International Journal of Pure and Applied Mathematics
Volume 112 No. 1 2017, 103-113
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v112i1.8

ijpam.eu

EIGENVALUES AND EIGENFUNCTIONS FOR REGULAR STURM-LIOUVILLE EQUATION WITH NON-LOCAL BOUNDARY CONDITIONS

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Abstract: In this paper, we study the existence and some general properties of eigenvalues and eigenfunctions of a nonlocal boundary value problem of the Sturm-Liouville differential equation.

AMS Subject Classification: 34A55, 34B10, 34B15, 34B18, 34L10, 34L40, 34K10
Key Words: Sturm-Liouville boundary value problem, nonlocal condition, eigenvalues, eigenfunctions

1. Introduction

Boundary value problems for various differential equations with nonlocal boundary conditions, were actively investigated during the last three decades. Re-

Received: August 28, 2016
Revised: October 23, 2016
Published: January 24, 2017

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search is motivated by both their interest to pure mathematics and new applications in physics, mechanics, biochemistry, ecology (see [1]-[5]and[8],[9]).

Recently, the authors studied in (see [6]) the existence of eigenvalues and eigenfunctions of the boundary value problem of the Sturm- Liouville differential equation

$$-y'' + q(x)y = \lambda^2 y, \quad 0 \le x \le \pi, \tag{1}$$

with each one of the two non-local conditions

$$y(0) = 0,\ y(\xi) = 0, \quad \xi \in (0, \pi], \tag{2}$$

and

$$y(\eta) = 0,\ y(\pi) = 0, \quad \eta \in [0, \pi) \tag{3}$$

Consider the non-local boundary value problem of the Sturm-Liouville equation (1) with the non-local conditions

$$y'(\eta) - Hy(\eta) = 0,\ y(\pi) = 0, \quad \xi \in (0, \pi] \tag{4}$$

where the non-negative real function $q(x)$ has a second piecewise integrable derivatives on $(0, \pi)$ and λ is spectral parameter.

Here we study the existence and some general properties of the eigenvalues and eigenfunctions of the two non-local boundary value problems (1) and (4). Comparison with the local boundary value problem problem of equation (1) with the local boundary value problem

$$y'(0) - Hy(0) = 0, \quad y(\pi) = 0$$

will be given.

2. General Properties

Here we prove some results concerning the eigenvalues and eigenfunctions of the non-local problem (1)-(4).

Lemma 1. The eigenvalues of the non-local boundary value problem (1) and (4) are real.

Proof. Let $y_0(x)$ be the eigenfunction that corresponds to the eigenvalue λ_0 of the problem (1) and (4), then

$$y_0'' + q(x)y_0 = \lambda_0^2 y_0 \quad (0 \le x \le \pi), \tag{5}$$

and

$$y_0(\eta) - Hy_0(\eta) = y_0(\pi) = 0 \tag{6}$$

Multiplying both sides of (5) by $\bar{y_0}$ and then integrating form 0 to ξ with respect to x, we have

$$-\bar{y_0}\acute{y_0}|_\eta^\pi + \int_\eta^\pi |\acute{y_0}|^2 dx + \int_\eta^\pi q(x)|y_0|^2 dx = \lambda_0^2 \int_\eta^\pi |y_0|^2 dx.$$

using the boundary condition (6), we have

$$\lambda_0^2 = \frac{\int_\eta^\pi [q(x)|y_0|^2 + |\acute{y_0}|^2]dx}{\int_\eta^\pi |y_0|^2 dx}.$$

From which it follows the reality of λ_0^2 . □

Lemma 2. The eigenfunctions that corresponds to two different eigenvalues of the non-local boundary value problem (1) and (4) are orthogonal.

