RADIUS OF STARLIKENESS OF CERTAIN CLASS OF CLOSE-TO-CONVEX FUNCTIONS

Abdullah Yahya^{1 §}, Shaharuddin C. Soh²

¹Department of Mathematics

Faculty of Computers and Mathematical Sciences
Universiti Teknologi MARA Cawangan Negeri Sembilan
Kampus Seremban, 70300 Seremban, MALAYSIA

²Department of Mathematics

Faculty of Computers and Mathematical Sciences
Universiti Teknologi MARA Malaysia
40450 Shah Alam, Selangor, MALAYSIA

Abstract: Let $G(\alpha, \delta)$ be the class of normalized analytic functions by $f(0) = f'(0) - 1 = 0$ and defined in the open unit disc, $E = \{z : |z| < 1\}$ satisfying

$$\operatorname{Re}\left\{e^{i\alpha} \frac{2zf'(z)}{f(z) - f(-z)}\right\} > \delta,$$

for $|\alpha| < \pi$, $\cos \alpha - \delta > 0$ and $0 \leq \delta < 1$. In this paper, we have determined the bound for $\frac{f(z)}{z}$ and have obtained result for radius of starlikeness, R_{St} of $G(\alpha, \delta)$.

AMS Subject Classification: 30C45

Key Words: analytic function, bounds, radius of starlikeness

1. Introduction

Let $\mathcal{A}$ be the class of functions given by

$$f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots = z + \sum_{n=2}^{\infty} a_nz^n \quad (1)$$

Received: September 15, 2016

Revised: November 3, 2016

Published: February 1, 2017

© 2017 Academic Publications, Ltd.

url: www.acadpubl.eu

[§]Correspondence author

which are analytic in $E = \{z : |z| < 1\}$. A function f analytic in E is said to be univalent in a domain D if for all $z_1, z_2 \in D$, $f(z_1) = f(z_2)$ implies $z_1 = z_2$ [2]. The class of all univalent f in E and have a form (1) will be denoted by S . We defined subclasses of S consisting of functions that are starlike, convex and close-to-convex denoted by St , K and C respectively.

Given $f \in A$ and satisfying $|a_n| \leq n$ for $n \geq 2$ in E , Gavrilov [1] showed that the radius of univalent of the function is real root $r_0 \simeq 0.164$ in the interval $(0, 1)$ of the equation $2(1-r)^3 - (1+r) = 0$ and the result is sharp for $f(z) = 2z - \frac{z}{(1-z)^2}$. Consequently, Yamashita [13] showed that the radius of univalence obtained by [1] is also the same as the radius of starlikeness for the function $f \in A$ satisfying $|a_n| \leq n$ for $n \geq 2$. In addition, the author also determined the radius of convexity for the functions $f \in A$ satisfying $|a_n| \leq n$ for $n \geq 2$ is the real root $r_0 \simeq 0.090$ of the equation $2(1-r)^4 - (1+4r+r^2) = 0$, while the radius of convexity for functions $f \in A$ satisfying $|a_n| \leq M$ for $n \geq 2$ is the real root of $(M+1)(1-r)^3 - M(1+r) = 0$

Kim and Nunokawa [4] investigated some inverse properties for the function $f(z)$ belonging to the class S and satisfying $|a_n| \leq n$ for $n \geq 2$. The authors have proved that $f(z)$ univalent in $|z| < r_0$, then r_0 is the unique solution of the equation $2r^3 - 6r^2 + 7r - 1 = 0$. Nunokawa and Owa [5] have extended results of [4] by showing that the radius of satrlikeness, $R_{St} > 0.08998\dots$. Besides that, Ikeda and Saigo [3] showed that the radius of univalent and the radius of convexity by using the Poisson formula satisfying $|a_n| \leq \frac{2}{n}$, $|a_n| \leq en$ and $|a_n| \leq n$ for all $n = 2, 3, \dots$. Recently, Yamakawa [12] showed that the radius of starlikeness, $R_{St} > 0.137788\dots$ and the radius of convexity, $R_K > 0.0075450\dots$

