GENERALIZATION OF θ -CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract: In this paper, we have introduced a new class of θ -closed sets called $\theta\omega\alpha$ -closed sets in topological spaces. The class of $\theta\omega\alpha$ -closed sets lies between the class of θ -closed and θg -closed sets. Further, we study the some of its properties.

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1. Introduction and Preliminary

The notion of θ -open sets introduced by Velicko [18] in 1968 for the purpose of studying the characterizations of H-closed Topological Spaces and these sets are stronger than the open sets. Noiri and Jafari [15], Joseph [10], Dickman and Porter [5], Long and Herrington [13] continued the work of Velicko and also they have obtained several interesting results related to these sets. The main purpose of this paper is to introduce new weaker forms of θ -closed sets called $\theta\omega\alpha$ -closed sets by using $\omega\alpha$ -open sets of Benchalli [1]. Further, we study the properties, characterizations of $\theta\omega\alpha$ -closed sets and also the relation of these sets with the existing closed sets in topological spaces.

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Throughout this paper the space (X, τ) (or simply X) represents the topological space on which no separation axioms are assumed unless otherwise mentioned. Let A be a subset of a space (X, τ) then $cl(A)$, $int(A)$ and A^c denote the θ -closure of A , θ -interior of A and θ -compliment of A respectively. A point $x \in X$ is called a θ -adherent point of A [18], if $U^c \cap A \neq \emptyset$ for every open set U of X containing x . The set of all θ -adherent points of A is called the θ -closure of A and is denoted by $Cl(A)$. A subset A of X is called θ -closed if $A = Cl(A)$. The complement of θ -closed set is called θ -open set. θ -interior (denoted by int) of a subset A of X is the union of all θ -open sets contained in A .

Definition 1.1. A subset A of a topological space X is called:

- (i) g -closed set [12] (resp. ω -closed [17], $g\omega\alpha$ -closed set [16]) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open (resp. semiopen, $\omega\alpha$ -open) in X .
- (ii) θ -generalized closed (briefly θg -closed) set [8] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .
- (iii) $\omega\alpha$ -closed [1] (resp. $g\omega\alpha$ -closed [2]) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ω -open (resp. $\omega\alpha$ -open) in X .

Remark 1.2. In a topological space X :

- (i) [8] Every θ -open set is an open.
- (ii)[1] Every open set is an $\omega\alpha$ -open.

Definition 1.3. A topological space (X, τ) is called a R_1 -space [7] if every two different points with distinct closures have disjoint neighborhoods.

Remark 1.4. [4] A topological space (X, τ) is R_1 -space if and only if $Cl(\{x\}) = Cl(\{x\})$ for each $x \in X$.

2. $\theta\omega\alpha$ -Closed Sets and $\theta\omega\alpha$ -Open Sets in Topological Spaces

Definition 2.1. A subset A of a topological space X is called a $\theta\omega\alpha$ -closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\omega\alpha$ -open in X .

The complement of $\theta\omega\alpha$ -closed set is called $\theta\omega\alpha$ -open in X . The collection of all $\theta\omega\alpha$ -closed sets and $\theta\omega\alpha$ -open sets of X are denoted by $\theta\omega\alpha C(X)$ and $\theta\omega\alpha O(X)$

Theorem 2.2. Every θ -closed set is $\theta\omega\alpha$ -closed.

Proof: Let A be a θ -closed and U be an $\omega\alpha$ -open set containing A in X . Since A is θ -closed, $A = cl(A)$ for every subset A of X , it implies that $cl(A) \subseteq U$. Hence A is a $\theta\omega\alpha$ -closed set.

Example 2.3. Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a, b\}, X\}$. In this topological space (X, τ) , a subset $A = \{b, c\}$ is $\theta\omega\alpha$ -closed but not θ -closed in X .

Theorem 2.4. Every $\theta\omega\alpha$ -closed set is θg -closed but not conversly.

Proof: Let A be any $\theta\omega\alpha$ -closed and U be an open set such that $A \subseteq U$. From Remark (ii) in 1.2 and A is $\theta\omega\alpha$ -closed set, it implies that $cl(A) \subseteq U$. Therefore A is θg -closed set in X .