Proof. Let $\lambda_1 \neq \lambda_2$ be two different eigenvalues of the non-local boundary value problem (1) and (4). Let $y_1(x), y_2(x)$ be the corresponding eigenfunctions, then

$$-y_1'' + q(x)y_1 = \lambda_1^2 y_1 \quad (0 \leq x \leq \pi), \tag{7}$$

$$y_1(\eta) - Hy_1(\eta) = y_1(\pi) = 0 \tag{8}$$

and

$$-y_2'' + q(x)y_2 = \lambda_2^2 y_2 \quad (0 \leq x \leq \pi), \tag{9}$$

$$y_2(\eta) - Hy_2(\eta) = y_2(\pi) = 0 \tag{10}$$

Multiplying both sides of (7) by $\bar{y}_2$ and integrating with respect to x, we obtain

$$-\int_\eta^\pi y''{}_1\bar{y}_2 dx + \int_\eta^\pi q(x)y_1\bar{y}_2 dx = \lambda_1^2 \int_\eta^\pi y_1\bar{y}_2 dx. \tag{11}$$

By taking the complex conjugate of (9) and multiply it by y_1 and integrate the resulting expression with respect to x, we have

$$-\int_\eta^\pi y_1\bar{y}''{}_2 dx + \int_\eta^\pi q(x)y_1\bar{y}_2 dx = \lambda_1^2 \int_\eta^\pi y_1\bar{y}_2 dx. \tag{12}$$

Subtracting (11) from (12) and using the boundary conditions of (8) and (10) we obtain

$$(\lambda_1^2 - \lambda_2^2)\int_\eta^\pi y_1\bar{y}_2 dx = 0, \quad \lambda_1^2 \neq \lambda_2^2.$$

which completes the proof. □

3. The Asymptotic Formulas for the Solution

Here we study the asymptotic formulas for the solutions of problem (1) and (4).

Lemma 1.1 deals with the nature the eigenvalues. Let be $\phi(x,\lambda)$ the solution of equation (1) and (4) satisfying the initial conditions

$$\phi(\eta,\lambda) = 1, \quad \phi'(\eta,\lambda) = H \tag{13}$$

and by $\vartheta(x,\lambda)$ the solution of the same equation, satisfying the initial conditions

$$\vartheta(\eta,\lambda) = 0, \quad \vartheta'(\eta,\lambda) = 1. \tag{14}$$

We notes that $\phi(x,\lambda)$ and $\vartheta(x,\lambda)$ are linearly independent if and only if $\omega(\lambda) \neq 0$.

$$\omega(\lambda) = \phi(x,\lambda)\vartheta'(x,\lambda) - \phi'(x,\lambda)\vartheta(x,\lambda).$$

The solution

$$Y(x,\lambda) = \alpha\phi(x,\lambda) + \beta\vartheta(x,\lambda), \text{ at least } \alpha \text{ or } \beta \neq 0.$$

$Y(x,\lambda)$ as eigenfunction must satisfy the first condition (4), we have

$$y'(\eta,\lambda) - Hy(\eta,\lambda) = 0,$$

and then,

$$\alpha\phi'(\eta,\lambda) + \beta\vartheta'(\eta,\lambda) - H(\alpha\phi(\eta,\lambda) + \beta\vartheta(\eta,\lambda)) = 0,$$

After using the condition (13), (14), we get

$$\alpha\phi(\pi,\lambda) = 0, \text{ where } \alpha \neq 0.$$

therefore, The characteristic equation will be

$$\omega(\lambda) = \phi(\pi,\lambda). \tag{15}$$

Lemma 3. The solution $\phi(x,\lambda)$ of problem (1) and (4) satisfy the integral equations

$$\phi(x,\lambda) = \cos\lambda(x-\eta) + \frac{H}{\lambda}\sin\lambda(x-\eta) + \int_{\eta}^{x} \frac{\sin\lambda(x-\tau)}{\lambda} q(\tau)\phi(\tau,\lambda)d\tau. \tag{16}$$

