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ON THE BOUNDS FOR THE NORMS OF R-CIRCULANT MATRICES WITH THE JACOBSTHAL AND JACOBSTHAL LUCAS NUMBERS

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Abstract: In this study, we have found upper and lower bounds for the spectral norms of circulant matrices in the forms $A = C_r(j_0, j_1, ..., j_{n-1})$ and $B = C_r(c_0, c_1, ..., c_{n-1})$. After that we obtain some bounds related to the spectral norms of Hadamard and Kronecker product of these matrices.

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Key Words: Jacobsthal number, Jacobsthal Lucas number, circulant matrix, norm

1. Introduction and Preliminaries

The Jacobsthal $\{j_n\}_{n\in\mathbb{N}}$, and the Jacobsthal Lucas $\{c_n\}_{n\in\mathbb{N}}$ sequences are defined recurrently by

$$j_n = j_{n-1} + 2j_{n-2}, \quad j_0 = 0, \quad j_1 = 1, \quad n \geq 2, \tag{1}$$

$$c_n = c_{n-1} + 2c_{n-2}, \quad c_0 = 2, \ c_1 = 1, \quad n \geq 2, \tag{2}$$

respectively. There have been several papers on the norms of special matrices [7-10]. Solak [8] has defined $A = [a_{ij}]$ and $B = [b_{ij}]$ as nxn circulant matrices, where $a_{ij} = F_{(\mathrm{mod}(j-i,n))}$ and $b_{ij} = L_{(\mathrm{mod}(j-i,n))}$, then he has given some bounds for the A and B matrices concerned with the spectral and Eu-

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clidean norms. In (9) Shen, Cen found the bounds for the norms of r-circulant matrices with the Fibonocci and Lucas numbers. Shen and Cen [10] have given upper and lower bounds for the spectral norms of r- circulant matrices $A = C_r(F_0^{(k,-1)}, F_1^{(k,-1)}, ..., F_{n-1}^{(k,-1)})$ and $B = C_r(L_0^{(k,-1)}, L_1^{(k,-1)}, ..., L_{n-1}^{(k,-1)})$. In addition, they also have obtained some bounds for the spectral norms of Hadamard and Kronecker products of these matrices. In (13) the authors gave the relations among k Fibonacci, $k-$Lucas and generalized $k-$ Fibonacci numbers and the spectral norms of the matrices of involving these numbers,

In this paper we give lower and upper bounds for the spectral norms of the $r-$circulant matrices $A = C_r(j_0, j_1, ..., j_{n-1})$ and $B = C_r(c_0, c_1, ..., c_{n-1})$. After that we obtain some bounds related to the spectral norms of Hadamard and Kronecker product of these matrices.

Recurrences (1) and (2) involve the characteristic equation

$$x^2 - x - 2 = 0,$$

with roots

$$\alpha = 2, \quad \beta = -1.$$

Their Binet's formulas are defined by

$$j_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad c_n = \alpha^n + \beta^n. \tag{3}$$

A matrix $C = [c_{ij}] \in M_{m,n}(C)$ is called a circulant matrix if it is of the form

$$c_{ij} = \begin{cases} c_{j-i}, & j \geq i \\ rc_{n+j-i}, & j < i \end{cases} \tag{4}$$

For any $A = [a_{ij}] \in M_{m,n}(C)$. The Frobenious(or Euclidean) norm of matrix A is

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} \tag{5}$$

and the spectral norm of matrix A is

$$\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(A^H A)}, \tag{6}$$

where $\lambda_i(A^H A)$ is eigenvalue of $A^H A$.

Lemma 1. *For any $A, B \in M_{m,n}(C)$, the Hadamard product of A, B is entrywise product and defined by (see [5,6])*

$$AoB = (a_{ij}b_{ij})$$

and have the following properties

$$\|AoB\|_2 \le \|A\|_2 \|B\|_2, \tag{7}$$

$$\|AoB\|_2 \le r_1(A) c_1(B). \tag{8}$$

Lemma 2. *Let $A \in M_{m,n}(C)$, $B \in M_{p,q}(C)$ be given, then the Kronecker product of A, B is defined by*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

and have the following property [11]