Example 2.5. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$. In this topological space (X, τ) , the subset $A = \{a, c\}$ is θg -closed but not $\theta\omega\alpha$ -closed set in X .

Theorem 2.6. Every $\theta\omega\alpha$ -closed set is g -closed [11] [12] and hence αg -closed, gs -closed and rg -closed set [6].

Proof: Let A be a $\theta\omega\alpha$ -closed and U be an open set such that $A \subseteq X$. Since A is $\theta\omega\alpha$ -closed set and also from remark (ii) in 1.2, it implies that $cl(A) \subseteq U$. But $cl(A) \subseteq cl(A) \subseteq U$ [4], it implies that $cl(A) \subseteq U$. Hence A is g -closed in X .

However the converse of the above theorem need not be true as seen from the following example.

Example 2.7. In Example 2.5, a subset $A = \{b\}$ is g -closed, αg -closed, gs -closed and rg -closed set but not $\theta\omega\alpha$ -closed in X .

Theorem 2.8. Every $\theta\omega\alpha$ -closed set is $g\omega\alpha$ -closed but converse need not be true.

Proof: Let A be a $\theta\omega\alpha$ -closed and U be an $\omega\alpha$ -open set in X such that $A \subseteq U$. Since A is a $\theta\omega\alpha$ -closed, then $cl(A) \subseteq U$. But $cl(A) \subseteq cl(A) \subseteq U$ [4], it implies that $cl(A) \subseteq U$. Hence A is $g\omega\alpha$ -closed set.

Remark 2.9. Every $\theta\omega\alpha$ -closed set is $g\omega\alpha$ -closed set but not conversly. It follows from the Theorem 2.8 and Remark 3.2 [16].

Example 2.10. Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. In this topological space the subset $A = \{c\}$ is $g\omega\alpha$ -closed and $g\omega\alpha$ -closed but not $\theta\omega\alpha$ -closed in X .

Remark 2.11. The class of $\theta\omega\alpha$ -closed set is independent of the class of closed, α -closed [14], semiclosed [11] and $\omega\alpha$ -closed [1]. This can be varified from following examples.

Example 2.12. Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. In this topological space (X, τ) , a subset $A = \{c\}$ is closed, α -closed, semiclosed and $\omega\alpha$ -closed set but not $\theta\omega\alpha$ -closed set.

Example 2.13. Let (X, τ) be a topological space where $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. The subset $A = \{b\}$ is $\theta\omega\alpha$ -closed set but not closed, α -closed, semiclosed and $\omega\alpha$ -closed in X .

From the above results, it follows that the $\theta\omega\alpha$ -closed set is properly placed between θ -closed and θg -closed set and also we obtain the following Figure 1.

The below implications are not reversible as shown in the above examples and also from the related papers[6] and [7].

Figure 1

Theorem 2.14. Union of two $\theta\omega\alpha$ -closed sets is a $\theta\omega\alpha$ -closed set in X .

Proof: Let A and B be two $\theta\omega\alpha$ -closed sets in X . Let U be an $\omega\alpha$ -open set in

X such that $A \cup B \subseteq U$. Then $A \subseteq U$ and $B \subseteq U$. Since A and B are $\theta\omega\alpha$ -closed sets, $cl(A) \subseteq U$ and $cl(B) \subseteq U$. Therefore $cl(A) \cup cl(B) = cl(A \cup B) \subseteq U$ [8]. Therefore $A \cup B$ is $\theta\omega\alpha$ -closed set in X .

Theorem 2.15. If A is a $\theta\omega\alpha$ -closed set in X and $A \subseteq Y \subseteq X$ then A is a $\theta\omega\alpha$ -closed set relative to Y .

Proof: Let $A \subseteq Y \cap U$ where U is an $\omega\alpha$ -open set in X . Then $A \subseteq Y$ and $A \subseteq U$. Since A is $\theta\omega\alpha$ -closed set in X , so $cl(A) \subseteq U$ which implies that $Y \cap cl(A) \subseteq Y \cap U$.