In this paper, we have defined $G(\alpha, \delta)$ as the class of normalized function $f \in \mathcal{A}$ satisfying the condition

$$Re\{e^{i\alpha} \frac{2zf'(z)}{f(z) - f(-z)}\} > \delta, \quad (z \in E) \quad (2)$$

where $|\alpha| < \pi$, $\cos \alpha - \delta > 0$ and $0 \leq \delta < 1$. These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [6]. Yahya et al., [8] introduced the generalization of (2) and have showed the coefficient bound, $|a_n| \leq \frac{1}{n}[1 + (n-1)A_{\alpha, \delta}]$ for $n \geq 2$. In addition, the authors also found others extremal properties in [7, 10, 11] and second Hankel determinant in [9]. Even though there are intensive study on this class of functions, but yet concentrete on the radius properties. Therefore, our purpose on this paper is to determine bound for $\frac{f(z)}{z}$ and obtain result for the radius of starlikeness, R_{St} of $G(\alpha, \delta)$.

2. Preliminary Results

Based on Yahya et al.,[8], we now find bound for $\frac{f(z)}{z}$ satisfying $|a_n| \leq \frac{1}{n}[1 + (n-1)A_{\alpha\delta}]$ for $n \geq 2$.

Theorem 1. Let $f \in S$ and $f \in G(\alpha, \delta)$, then

$$2 - \frac{A_{\alpha\delta}}{1-r} + \left(\frac{1-A_{\alpha\delta}}{r}\right) \log(1-r) \leq \left|\frac{f(z)}{z}\right| \leq \frac{A_{\alpha\delta}}{1-r} + \left(\frac{A_{\alpha\delta}-1}{r}\right) \log(1-r)$$

for $|z| = r < 1$. This result is sharp for $\operatorname{Re}\{e^{i\alpha} \frac{2zf'(z)}{f(z)-f(-z)}\} > \delta$ where $|\alpha| < \pi$, $\cos \alpha - \delta > 0$ and $0 \leq \delta < 1$.

Proof. Let $\mathcal{A}$ and since $f \in S$ satisfying $|a_n| \leq \frac{1}{n}[1 + (n-1)A_{\alpha\delta}]$ for $n \geq 2$, from (1) we have

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n.$$

Based on Yahya et al.,[8], $|f(z)|$ is given

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} \left(\frac{1}{n}[1 + (n-1)A_{\alpha\delta}]\right) |z|^n.$$

We simplify

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} \frac{|z|^n}{n} + A_{\alpha\delta} \sum_{n=2}^{\infty} |z|^n - A_{\alpha\delta} \sum_{n=2}^{\infty} \frac{|z|^n}{n}.$$

Since $\log(1-|z|) = -\sum_{n=2}^{\infty} \frac{|z|^n}{n}$ and $\frac{1}{1-|z|} = \sum_{n=2}^{\infty} |z|^n$, thus

$$|f(z)| \leq |z| + (-|z| - \log(1-|z|)) + A_{\alpha\delta} \left(-1 - |z| + \frac{1}{1-|z|}\right) - A_{\alpha\delta} (-|z| - \log(1-|z|))$$

which gives

$$\left|\frac{f(z)}{z}\right| \leq \frac{A_{\alpha\delta}}{1-r} + \left(\frac{A_{\alpha\delta}-1}{r}\right) \log(1-r),$$

for $|z| = r < 1$. Therefore, $f(z)$ absolutely converge in E , and so, $f(z)$ analytic in E . On the other hand, we have

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n$$

and

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} \left(\frac{1}{n} [1 + (n-1)A_{\alpha\delta}] \right) |z|^n.$$