Proof. First we obtain formula (16) Indeed,with solution of the form $q(x) = 0$. (1) becomes becomes $-y'' = \lambda^2 y$ by means of variation of parameter method, we have

$$\phi(x,\lambda) = C_1(x,\lambda)\cos\lambda x + C_2(x,\lambda)\sin\lambda x \tag{17}$$

and the direct calculation of $C_1(x,s)$ and $C_2(x,s)$, we have

$$\begin{aligned} C_1(x,\lambda) &= \cos\lambda\eta - \frac{H}{\lambda}\sin\lambda\eta - \int_\eta^x \frac{\sin\lambda\tau}{\lambda} q(\tau)\phi(\tau,\lambda)d\tau, \\ C_2(x,\lambda) &= \sin\lambda\eta + \frac{H}{\lambda}\cos\lambda\eta + \int_\eta^x \frac{\cos\lambda\tau}{\lambda} q(\tau)\phi(\tau,\lambda)d\tau. \end{aligned} \tag{18}$$

substituting from (18) into (17) equation (16) follows. Second we show that the integral representation (16) satisfies the problem (1) and (13). Let $\varphi(x,\lambda)$ be the solution of (1), so that

$$q(x)\phi(x,\lambda) = \phi''(x,\lambda) + \lambda^2\phi(x,\lambda).$$

We multiply both sides by

$$\frac{\sin\lambda(x-\tau)}{\lambda}$$

and integrating with respect to τ from η to x we obtain

$$\int_\eta^x \frac{\sin\lambda(x-\tau)}{\lambda} q(\tau)\phi(\tau,\lambda)d\tau = \int_\eta^x \frac{\sin\lambda(x-\tau)}{\lambda}\phi''(\tau,\lambda)d\tau \tag{19}$$

$$+\lambda^2 \int_\eta^x \frac{\sin\lambda(x-\tau)}{\lambda}\phi(\tau,\lambda)d\tau.$$

Integrating by parts twice and using the condition (13), we have

$$\int_\eta^x \frac{\sin\lambda(x-\tau)}{\lambda}\phi''(\tau,\lambda)d\tau = \phi(x,\lambda) - \frac{H}{\lambda}\sin\lambda(x-\eta)$$

$$-\cos\lambda(x-\eta) - \lambda\int_\eta^x \sin\lambda(x-\tau)\phi(\tau,\lambda)d\tau. \tag{20}$$

By substituting from (20) into (19) we get the required formula (16). □

Lemma 4. Let $\lambda = \sigma + it$. Then there exists $\lambda_0 > 0$, such that $|\lambda| > \lambda_0$ the following inequalities for the solutions $\phi(x,\lambda)$ of boundary value problem (1) and (4) hold true

$$\phi(x,\lambda) = cos\lambda(x-\eta) + O\left(\frac{e^{|t|(x-\eta)}}{|\lambda|}\right). \tag{21}$$

Proof. We show first that

$$\phi(x,\lambda) = O\left(e^{|t|(x-\eta)}\right).$$

where the inequality is uniformly with respect to x. Form the integral equation (16) we have

$$|\phi(x,\lambda)| \leq e^{|t|(x-\eta)} + \frac{|H|}{|\lambda|}e^{|t|(x-\eta)} + \frac{1}{\lambda}\int_\eta^x e^{|t|(x-\eta)}|q(\tau)||\phi(\tau,\lambda)|d\tau. \tag{22}$$

By using the notation $\phi(x,\lambda)e^{-|t|(x-\eta)} = F(x,\lambda)$, equation (22) takes the form

$$|F(x,\lambda)| \leq 1 + \frac{|H|}{|\lambda|} + \int_0^\pi \frac{|q(\tau)|}{|\lambda|}|F(\tau,\lambda)|d\tau. \tag{23}$$

Let $\mu = \max_{0\leq x\leq\pi} F(x,\lambda)$, so that from (23) it follows that

$$\mu \leq \frac{1+\frac{|H|}{|\lambda|}}{1-\frac{1}{\lambda}\int_0^\pi |q(\tau)|d\tau}.$$