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2. \tag{9}$$

Lemma 3. *Let $A \in M_{m,n}(C)$ be given, then the inequality is hold (see [4])*

$$\frac{1}{\sqrt{n}} \|A\|_F \le \|A\|_2 \le \|A\|_F. \tag{10}$$

Lemma 4. *The sum of squares of the first n elements of Jacobsthal and Jacobsthal Lucas sequences are given as*

$$\sum_{k=0}^{n-1} j_k^2 = \frac{j_{2n} + 2(-1)^n j_n + n}{9}, \tag{11}$$

$$\sum_{k=0}^{n-1} c_k^2 = j_{2n} - 2(-1)^n j_n + n. \tag{12}$$

2. Lower and Upper Bounds of r-Circulant Matrices Involving Jacobsthal and Jacobsthal-Lucas Numbers

Theorem 5. *Let $A = C_r(j_0, j_1, ..., j_{n-1})$ be r-circulant matrix, then we obtain where $r \in \mathbb{C}$.*

(i) If $|r| \geqslant 1$, then

$$\sqrt{\frac{j_{2n} + 2(-1)^n j_n + n}{9}} \leq \|A\|_2 \leq |r| \frac{j_{2n} + 2(-1)^n j_n + n}{9}. \quad (13)$$

(ii) If $|r| \leq 1$, then

$$|r| \sqrt{\frac{j_{2n} + 2(-1)^n j_n + n}{9}} \leq \|A\|_2 \leq \sqrt{\frac{(n-1)(j_{2n} + 2(-1)^n j_n + n)}{9}}. \quad (14)$$

Proof. The matrix A is of the form

$$A = \begin{bmatrix} j_0 & j_1 & j_2 & \cdots & j_{n-1} \\ rj_{n-1} & j_0 & j_1 & \cdots & j_{n-2} \\ rj_{n-2} & rj_{n-1} & j_0 & \cdots & j_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rj_1 & rj_2 & rj_3 & \cdots & j_0 \end{bmatrix}.$$

For $|r| \geqslant 1$, by using (4), (11) we have

$$\begin{aligned} \|A\|_F^2 &= \sum_{k=0}^{n-1} (n-k) j_k^2 + \sum_{k=1}^{n-1} k |r|^2 j_k^2 \\ &\geqslant \sum_{k=0}^{n-1} (n-k) j_k^2 + \sum_{k=1}^{n-1} k j_k^2 = n \sum_{k=0}^{n-1} j_k^2 \\ &= \frac{n}{9} [j_{2n} + 2(-1)^n j_n + n] \end{aligned}$$

From (10),

$$\frac{1}{3} \sqrt{j_{2n} + 2(-1)^n j_n + n} \leq \frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2$$

On the other hand, let $A = BoC$ where as

$$B = \begin{bmatrix} rj_0 & 1 & 1 & \cdots & 1 \\ rj_{n-1} & rj_0 & 1 & \cdots & 1 \\ rj_{n-2} & rj_{n-1} & rj_0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rj_1 & rj_2 & rj_3 & \cdots & rj_0 \end{bmatrix}, \quad C = \begin{bmatrix} j_0 & j_1 & j_2 & \cdots & j_{n-1} \\ 1 & j_0 & j_1 & \cdots & j_{n-2} \\ 1 & 1 & j_0 & \cdots & j_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & j_0 \end{bmatrix}$$

then

$$\begin{aligned} r_1(B) &= \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}|b_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|b_{nj}|^2} \\ &= \sqrt{|r|^2\sum_{k=0}^{n-1} j_k^2} = \tfrac{|r|}{3}\sqrt{j_{2n}+2(-1)^n j_n+n}, \\ c_1(C) &= \max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}|c_{ij}|^2} = \sqrt{\sum_{i=1}^{n}|c_{in}|^2} \\ &= \sqrt{\sum_{k=0}^{n-1} j_k^2} = \tfrac{1}{3}\sqrt{j_{2n}+2(-1)^n j_n+n}. \end{aligned}$$

By using (8) we obtain

$$\|A\|_2 \le r_1(B)\,c_1(C) = \frac{|r|}{9}\left[j_{2n}+2(-1)^n j_n+n\right].$$

The proof is completed for the first part.

(*ii*) For $|r|\le 1$ by using (11) we have,

$$\begin{aligned} \|A\|_F^2 &= \sum_{k=0}^{n-1}(n-k)\,j_k^2 + \sum_{k=1}^{n-1}k\,|r|^2 j_k^2 \\ &\geqslant \sum_{k=0}^{n-1}(n-k)\,|r|^2 j_k^2 + \sum_{k=1}^{n-1}k\,|r|^2 j_k^2 = n\,|r|^2\sum_{k=0}^{n-1} j_k^2 \\ &= \frac{n\,|r|^2}{9}\left[j_{2n}+2(-1)^n j_n+n\right]. \end{aligned}$$

From (10),

$$\frac{|r|}{3}\sqrt{j_{2n}+2(-1)^n j_n+n} \le \frac{1}{\sqrt{n}}\|A\|_F \le \|A\|_2.$$