We simplify

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} \frac{|z|^n}{n} - A_{\alpha\delta} \sum_{n=2}^{\infty} |z|^n + A_{\alpha\delta} \sum_{n=2}^{\infty} \frac{|z|^n}{n}.$$

and

$$\begin{aligned} |f(z)| &\geq |z| - \left(-|z| + \sum_{n=2}^{\infty} \frac{|z|^n}{n} \right) \\ &\quad - A_{\alpha\delta} \left(1 - |z| + \sum_{n=2}^{\infty} |z|^n \right) + A_{\alpha\delta} \left(1|z| + \sum_{n=2}^{\infty} \frac{|z|^n}{n} \right). \end{aligned}$$

So that

$$|f(z)| \geq 2|z| + \log(1 - |z|) + A_{\alpha\delta} - \frac{A_{\alpha\delta}}{1 - |z|} - A_{\alpha\delta} \log(1 - |z|),$$

which gives

$$\left| \frac{f(z)}{z} \right| \geq 2 - \frac{A_{\alpha\delta}}{1 - r} + \left(\frac{1 - A_{\alpha\delta}}{r} \right) \log(1 - r),$$

for $|z| = r < 1$ as required. The following corollary is immediate. □

Corollary 2. *Let $f \in G(\alpha, \delta)$ and for $|z| = r < 1$, then*

$$|z| < R_1 = 0.7475842618$$

for $0 \leq R_1 \leq r < 1$. Thus Theorem 1 is sharp.

Proof. By means of Theorem 1, we have $\left| \frac{f(z)}{z} \right| > 0$, thus

$$\left| \frac{f(z)}{z} \right| \geq 2 - \frac{A_{\alpha\delta}}{1 - R_1} + \left(\frac{1 - A_{\alpha\delta}}{R_1} \right) \log(1 - R_1) = 0.$$

Based on Yahya et al.,[9], from $A_{\alpha\delta} = \cos \alpha - \delta \leq 0.9433$, we have

$$2 - \frac{0.9433}{1 - R_1} + \left(\frac{1 - 0.9433}{R_1} \right) \log(1 - R_1) = 0$$

and yield $R_1 = 0.7475842618$ as required. □

3. Main Result

We now will estimate radius of starlikeness, R_{St} that satisfying the class $G(\alpha, \delta)$ in E . We begin with the following corollary.

Corollary 3. *Let $u(z)$ be harmonic in $|z| \leq \rho$ and continuous in $|z| \leq \rho$. Then $u(z)$ is given by the equation*

$$u(z) = u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \Re\{u(\rho e^{i\phi})\} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\phi - \theta) + r^2} d\phi, \quad (3)$$

where $0 \leq r < \rho$.

The (3) is called the Poisson formula.[3]

Theorem 4. *Let $f \in \mathcal{A}$ and $f \in G(\alpha, \delta)$, then f is univalent and starlike in $|z| < R_{St}$, where*

$$R_{St} = 0.2655158289.$$

Proof. By means of Theorem 1, we have $\left|\frac{f(z)}{z}\right| > 0$ in $|z| < R_1 = 0.7475842618$, and therefore $\log\left(\frac{f(z)}{z}\right)$ is harmonic in $|z| < R_1$. From Corollary 3 and based on the harmonic function theory, we defined

$$\log \frac{f(z)}{z} = \frac{1}{2\pi} \int_0^{2\pi} \left(\log \left| \frac{f(\zeta)}{\zeta} \right| \right) \frac{\zeta + z}{\zeta - z} d\phi,$$

where $\zeta = \rho e^{i\phi}$, ($0 \leq \phi \leq 2\pi$), $z = re^{i\theta}$, ($0 \leq \theta \leq 2\pi$) and $0 \leq r < \rho \leq R_1 = 0.7475842618$.