For $|\lambda| > \lambda_0 = \int_0^\pi |q(\tau)|d\tau$ it follows from the last inequality that $F(x,\lambda) \leq$ constant $/|\lambda|$ and this implies that

$$\phi(x,\lambda) = O\left(e^{|t|(x-\eta)}\right), \tag{24}$$

By the aid of (23) we find that

$$\int_\eta^x \frac{\sin\lambda(x-\tau)}{\lambda}q(\tau)\phi(\tau,\lambda)d\tau = O\left(\frac{e^{|t|(x-\eta)}}{|\lambda|}\right). \tag{25}$$

From (16) and (23) it follows that, $\varphi(x,\lambda)$ has the asymptotic formula (21). □

Theorem 5. *Let $\lambda = \sigma + it$ and suppose that $q(x)$ has a second order piecewise differentiable derivatives on $[0,\pi]$. Then the solution $\phi(x,\lambda)$ of non-local boundary value (1) and (4) have the following asymptotic formula*

$$\phi(x,\lambda) = cos\lambda(x-\eta) + \frac{\alpha_1(x)}{\lambda}\sin\lambda(x-\eta) + \frac{\alpha_2(x)}{\lambda^2}\cos\lambda(x-\eta)$$

$$+\frac{\alpha_3(x)}{\lambda^3}\sin\lambda(x-\eta) + O\left(\frac{e^{|t|(x-\eta)}}{|\lambda^4|}\right) \tag{26}$$

where

$$\begin{aligned}
\alpha_1(x) &= \frac{1}{2}\int_\eta^x q(t)dt + H, \\
\alpha_2(x) &= -\frac{1}{4}\left(\int_\eta^x q(t)dt\right)^2 + H + \frac{1}{4}[q(x) - q(\eta)], \\
\alpha_3(x) &= \frac{1}{8}\left(\int_\eta^x q(t)dt\right)^3 + H + \frac{1}{8}[q'(x) - q'(\eta)] \\
&- \left(\frac{1}{8}\int_\eta^x q(t)dt + H\right)[q(\eta) - q(x)].
\end{aligned} \tag{27}$$

Proof. By substituting from (21) into the integral equation (16), we have

$$\phi(x,\lambda) = \cos\lambda(x-\eta) + \frac{H}{\lambda}\sin\lambda(x-\eta) + \frac{\sin\lambda(x-\eta)}{2\lambda}\int_\eta^x q(t)dt$$

$$+\frac{1}{2\lambda}\int_\eta^x \sin\lambda(x-2t-\eta)q(t)dt + O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^2}\right). \tag{28}$$

Integrating the last integration of (28) by parts and noticing that there exists $q'(x)$ such that $q' \in L_1[0,\pi]$

$$\frac{1}{2\lambda}\int_0^x \sin\lambda(x-2t+\eta)q(t)dt$$

$$= \frac{1}{4}[q(x) - q(\eta)]\frac{\cos\lambda x}{\lambda^2} - \frac{1}{4\lambda^2}\int_\eta^x \cos\lambda(x-2t+\eta)q'(t)dt$$

$$= O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^2}\right). \tag{29}$$

substituting from (29) into (28) , we get

$$\phi(x,\lambda) = \cos\lambda(x-\eta) + \frac{\alpha_1(x)}{\lambda}\sin\lambda(x-\eta) + O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^2}\right). \tag{30}$$

where $\alpha_1(x)$ is defined by (27) . In order to make $\phi(x,\lambda)$ more precise we repeat this procedure again by substituting from the last result (30) into the same integral equation (16), we have

$$\phi(x,\lambda) = \cos\lambda(x-\eta) + \frac{H}{\lambda}\sin\lambda(x-\eta) + \int_\eta^x \frac{\sin\lambda(x-t)\cos\lambda(t-\eta)}{\lambda}$$

$$q(t)dt + \int_{\eta}^{x} \frac{\sin\lambda(x-t)\sin\lambda(x-\eta)}{\lambda^2} q(t)\alpha_1(t)dt$$

$$+ \int_{\eta}^{x} \frac{\sin\lambda(x-t)}{\lambda} q(t) O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^2}\right) dt. \tag{31}$$