Theorem 2.16. If A is $\omega\alpha$ -open and $\theta\omega\alpha$ -closed set in X then A is θ -closed in X .

Proof: By hypothesis A is $\omega\alpha$ -open and $\theta\omega\alpha$ -closed in X then $cl(A) \subseteq A$. But we have $A \subseteq cl(A)$, it implies that $cl(A) = A$. Therefore A is θ -closed in X .

Theorem 2.17. If a subset A of X is $\theta\omega\alpha$ -closed then $cl(A)-A$ doesnot contain any non empty θ -closed set in X .

Proof: Assume that A is $\theta\omega\alpha$ -closed set and U be a non empty θ -closed subset of $cl(A)-A$. Now $A \subseteq (X - U)$ where $(X - U)$ is θ -open set and also from Remark 1.2 [(i) and (ii)], which implies that $(X - U)$ is $\omega\alpha$ -open set in X . By hypothesis, A is $\theta\omega\alpha$ -closed, $cl(A) \subseteq X - cl(A)$ that is $U \subseteq X - cl(A)$. Now we have $U \subseteq cl(A)$ and $U \subseteq X - cl(A)$, then $U \subseteq cl(A) \cap X - cl(A) = \phi$. Therefore $U = \phi$.

The converse of the above theorem need not be true as seen in the following example.

Example 2.18. In the Example 2.13, a subset $A = \{a\}$ of X then $cl(A)-A = \{b, c\}$ doesnot contain any non empty θ -closed set in X but A is not a $\theta\omega\alpha$ -closed in X .

Theorem 2.19. A subset A of a space X is $\theta\omega\alpha$ -closed if and only if $cl(A)-A$ contains no non empty $\omega\alpha$ -closed set in X .

Proof: Suppose A is $\theta\omega\alpha$ -closed in X . Let V be a $\omega\alpha$ -closed subset of $cl(A)-A$. Since A is $\theta\omega\alpha$ -closed set, $A \subseteq (X - V)$ where $(X - V)$ is an $\omega\alpha$ -open set in X . It

gives that $cl(A) \subseteq (X-V)$ then $V \subseteq X-cl(A)$. Now we have $V \subseteq cl(A)$ and $V \subseteq X-cl(A)$ then $V \subseteq cl(A) \cap X-cl(A) = \phi$. Therefore $V = \phi$.

Conversely, assume that $cl(A)-A$ contains no non empty $\omega\alpha$ -closed. Let $A \subseteq V$ where V is $\omega\alpha$ -open. If $cl(A)$ does not contain in V then $cl(A) \cap (X-V)$ is non empty $\omega\alpha$ -closed subset of $cl(A)-A$ that is $cl(A) \cap (X-V) \neq \phi$ which is a contradiction to hypothesis. Therefore A is a $\theta\omega\alpha$ -closed set in X .

Theorem 2.20. A subset A of a topological space X is $\theta\omega\alpha$ -open if and only if $V \subseteq int(A)$ where V is $\omega\alpha$ -closed set in X .

Proof: Suppose A is $\theta\omega\alpha$ -open in X . Let $V \subseteq A$ where V is $\omega\alpha$ -closed set in X , then $(X-A) \subseteq (X-V)$. Then $cl(X-A) = X-int(A) [3] \subseteq (X-A)$. Therefore $V \subseteq int(A)$.

Conversely, Assume that V is $\omega\alpha$ -closed set with $V \subseteq int(A)$ whenever $V \subseteq A$. Then $(X-A) \subseteq (X-V)$ and $X-int(A) = cl(X-A) [3] \subseteq (X-V)$. It shows that the $(X-V)$ is a $\theta\omega\alpha$ -closed and subset A of X becomes $\theta\omega\alpha$ -open in X .

Theorem 2.21. A subset A of X is $\theta\omega\alpha$ -open in X if and only if $U = X$ where $int(A) \cup (X-A) \subseteq U$ and U is an $\omega\alpha$ -open in X .