By using the logarithmic differentiation, we have

$$\frac{d}{dz} \left(\log \frac{f(z)}{z} \right) = \frac{1}{2\pi} \int_0^{2\pi} \left(\log \left| \frac{f(\zeta)}{\zeta} \right| \right) \frac{d}{dz} \left(\frac{\zeta + z}{\zeta - z} \right) d\phi,$$

and yield

$$\frac{z}{f(z)} \left(\frac{f'(z)}{z} - \frac{f(z)}{z^2} \right) = \frac{1}{2\pi} \int_0^{2\pi} \left(\log \left| \frac{f(\zeta)}{\zeta} \right| \right) \left(\frac{(\zeta - z) - (\zeta + z)}{(\zeta - z)^2} \right) d\phi. \quad (4)$$

We can simplify (4) as

$$\frac{1}{z} \left(\frac{zf'(z)}{f(z)} - 1 \right) = \frac{1}{2\pi} \int_0^{2\pi} \left(\log \left| \frac{f(\zeta)}{\zeta} \right| \right) \left(\frac{2\zeta}{(\zeta - z)^2} \right) d\phi$$

and

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{2\pi} \int_0^{2\pi} \left(\log \left| \frac{f(\zeta)}{\zeta} \right| \right) \left(\frac{2z\zeta}{(\zeta - z)^2} \right) d\phi. \quad (5)$$

From (5), we have

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{2\pi} \int_0^{2\pi} \left(\left| \log \left| \frac{f(\zeta)}{\zeta} \right| \right| \right) \left| \frac{2z\zeta}{(\zeta - z)^2} \right| d\phi,$$

and yield

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{2\pi} \int_0^{2\pi} \left(\left| \log \left| \frac{f(\zeta)}{\zeta} \right| \right| \right) \frac{2|re^{i\theta}||\rho e^{i\phi}|}{|\rho e^{i\phi} - re^{i\theta}|^2} d\phi.$$

From $|\rho e^{i\phi}| = \rho$ and $|re^{i\theta}| = r$, we have

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{2\pi} \int_0^{2\pi} \left(\left| \log \left| \frac{f(\zeta)}{\zeta} \right| \right| \right) \frac{2r\rho}{|(\rho \cos \phi - r \cos \theta) + i(\rho \sin \phi - r \sin \theta)|^2} d\phi,$$

which gives

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{2\pi} \int_0^{2\pi} \left(\left| \log \left| \frac{f(\zeta)}{\zeta} \right| \right| \right) \frac{2r\rho}{(\rho \cos \phi - r \cos \theta)^2 + (\rho \sin \phi - r \sin \theta)^2} d\phi.$$

So that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{2\pi} \int_0^{2\pi} \left(\left| \log \left| \frac{f(\zeta)}{\zeta} \right| \right| \right) \frac{2r\rho}{\rho^2 - 2r\rho \cos(\rho - \theta) + r^2} d\phi.$$

From Theorem 1, we have

$$2 - \frac{A_{\alpha\delta}}{1-r} + \left(\frac{1-A_{\alpha\delta}}{r} \right) \log(1-r) < \frac{A_{\alpha\delta}}{1-r} + \left(\frac{A_{\alpha\delta}-1}{r} \right) \log(1-r) \quad (6)$$

for $|z| = r < 1$. We derive

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \geq 1 - \frac{1}{2\pi} \int_0^{2\pi} \max_{|\phi|=\rho} \left(\left| \log \left| \frac{f(\zeta)}{\zeta} \right| \right| \right) \frac{2r\rho}{\rho^2 - 2r\rho \cos(\rho - \theta) + r^2} d\phi$$

and from (6) and integrating with respect to ϕ yield

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \geq 1 - \frac{2r\rho}{\rho^2 - r^2} \log \left(\frac{A_{\alpha\delta}}{1-\rho} + \left(\frac{A_{\alpha\delta}-1}{\rho} \right) \log(1-\rho) \right) \quad (7)$$

where $0 \leq r < \rho \leq R_1 = 0.7475842618$.