Now we estimate each term in (31). Integrating by parts twice the first term of (31), and noticing that $q'' \in L_1[0,\pi]$, we have

$$\int_{\eta}^{x} \frac{\sin\lambda(x-t)\cos\lambda(t-\eta)}{\lambda} q(t)dt = \frac{\sin\lambda(x-\eta)}{2\lambda} \int_{\eta}^{x} q(t)dt$$

$$+ \frac{[q(x)-q(\eta)]}{4\lambda^2} \cos\lambda(x-\eta) + O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^3}\right). \tag{32}$$

Similarly, we have

$$\int_{\eta}^{x} \frac{\sin\lambda(x-t)\sin\lambda(t-\eta)}{\lambda^2} q(t)\alpha_1(t)dt$$

$$= -\frac{1}{2}\int_{\eta}^{x} \alpha_1(t)q(t)dt \frac{\cos\lambda(x-\eta)}{\lambda^2} + O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^3}\right) \tag{33}$$

Substituting from (32) and (33) into (31), we get

$$\phi(x,\lambda) = \cos\lambda(x-\eta) + \frac{\alpha_1(x)}{\lambda}\sin\lambda(x-\eta)$$

$$+ \frac{\alpha_2}{\lambda^2}\cos\lambda(x-\eta) + O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^3}\right). \tag{34}$$

where $\alpha_1(x)$ and $\alpha_2(x)$ is defined by (27) . In order to make $\phi(x,\lambda)$ more precise we repeat this procedure again by substituting from the last result (34) into the same integral equation (16), we have

$$\begin{aligned}
\phi(x,\lambda) &= \cos\lambda(x-\eta) + \frac{H}{\lambda}\sin\lambda(x-\eta) \\
&+ \int_{\eta}^{x} \frac{\sin\lambda(x-t)\cos\lambda(t-\eta)}{\lambda} q(t) \\
&+ \int_{\eta}^{x} \frac{\sin\lambda(x-t)\sin\lambda(t-\eta)}{\lambda^2} q(t)\alpha_1(t)dt \\
&+ \int_{\eta}^{x} \frac{\sin\lambda(x-t)\cos\lambda(t-\eta)}{\lambda^3} q(t)\alpha_2(t)dt
\end{aligned}$$

$$+ \int_\eta^x \frac{\sin \lambda(x-t)}{\lambda} q(t) O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^4}\right) dt. \tag{35}$$

Now we estimate each term in (35). Integrating by parts twice the first term of (35), and noticing that $q'' \in L_1[0,\pi]$, we have

$$\int_\eta^x \frac{\sin \lambda(x-t)\cos\lambda(t-\eta)}{\lambda} q(t)dt$$

$$= \frac{\sin\lambda(x-\eta)}{2\lambda}\int_\eta^x q(t)dt + \frac{[q(x)-q(\eta)]}{4\lambda^2}\cos\lambda(x-\eta)$$

$$+ \frac{\sin\lambda(x-\eta)}{8\lambda^3}[q'(x)-q'(\eta)] + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^4}\right). \tag{36}$$

Further,

$$\int_\eta^x \frac{\sin\lambda(x-t)\sin\lambda(t-\eta)}{\lambda^2} q(t)\alpha_1(t)dt$$

$$\begin{aligned} &= -\frac{\cos\lambda(x-\eta)}{2\lambda^2}\int_\eta^x \alpha_1(t)q(t)dt \\ &+ \frac{\sin\lambda(x-\eta)}{4\lambda^3}[q(\eta)\alpha_1(\eta)-q(x)\alpha_1(x)] \\ &+ \frac{\sin\lambda(x-\eta)}{4\lambda^3}[q(\eta)\alpha_1(\eta)-q(x)\alpha_1(x)] \\ &+ O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^3}\right) \end{aligned} \tag{37}$$

and

$$\int_\eta^x \frac{\sin\lambda(x-t)\cos\lambda(t-\eta)}{\lambda^3} q(t)\alpha_2(t)dt$$

$$= \frac{\sin\lambda(x-\eta)}{2\lambda^3}\int_\eta^x \alpha_2(t)q(t)dt + O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^3}\right) \tag{38}$$

Substituting from (36)-(38) into (35) we get the required formula (26).