Proof: Suppose a subset A is an $\theta\omega\alpha$ -open set in X . Let U be an $\omega\alpha$ -open set and $int(A) \cup (X-A) \subseteq U$. Then $(X-U) \subseteq X-[int(A) \cup (X-A)] = (X-int(A)) - (X-A) \Rightarrow (X-V) \subseteq cl(X-A) - (X-A)$. Then by Theorem 2.19, $(X-U) = \phi$. Therefore $X = U$.

Conversely, suppose V is an $\omega\alpha$ -closed and $V \subseteq A$, then $(X-A) \subseteq (X-V) \Rightarrow int(A) \cup (X-A) \subseteq int(A) \cup (X-V)$. By hypothesis, we get $int(A) \cup (X-V) = X$ and $V \subseteq int(A)$. Hence by theorem 2.20, A is $\theta\omega\alpha$ -open set in X .

Theorem 2.22. If $A \subseteq B \subseteq X$ where A is $\theta\omega\alpha$ -open relative to B and B is $\theta\omega\alpha$ -open in X then A is $\theta\omega\alpha$ -open set in X .

Proof: Let V be a $\omega\alpha$ -closed set in X . Suppose $V \subseteq A$ then $V = V \cap B$ is $\omega\alpha$ -closed in B . But A is $\theta\omega\alpha$ -open relative to B and from Theorem 2.20 it implies that $V \subseteq int_B(A)$. Since $int_B(A)$ is an θ -open set relative B , $V \subseteq U \cap B \subseteq A$ for some θ -open set U in X . Since B is a $\theta\omega\alpha$ -open set in X . We have $V \subseteq int(B) \subseteq B$. Therefore $V \subseteq int(B) \cap U \subseteq B \cap U \subseteq A$. It implies that $V \subseteq int(A)$. Therefore by Theorem 2.20, A is $\theta\omega\alpha$ -open set in X .

Theorem 2.23. In a topological space (X, τ) , $\omega\alpha O(X, \tau) = \theta C(X, \tau)$ if and only if every subset of X is a $\theta\omega\alpha$ -closed in (X, τ) .

Proof: Assume that every subset of X is $\theta\omega\alpha$ -closed. Let $U \in \omega\alpha O(X, \tau)$. Since $U \subseteq U$ and U is $\theta\omega\alpha$ -closed set, so $Cl(U) \subseteq U$ then $Cl(U) = U$ and $U \in \theta C(X, \tau)$. Therefore $\omega\alpha O(X, \tau) \subseteq \theta C(X, \tau)$. Let $V \in \theta C(X, \tau)$ then $(X - V)$ is θ -open and hence it is $\omega\alpha$ -open subset of (X, τ) . Therefore $(X - V) \in \omega\alpha O(X, \tau)$. Hence $\omega\alpha O(X, \tau) = \theta C(X, \tau)$

Conversely, Assume that $\omega\alpha O(X, \tau) = \theta C(X, \tau)$. Let A be a subset of X such that $A \subseteq U$ where $U \in \omega\alpha O(X, \tau)$, then $U \in \theta C(X, \tau)$. It implies that $cl(U) = U$ and also $cl(A) \subseteq cl(U) = U$, then $cl(A) \subseteq U$. Hence A is $\theta\omega\alpha$ -closed set in X .

Theorem 2.24. Let A and B be any two subsets of X . If A is a $\theta\omega\alpha$ -closed and $A \subseteq B \subseteq cl(A)$, then B is $\theta\omega\alpha$ -closed in X .

Proof: By hypothesis, $A \subseteq B$ and $B \subseteq cl(A)$, then $cl(B) \subseteq cl(A)$ and $cl(B) - B \subseteq cl(A) - A$. Since A is $\theta\omega\alpha$ -closed set in X , then $cl(A) - A$ has no non empty $\omega\alpha$ -closed set in X from Theorem 2.19 and also $cl(B) - B$ has no non empty $\omega\alpha$ -closed set in X . Again by Theorem 2.19 B is a $\theta\omega\alpha$ -closed set in X .

Theorem 2.25. Let A and B be any two subsets of X . If $int(A) \subseteq B \subseteq A$ and A is $\theta\omega\alpha$ -open in X then B is an $\theta\omega\alpha$ -open set in X .