On the other hand, let $A = BoC$, where

$$B = \begin{bmatrix} j_0 & 1 & 1 & \cdots & 1 \\ r & j_0 & 1 & \cdots & 1 \\ r & r & j_0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \cdots & j_0 \end{bmatrix}, \quad C = \begin{bmatrix} j_0 & j_1 & j_2 & \cdots & j_{n-1} \\ j_{n-1} & j_0 & j_1 & \cdots & j_{n-2} \\ j_{n-2} & j_{n-1} & j_0 & \cdots & j_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ j_1 & j_2 & j_3 & \cdots & j_0 \end{bmatrix},$$

$$r_1(B) = \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}|b_{ij}|^2} = \sqrt{j_0^2+(n-1)} = \sqrt{(n-1)},$$

$$c_1(C) = \max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}|c_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1}j_k^2} = \frac{1}{3}\sqrt{j_{2n}+2(-1)^n j_n+n}.$$

By using (8) we obtain

$$\|A\|_2 \le r_1(B)\,c_1(C) = \sqrt{\frac{(n-1)(j_{2n}+2(-1)^n j_n+n)}{9}}.$$

Therefore the proof is completed .

$$\frac{|r|}{3}\sqrt{j_{2n}+2(-1)^n j_n+n} \le \|A\|_2 \le \frac{1}{3}\sqrt{(n-1)(j_{2n}+2(-1)^n j_n+n)}. \quad \square$$

Theorem 6. *Let $A = C_r(c_0, c_1, ..., c_{n-1})$ be r- circulant matrix, then we obtain:*

(i) If $|r| \ge 1$, then

$$\sqrt{j_{2n}-2(-1)^n j_n+n} \le \|A\|_2$$
$$\le \sqrt{\left(4+|r|^2(j_{2n}-2(-1)^n j_n+n-4)\right)(j_{2n}-2(-1)^n j_n+n-3)}. \quad (15)$$

(ii) If $|r| \le 1$, then

$$|r|\sqrt{j_{2n}-2(-1)^n j_n+n} \le \|A\|_2 \le \sqrt{n(j_{2n}-2(-1)^n j_n+n)}. \quad (16)$$

Proof. The matrix A is of the form

$$A = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & c_0 \end{bmatrix}.$$

For $|r| \geqslant 1$, by using (4), (12) we have

$$\|A\|_F^2 = \sum_{k=0}^{n-1}(n-k)\,c_k^2 + \sum_{k=1}^{n-1}k\,|r|^2\,c_k^2$$

$$\geq \sum_{k=0}^{n-1}(n-k)\,c_k^2+\sum_{k=1}^{n-1}kc_k^2=n\sum_{k=0}^{n-1}c_k^2=n\left(j_{2n}-2\left(-1\right)^n j_n+n\right)$$

From (10),

$$\|A\|_2\geq\frac{1}{\sqrt{n}}\|A\|_F\geq\sqrt{j_{2n}-2\left(-1\right)^n j_n+n}$$

On the other hand, let $A=BoC$ where

$$B=\begin{bmatrix} c_0 & 1 & 1 & \cdots & 1\\ rc_{n-1} & c_0 & 1 & \cdots & 1\\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ rc_1 & rc_2 & rc_3 & \cdots & c_0\end{bmatrix},\quad C=\begin{bmatrix} 1 & c_1 & c_2 & \cdots & c_{n-1}\\ 1 & 1 & c_1 & \cdots & c_{n-2}\\ 1 & 1 & 1 & \cdots & c_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \cdots & 1\end{bmatrix}$$

$$\begin{aligned} r_1\left(B\right) &=\max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}\left|b_{ij}\right|^2}=\sqrt{\sum_{j=1}^{n-1}\left|b_{nj}\right|^2}\\ &=\sqrt{c_0^2+\left|r\right|^2\sum_{k=1}^{n-1}c_k^2}=\sqrt{4+\left|r\right|^2\sum_{k=1}^{n-1}c_k^2}\\ &=\sqrt{4+\left|r\right|^2\left(j_{2n}-2\left(-1\right)^n j_n+n-4\right)}\\ c_1\left(C\right) &=\max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}\left|c_{ij}\right|^2}=\sqrt{\sum_{i=1}^{n}\left|c_{in}\right|^2}=\sqrt{1+\sum_{k=1}^{n-1}c_k^2}\\ &=\sqrt{j_{2n}-2\left(-1\right)^n j_n+n-3}\end{aligned}$$

