>( \mathcal{W}_0^* )-curvature tensor</td>
<td>0</td>
</tr>
<tr>
<td>( \mathcal{W}_1 )-curvature tensor</td>
<td>( \frac{1}{2n} )</td>
</tr>
<tr>
<td>( \mathcal{W}_1^* )-curvature tensor</td>
<td>( \frac{1}{2n} )</td>
</tr>
<tr>
<td>( \mathcal{W}_3 )-curvature tensor</td>
<td>( \frac{1}{2n} )</td>
</tr>
<tr>
<td>( \mathcal{W}_4 )-curvature tensor</td>
<td>( \frac{1}{2n} )</td>
</tr>
<tr>
<td>( \mathcal{W}_5 )-curvature tensor</td>
<td>( \frac{1}{2n} )</td>
</tr>
<tr>
<td>( \mathcal{W}_6 )-curvature tensor</td>
<td>( \frac{1}{2n} )</td>
</tr>
<tr>
<td>( \mathcal{W}_7 )-curvature tensor</td>
<td>0</td>
</tr>
<tr>
<td>( \mathcal{W}_8 )-curvature tensor</td>
<td>( \frac{2na^2}{1+4na^2} )</td>
</tr>
<tr>
<td>( \mathcal{W}_9 )-curvature tensor</td>
<td>( \frac{2na^2}{1+4na^2} )</td>
</tr>
<tr>
<td>( \mathcal{W}_2 )-curvature tensor</td>
<td>0</td>
</tr>
<tr>
<td>Quasi conformal curvature tensor ( \mathcal{C}_* )</td>
<td>( \frac{-B}{A} )</td>
</tr>
<tr>
<td>Pseudo-projective curvature tensor ( \mathcal{P}_* )</td>
<td>( \frac{-B}{A} )</td>
</tr>
</tbody>
</table>

Here \( A \) and \( B \) are some smooth functions on \( M \).

References