Proof: By hypothesis, $int(A) \subseteq (B) \subseteq A$ then $(X - A) \subseteq (X - B) \subseteq X - int(A) = cl(X - A)$. Since A is $\theta\omega\alpha$ -closed set, then $(X - A)$ is $\theta\omega\alpha$ -closed set. From Theorem 2.24, $(X - B)$ is $\theta\omega\alpha$ -closed set in X . Hence B is $\theta\omega\alpha$ -open set in X .

Definition 2.26. We recall that a topological space X is said to be a regular space if $\forall$ closed subset F of X and for each point $x \in X$ and $x \notin F$ there exist disjoint open sets G, H of X such that $F \subset G$ and $x \in H$.

Theorem 2.27. A subset A of a regular space X is $\theta\omega\alpha$ -closed then A is $\theta\omega\alpha$ -closed set.

Proof: Since the space X is regular, then $cl(A) = cl(A)$ [3]. By hypothesis, A is $\theta\omega\alpha$ -closed set, it implies that $cl(A) = cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\omega\alpha$ -open set. Hence A is $\theta\omega\alpha$ -closed in X .

Theorem 2.28. Let X be a R_1 -space then for a singleton subset A of X the following results are equivalent:

- (i) A is $\theta\omega\alpha$ -closed
- (ii) A is $g\omega\alpha$ -closed.

Proof: (i) $\Rightarrow$ (ii) Follows from the Theorem 2.8 .

(ii) $\Rightarrow$ (i). From Remark 1.4, the concepts of θ -closure and closure coincide for singleton sets. Then replace $cl(A)$ by $cl(A)$ in the Theorem 3.8.

3. $\theta\omega\alpha$ -Closure and $\theta\omega\alpha$ -Interior in Topological Spaces

In this section, we introduce the notion of $\theta\omega\alpha$ -Closure and $\theta\omega\alpha$ -Interior of topological spaces.

Definition 3.1. The $\theta\omega\alpha$ -closure of a subset A of X is denoted by $\theta\omega\alpha cl(A)$ and is defined as the intersection of all $\theta\omega\alpha$ -closed sets containing A .

Therefore $\theta\omega\alpha cl(A) = \bigcap \{ F \subseteq X : A \subseteq F \text{ and } F \text{ is } \theta\omega\alpha\text{-closed set} \}$.

Definition 3.2. The $\theta\omega\alpha$ -interior of a subset A of X is denoted by $\theta\omega\alpha int(A)$ and is defined as the union of all $\theta\omega\alpha$ -open sets contained in A .

Therefore $\theta\omega\alpha int(A) = \bigcup \{ G \subseteq X : G \subseteq A \text{ and } G \text{ is } \theta\omega\alpha\text{-open set} \}$.

Remark 3.3. If $A \subseteq X$ then,

- (i) $A \subseteq \theta\omega\alpha cl(A) \subseteq cl(A)$.
- (ii) $int(A) \subseteq \theta\omega\alpha int(A) \subseteq A$.

Theorem 3.4. Let A and B be any two subsets of a space X then the following properties are true

- (i) A is $\theta\omega\alpha$ -closed if and only if $\theta\omega\alpha cl(A) = A$.
- (ii) A is $\theta\omega\alpha$ -closed set in X then $\theta\omega\alpha cl(A)$ is the smallest $\theta\omega\alpha$ -closed subset of X containing A .
- (iii) $\theta\omega\alpha cl(\phi) = \phi$ and $\theta\omega\alpha cl(X) = X$
- (iv) $\theta\omega\alpha cl(A)$ is a $\theta\omega\alpha$ -closed set in X .
- (v) If $A \subseteq B$, then $\theta\omega\alpha cl(A) \subseteq \theta\omega\alpha cl(B)$
- (vi) $\theta\omega\alpha cl(A \cup B) = \theta\omega\alpha cl(A) \cup \theta\omega\alpha cl(B)$
- (vii) $\theta\omega\alpha cl(A \cap B) \subseteq \theta\omega\alpha cl(A) \cap \theta\omega\alpha cl(B)$
- (viii) $\theta\omega\alpha cl(\theta\omega\alpha cl(A)) = \theta\omega\alpha cl(A)$