Based on $\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0$ in $|z| < R_{St}$ and from (7), we see that

$$\frac{2r\rho}{\rho^2 - r^2} \log \left(\frac{A_{\alpha\delta}}{1-\rho} + \left(\frac{A_{\alpha\delta}-1}{\rho} \right) \log(1-\rho) \right) = 1$$

and

$$\log \left(\frac{A_{\alpha\delta}}{1-\rho} + \left(\frac{A_{\alpha\delta}-1}{\rho} \right) \log(1-\rho) \right) = \frac{\rho^2 - r^2}{2r\rho}. \quad (8)$$

Substituting $\rho = (1 + \sqrt{2})r$ in (8), we have

$$\log \left(\frac{A_{\alpha\delta}}{1 - (1 + \sqrt{2})r} + \left(\frac{A_{\alpha\delta}-1}{(1 + \sqrt{2})r} \right) \log(1 - (1 + \sqrt{2})r) \right) = 1,$$

which gives

$$\frac{A_{\alpha\delta}}{1 - (1 + \sqrt{2})r} + \left(\frac{A_{\alpha\delta}-1}{(1 + \sqrt{2})r} \right) \log(1 - (1 + \sqrt{2})r) = e,$$

in $|z| < R_{St}$. Based on Yahya et al.,[9], from $A_{\alpha\delta} = \cos \alpha - \delta \leq 0.9433$, we have

$$R_{St} = 0.2655158289$$

as required. □

Acknowledgments

This research was supported by the Universiti Teknologi MARA (UiTM) under the Geran Dana Pembudayaan Penyelidikan (iRAGS), 600-RMI/DANA 5/3/IRAGS(22/2015). The author(s) would like to acknowledge UiTM and Institut Pengurusan Penyelidikan dan Inovasi (IRMI) for the support and contributions.

References

- [1] V.I. Gavrilov, Remarks on the radius of univalence of holomorphic functions, *Matematicheskie Zametki*, **7** (1970), 295-298.
- [2] A.W. Goodman, *Univalent Function Volume 1*, Mariner Publishing Co. Inc. Tampa: University of South Florida, (1983).
- [3] A. Ikeda and M. Saigo, Various inverse problems for univalent functions, *RIMS Kokyū Record*, **1164** (2000), 22-30.
- [4] Y.C. Kim and M. Nunokawa, On some radius results for certain analytic functions, *Kyungpook Mathematical Journal*, **37** (1997), 61-65.
- [5] M. Nunokawa and S. Owa, On some inverse properties for univalent functions, *Kyoto University Koukyuroku*, **1164** (2000), 73-76.

- [6] K. Sakaguchi, On a Certain Univalent Mapping, *J. Mathematical Society of Japan*, **11** (1959).
- [7] A. Yahya, S.C. Soh and D. Mohamad, Arc length and Area of certain class of close-to-convex functions, *Proceeding of The 10TH IMT-GT International Conference on Mathematics, Statistics and Its Application 2014*, **10** (2014), 24-32.
- [8] A. Yahya, S.C. Soh and D. Mohamad, Coefficient bound of a generalized close-to-convex function, *International Journal of Pure and Applied Mathematics*, **83** (2013), 287-293.
- [9] A. Yahya, S.C. Soh and D. Mohamad, Second Hankel determinant for a class of a generalised Sakaguchi class of analytic functions, *International Journal of Mathematical Analysis*, **33** (2013), 1601-1608.
- [10] A. Yahya, S.C. Soh and D. Mohamad, Some extremal properties of a generalized close-to-convex functions, *International Journal of Mathematical Analysis*, **39** (2014), 1931-1936.
- [11] A. Yahya, S.C. Soh and D. Mohamad, Some extremal properties of a generalized Sakaguchi class of analytic functions, *Proceeding of the 20th National Symposium on Mathematical Sciences*, **1522** (2013), 920-925.
- [12] R. Yamakawa, Radius of starlikeness and convexity 2, *RIMS Kokyu Record*, **1363** (2004), 131-135.
- [13] S. Yamashita, Radii of univalence, starlike and convexity, *Bulletin of the Australian Mathematical Society*, **25** (1982), 453-457.