By using (8) we obtain

$$\begin{aligned}\|A\|_2 &=\|BoC\|_2\\ &\leq\sqrt{\left(4+\left|r\right|^2\left(j_{2n}-2\left(-1\right)^n j_n+n-4\right)\right)\left(j_{2n}-2\left(-1\right)^n j_n+n-3\right)}.\end{aligned}$$

(ii) For $|r|\leq 1$,

$$\|A\|_F^2\geq\sum_{k=0}^{n-1}(n-k)\left|r\right|^2c_k^2+\sum_{k=1}^{n-1}k\left|r\right|^2c_k^2=n\left|r\right|^2\sum_{k=0}^{n-1}c_k^2$$

From (10),

$$\begin{aligned}\|A\|_2 &\geq \frac{1}{\sqrt{n}}\|A\|_F \geq \sqrt{|r|^2\sum_{k=0}^{n-1}c_k^2}\\ &= |r|\sqrt{(j_{2n}-2(-1)^n j_n+n)}\end{aligned}$$

On the other hand, let $A = BoC$ where

$$B=\begin{bmatrix} r & 1 & 1 & \cdots & 1\\ r & r & 1 & \cdots & 1\\ r & r & r & \cdots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ r & r & r & \cdots & r\end{bmatrix},\quad C=\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1}\\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2}\\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ c_1 & c_2 & c_3 & \cdots & c_0\end{bmatrix},$$

$$\begin{aligned}r_1(B) &= \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|b_{ij}|^2}=\sqrt{n},\\ c_1(C) &= \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|c_{ij}|^2}=\sqrt{\sum_{k=0}^{n-1}c_k^2}=\sqrt{j_{2n}-2(-1)^n j_n+n}.\end{aligned}$$

By using (8) we obtain

$$\|A\|_2=\|BoC\|_2\leq r_1(B)\,c_1(C)\leq\sqrt{n\,(j_{2n}-2(-1)^n j_n+n)},$$

$$|r|\sqrt{j_{2n}-2(-1)^n j_n+n}\leq\|A\|_2\leq\sqrt{n\,(j_{2n}-2(-1)^n j_n+n)}.$$

So the proof is completed. □

Corollary 7. *Let $A = C_r(j_0, j_1, ..., j_{n-1})$ and $A = C_r(c_0, c_1, ..., c_{n-1})$ be $r-$ circulant matrices, where $r \in \mathbb{C}$.*

(i) If $|r| \geq 1$, then

$$\begin{aligned}\|AoB\|_2 \leq |r|\,&\frac{j_{2n}+2(-1)^n j_n+n}{9}\\ &\times\sqrt{\left(4+|r|^2\left[j_{2n}-2(-1)^n j_n+n-4\right]\right)(j_{2n}-2(-1)^n j_n+n-3)}.\end{aligned}$$

(ii) If $|r| \leq 1$, *then*

$$\|AoB\|_2 \leq \sqrt{\left(\frac{(n-1)\left(j_{2n}+2\left(-1\right)^n j_n+n\right)}{9}\right)\left(n\left(j_{2n}-2\left(-1\right)^n j_n+n\right)\right)}.$$

Proof. Since $\|AoB\|_2 \leq \|A\|_2 \|B\|_2$,the proof is trivial by (7). □

Corollary 8. *Let* $A = C_r(j_0, j_1, ..., j_{n-1})$ *and* $A = C_r(c_0, c_1, ..., c_{n-1})$ *be* $r-$ *circulant matrices, where* $r \in \mathbb{C}$.

(i) If $|r| \geq 1$, *then*

$$\|A \otimes B\|_2 \leq |r| \frac{j_{2n}+2\left(-1\right)^n j_n+n}{9} \times \sqrt{\left(4+|r|^2\left[j_{2n}-2\left(-1\right)^n j_n+n-4\right]\right)\left(j_{2n}-2\left(-1\right)^n j_n+n-3\right)},$$

$$\|A \otimes B\|_2 \geq \sqrt{\left(\frac{j_{2n}+2\left(-1\right)^n j_n+n}{9}\right)\left(j_{2n}-2\left(-1\right)^n j_n+n\right)}.$$

(ii) If $|r| \leq 1$, *then*

$$\|A \otimes B\|_2 \leq \sqrt{n\left(\frac{(n-1)\left(j_{2n}+2\left(-1\right)^n j_n+n\right)}{9}\right)\left(j_{2n}-2\left(-1\right)^n j_n+n\right)},$$

$$\|A \otimes B\|_2 \geq |r|^2 \sqrt{\left(j_{2n}-2\left(-1\right)^n j_n+n\right)\left(j_{2n}-2\left(-1\right)^n j_n+n\right)}.$$

Proof. By using (9) and theorems 5 and 6, the proof is easily seen. □

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