Proof: (i) Let A be a $\theta\omega\alpha$ -closed set in X . Since $A \subseteq A$ and $A \in \{ F \subseteq X: A \subseteq F \text{ and } F \text{ is } \theta\omega\alpha\text{-closed set} \}$ it implies that $A = \cap \{ F \subseteq X: A \subseteq F \text{ and } F \text{ is } \theta\omega\alpha\text{-closed set} \} \subseteq A$. Then $\theta\omega\alpha cl(A) \subseteq A$. But $A \subseteq \theta\omega\alpha cl(A)$ is always true. Hence $A = \theta\omega\alpha cl(A)$.

Conversely, suppose $A = \theta\omega\alpha cl(A)$, then $\theta\omega\alpha cl(A)$ is $\theta\omega\alpha$ -closed in X which implies that A is $\theta\omega\alpha$ -closed set in space X .

(ii) Suppose A is $\theta\omega\alpha$ -closed set in X then $\theta\omega\alpha cl(A) = \cap \{ F \subseteq X: A \subseteq F \text{ and } F \text{ is } \theta\omega\alpha\text{-closed} \}$. Since $A \subseteq A$ and A is $\theta\omega\alpha$ -closed in X , from i^{st} , $A = \theta\omega\alpha cl(A)$ is the smallest $\theta\omega\alpha$ -closed set in X containing A .

(iii) and (iv) - Follows from the Definition 3.1

(v) We know that $B \subseteq \theta\omega\alpha cl(B)$, then $A \subseteq B \subseteq \theta\omega\alpha cl(B)$. So $\theta\omega\alpha cl(B)$ is the smallest $\theta\omega\alpha$ -closed set containing A . $\rightarrow$ (a). But $\theta\omega\alpha cl(A)$ is the smallest $\theta\omega\alpha$ -closed set containing A . $\rightarrow$ (b) From (a) and (b) we get, $\theta\omega\alpha cl(A)$ is smaller than $\theta\omega\alpha cl(B)$. Therefore $\theta\omega\alpha cl(A) \subseteq \theta\omega\alpha cl(B)$.

(vi) We know that $A \subseteq A \cup B$ and $B \subseteq A \cup B$, from (v), $\theta\omega\alpha cl(A) \subseteq \theta\omega\alpha cl(A \cup B)$ and $\theta\omega\alpha cl(B) \subseteq \theta\omega\alpha cl(A \cup B)$. Then $\theta\omega\alpha cl(A) \cup \theta\omega\alpha cl(B) \subseteq \theta\omega\alpha cl(A \cup B) \rightarrow$ (a). Let $x \in X$ be any point such that $x \notin \theta\omega\alpha cl(A) \cup \theta\omega\alpha cl(B)$, then there exist $\theta\omega\alpha$ -closed sets G and H such that $A \subseteq G$ and $B \subseteq H$, $x \notin G$ and $x \notin H$, it implies that $x \notin G \cup H$ and $A \cup B \subseteq G \cup H$. By the Theorem 2.14 $G \cup H$ is $\theta\omega\alpha$ -closed, then $x \notin \theta\omega\alpha cl(A \cup B)$. Therefore $\theta\omega\alpha cl(A \cup B) \subseteq \theta\omega\alpha cl(A) \cup \theta\omega\alpha cl(B) \rightarrow$ (b). From (a) and (b) we have, $\theta\omega\alpha cl(A \cup B) = \theta\omega\alpha cl(A) \cup \theta\omega\alpha cl(B)$.

(vii) We know that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. From (v) we have, $\theta\omega\alpha cl(A \cap B) \subseteq \theta\omega\alpha cl(A)$ and $\theta\omega\alpha cl(A \cap B) \subseteq \theta\omega\alpha cl(B)$. Then $\theta\omega\alpha cl(A \cap B) \subseteq \theta\omega\alpha cl(A) \cap \theta\omega\alpha cl(B)$.