Now inserting the values of the functions $\varphi(x,\lambda)$ from the estimate (26) into the second of the boundary conditions in (4), we obtain the following equation for the determination of the eigenvalues: Equation (21) is the characteristic equation which gives roots of λ

$$\lambda_n^0 = (n+\frac{1}{2})\frac{\pi}{\pi-\eta}, \quad n = 0, \pm1, \pm2, ...$$

Then the $\omega(\lambda)$ has the same root of the function $sin\lambda\xi$ (By Rouche's theorem)

$$\lambda_n = \lambda_n^0 + \varepsilon_n, \quad n = 0, 1, 2, ... \tag{39}$$

□

Theorem 6. *Let $q \in L_1(0,\pi)$, then we have the following asymptotic formulas for λ_n of non-local boundary value (1) and (4)*

$$\lambda_n = (n + \frac{1}{2})\frac{\pi}{\pi - \eta} + \frac{\alpha_1}{n\pi} + O\left(\frac{1}{n^2}\right). \tag{40}$$

where $\alpha_1(x)$ defined in (27).

Proof.

$$\omega(x,\lambda) = \cos\lambda(\pi-\eta) + \frac{\alpha_1}{\lambda}\sin\lambda(\pi-\eta)$$

$$+\frac{\alpha_2}{\lambda^2}\cos\lambda(\pi-\eta) + \frac{\alpha_3}{\lambda^3}\sin\lambda(\pi-\eta) + O\left(\frac{e^{|Im\lambda|(\pi-\eta)}}{|\lambda^4|}\right) \tag{41}$$

It follows from (41) that

$$\cos\lambda(\pi-\eta) + \frac{\alpha_1}{\lambda}\sin\lambda(\pi-\eta) + \frac{\alpha_2}{\lambda^2}\cos\lambda(\pi-\eta)$$

$$+\frac{\alpha_3}{\lambda^3}\sin\lambda(\pi-\eta) + O\left(\frac{e^{|Im\lambda|(\pi-\eta)}}{|\lambda^4|}\right) = 0 \tag{42}$$

From equation (42), we have

$$\left[1 + \frac{\alpha_2}{\lambda^2}\right]\cos\lambda(\pi-\eta) + \left[\frac{\alpha_1}{\lambda} + \frac{\alpha_3}{\lambda^3}\right]\sin\lambda(\pi-\eta) = 0 \tag{43}$$

Dividing (43) by $\sin\lambda(\pi-\eta)$ we obtain

$$\left[1 + \frac{\alpha_2}{\lambda^2}\right]\cot\lambda(\pi-\eta) = -\left[\frac{\alpha_1}{\lambda} + \frac{\alpha_3}{\lambda^3}\right]$$

since imaginary $\lambda = O\left(\frac{1}{n}\right)$, then

$$\cot\lambda_n(\pi-\eta) = -\frac{\alpha_1}{\lambda_n} + \frac{\alpha_1\alpha_2}{\lambda_n^3} - \frac{\alpha_3}{\lambda_n^3} + O\left(\frac{1}{n^4}\right) \tag{44}$$

From (39), (44) after elementary calculation, we obtain

$$\varepsilon_n = \frac{\alpha_1}{n\pi} + O\left(\frac{1}{n^2}\right) \tag{45}$$

From (39) and (45) , we have

$$\lambda_n = (n + \frac{1}{2})\frac{\pi}{\pi - \eta} + \frac{\alpha_1}{n\pi} + O\left(\frac{1}{n^2}\right).$$

□

Corollary 7. *If $\eta = 0$,then the eigenvalues of (40), we obtain*

$$\lambda_n = n + \frac{1}{2} + \frac{\alpha_1}{n\pi} + O\left(\frac{1}{n^2}\right).$$

Which meets with the result obtained in (see [7]).

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