(viii) From Definition 3.1 we have, $\theta\omega\alpha cl(A)$ is a $\theta\omega\alpha$ -closed set in X . Let $\theta\omega\alpha cl(A) = G$, then G is a $\theta\omega\alpha$ -closed set in X . From (i) we have, $\theta\omega\alpha cl(G) = G$. It implies that $\theta\omega\alpha cl(\theta\omega\alpha cl(A)) = \theta\omega\alpha cl(A)$.

The converse of property (ii) and property (vii) are need not be true as seen in the following examples.

Example 3.5. Let $X = \{ a, b, c \}$ with $\tau = \{ \phi, \{ a \}, \{ b \}, \{ a, b \}, X \}$. In this topological space, a subset $A = \{ a \}$, then $\theta\omega\alpha cl(A) = \{ a, c \}$ which is the smallest $\theta\omega\alpha$ -closed set in X containing A but A is not a $\theta\omega\alpha$ -closed in X .

Example 3.6. In the Example 3.5, subsets $A = \{ a \}$ and $B = \{ b \}$, then $\theta\omega\alpha cl(A) = \{ a, c \}$, $\theta\omega\alpha cl(B) = \{ b, c \}$ and $\theta\omega\alpha cl(A \cap B) = \phi$. Therefore $\{ c \} = \theta\omega\alpha cl(A) \cap \theta\omega\alpha cl(B) \not\subseteq \theta\omega\alpha cl(A \cap B) = \phi$.

Definition 3.7. [9] Let A be any subset of a space X . Then $\omega\alpha$ -kernel of

A is denoted by $\omega\alpha\text{-ker}(A)$ and is defined as intersection of all $\omega\alpha$ -open sets containing A . Then $\omega\alpha\text{-ker}(A) = \cap \{ U \subseteq X: A \subseteq U \text{ and } U \text{ is } \omega\alpha\text{-open set} \}$

Theorem 3.8. A subset A of a space X is $\theta\omega\alpha$ -closed if and only if $cl(A) \subseteq \omega\alpha\text{-ker}(A)$.

Proof: Let U be an $\omega\alpha$ -open set containing A then $cl(A) \subseteq \omega\alpha\text{-ker}(A) \subseteq U$. Therefore A is $\theta\omega\alpha$ -closed set.

Conversely, suppose A is $\theta\omega\alpha$ -closed in X . Then $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\omega\alpha$ -open in X . Let $x \in cl(A)$. If $x \notin \omega\alpha\text{-ker}(A)$ then there exists an $\omega\alpha$ -open set U containing A such that $x \notin U$. Then $x \notin cl(A)$, which is a contradiction to the hypothesis. Hence $cl(A) \subseteq \omega\alpha\text{-ker}(A)$.

Definition 3.9. Let N be any subset of topological space X then N is said to be $\theta\omega\alpha$ -neighborhood (denoted by $\theta\omega\alpha\text{-nbd}$) of point $x \in X$ if there exist an $\theta\omega\alpha$ -open set U such that $x \in U \subseteq N$.

Theorem 3.10. A subset A of topological space X is $\theta\omega\alpha$ -closed and $x \in \theta\omega\alpha cl(A)$ if and only if $N \cap A \neq \phi$ for any $\theta\omega\alpha\text{-nbd}$ N of x in X .

Proof: If $x \notin \theta\omega\alpha cl(A)$. Then there exists $\theta\omega\alpha$ -closed set F of X such that $A \subseteq F$ and $x \notin F$. Thus $x \in (X - F)$ and $(X - F)$ is $\theta\omega\alpha$ -open in X . But $A \cap (X - F) = \phi$ which is a contradiction. Hence $x \in \theta\omega\alpha cl(A)$.

Conversely, suppose that there exists an $\theta\omega\alpha\text{-nbd}$ N of a point $x \in X$ such that $N \cap A = \phi$. Then there exists an $\theta\omega\alpha$ -open set F of X such that $x \in F \subseteq N$. Therefore we have $F \cap A = \phi$ and $x \in (X - F)$. Then $\theta\omega\alpha cl(A) \subseteq (X - F)$ and $x \notin \theta\omega\alpha cl(A)$, which is a contradiction to hypothesis that $x \in \theta\omega\alpha cl(A)$. Therefore $N \cap A \neq \phi$.

Remark 3.11. The intersection of any two members of $\theta\omega\alpha N(x)$ is again a member of $\theta\omega\alpha N(x)$.

Definition 3.12. Let A be a subset of topological space X . Then a point $x \in X$ is said to be a $\theta\omega\alpha$ -limit point of A if every $\theta\omega\alpha$ -open set of x contains a point of A other than x , that is $(G - \{x\}) \cap A \neq \phi$ [or $G \cap (A - \{x\}) \neq \phi$] for every $\theta\omega\alpha$ -open set G of x .

In a topological space X , the set of all $\theta\omega\alpha$ -limit points of a given subset A of X is called $\theta\omega\alpha$ -derived set of A and is denoted by $\theta\omega\alpha d(A)$.

Theorem 3.13. Let A and B be any two subsets of a space X then the

following properties are true:

- (i) $\theta\omega\alpha d(\phi) = \phi$
- (ii) If $A \subseteq B$, then $\theta\omega\alpha d(A) \subseteq \theta\omega\alpha d(B)$
- (iii) $\theta\omega\alpha d(A \cup B) = \theta\omega\alpha d(A) \cup \theta\omega\alpha cl(B)$
- (iv) $\theta\omega\alpha d(A \cap B) \subseteq \theta\omega\alpha d(A) \cap \theta\omega\alpha d(B)$.

Proof: (i) Let $x \in X$ and $x \in \theta\omega\alpha d(\phi)$. Then for every $\theta\omega\alpha$ -open set G containing x , we should have $[G - \{x\}] \cap A \neq \phi$, which is impossible. Therefore $\theta\omega\alpha d(\phi) = \phi$.

(ii) Let $x \in \theta\omega\alpha d(A)$ then x is a limit point of A . From definition 3.12, $[G - \{x\}] \cap A \neq \phi$ for every $\theta\omega\alpha$ -nbd G containing x . Since $A \subseteq B$, then $[G - \{x\}] \cap A \subseteq [G - \{x\}] \cap B$. Therefore $x \in \theta\omega\alpha d(B)$. Hence $\theta\omega\alpha d(A) \subseteq \theta\omega\alpha d(B)$.

(iii) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ then from property (ii), $\theta\omega\alpha d(A) \cup \theta\omega\alpha d(B) \subseteq \theta\omega\alpha d(A \cup B) \rightarrow (a)$. On the other hand if $x \notin \theta\omega\alpha d(A) \cup \theta\omega\alpha d(B)$, then $x \notin \theta\omega\alpha d(A)$ and $x \notin \theta\omega\alpha d(B)$. Therefore there exist $\theta\omega\alpha$ -nbds G_1 and G_2 of x such that $G_1 \cap (A - \{x\}) = \phi$ and $G_2 \cap (B - \{x\}) = \phi$. Since $G_1 \cap G_2$ is $\theta\omega\alpha$ -nbd of x , then we get $(G_1 \cap G_2) \cap [A - \{x\}] = \phi$. Therefore x not in $\theta\omega\alpha d(A \cup B)$. Therefore $\theta\omega\alpha d(A \cup B) \subseteq \theta\omega\alpha d(A) \cup \theta\omega\alpha d(B) \rightarrow (b)$. Therefore from (a) and (b) we get, $\theta\omega\alpha d(A \cup B) = \theta\omega\alpha d(A) \cup \theta\omega\alpha cl(B)$.

(iv)- Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then from property (ii) we have $\theta\omega\alpha d(A \cup B) \subseteq \theta\omega\alpha d(A)$ and $\theta\omega\alpha d(A \cap B) \subseteq \theta\omega\alpha d(B)$. Consequently, $\theta\omega\alpha d(A \cap B) \subseteq \theta\omega\alpha d(A) \cap \theta\omega\alpha d(B)$

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