NORMED Z-MODULES

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Abstract: In this study, norm is defined on algebraic structures which are not vector spaces. We introduce for the first time the concept of normed Z-modules is introduced and some basic properties are established. Then the cartesian product of normed Z-modules, Cauchy sequence of normed Z-modules, metric of normed Z-modules, are defined and verified by suitable examples.

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1. Introduction

Shilov[11] defined on decomposition of a commutative normed ring in a direct sums of ideals. Gel'fand et al. [12] defined commutative normed rings. Freundlich [13] introduced completely continuous elements of a normed ring. Raikov [14] defined to the theory of normed rings with involution. Naimark et al.[15] defined normed rings. Shilov [16] defined analytic functions in a normed ring. Jarden [17] defined normed rings in 2011 and is studied norms $\|.\|$ of associative rings are generalizations of absolute values $|\cdot|$ of integral domains.

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Dunkl[2] defined modules the theory of submodules is the analogue of the ideal theory of Banach algebra. He will develop some theory for closed maximal submodules and prove that with certain finiteness conditions every proper submodule is contained in a closed maximal submodule. Takasahi[4] give an extension of the Helson-Edwards theorem for the group algebras to Banach modules. He also observed that this extension can be viewed as a generalization of Liu-RooiJ-Wang's result for Banach modules over the group algebras. Kitchen et al.[5] are concerned with the representation of Banach algebras and Banach modules as sections of bundles of Banach spaces. They also investigate projective tensor products of bundles of Banach algebras and bundles of Banach modules. Misra et al. [6] show that these modules are always bounded, where the bound is related to the solution of an extremal problem. In particular, they obtain necessary sufficient conditions for such a module to be contractive. Venkataraman et al. [7] introduced normed module. They also the existence of ultracomplete immediate extensions for a special class of normed R -modules, which we shall call pure, closed. Their methods yield an application to the theory of linearly ordered Abelian groups. It can be deduced that every linearly ordered Abelian group admits a linearly ordered ultracomplete immediate extension. Alizadeh et al. [8] proved that each norm-continuous local derivation of von Neumann algebra A into a dual A -bimodule M is a derivation. Wu et al. [9] embed each L^∞ -normed module E into an appropriate and unique complete random normed module E_0 - so that the properties of E are closely related to the properties of E_0 . They try to connect Eisele and Taieb's study for L^∞ -modules and Guo's study for random normed modules. Kitchen [10] the fundamental theorem on almost periodic functions is placed in a more general setting in which a commutative Banach algebra A and its "spectrum", that is, its maximal ideal space $\Delta(A)$, take over the roles played by G and $\hat{G}$ respectively.

The present work, we the concept of normed Z -modules is introduced and some basic properties are established. Its organization is as follows, In Section 2, well-known results of some preliminaries are given. In Section 3, a notion Z -normed module is given and various properties are studied. Finally Section 4, presents the conclusion of our work.

2. Preliminaries

Definition 1. [1] A vector space V is a set of vectors $v \in V$ which is closed under addition and closed under multiplication by scalars $\alpha \in R$. This mean $u + v \in V$ and $\alpha v \in V$ and the following five axioms must hold for all

$u, v, w \in V$ and all $\alpha, \beta \in R$:

1. $u + (v + w) = (u + v) + w$
2. $\exists 0 \in V$ s.t. $0 + v = v$
3. $\forall v \in V, \exists v' \in V$ s.t. $v + v' = 0$
4. $u + v = v + u$
5. $1v = v, \alpha(\beta v) = (\alpha\beta)v, (\alpha + \beta)v = \alpha v + \beta v, \alpha(u + v) = \alpha u + \alpha v.$

Definition 2. [12] Let A be a ring. A is called a normed ring if a non-negative real-valued function, namely a norm, $\|\cdot\| : A \rightarrow R$ is defined for all $a, b \in A$ which satisfies the following conditions:

1. $\|a\| = 0 \Leftrightarrow a = 0,$
2. $\|a + b\| \leq \|a\| + \|b\|,$
3. $\|-a\| = \|a\|$; if A is a ring with unity then $\|1\| = \|1\| = 1$;
4. $\|a.b\| \leq \|a\|.\|b\|.$

Definition 3. [1] A vector space N is a normed vector space if there is a norm function mapping N to the non-negative real numbers, written $\|x\|$, for $\forall x, y \in N$ and satisfying the following three axioms:

1. $\|x\| = 0 \Leftrightarrow x = 0,$
2. $\|\alpha x\| = |\alpha|\|x\|$ ($\alpha \in F$),
3. $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality).

Definition 4. [3] Suppose that R is a ring and 1_R is its multiplicative identity. A left R -module M consists of an abelian group $(M, +)$ and an operation $\cdot : R \times M \rightarrow M$ such that $\forall r, s \in R$ and $x, y \in M$, we have:

1. $r + (x + y) = r.x + r.y$
2. $(r + s).x = r.x + s.x$
3. $(r.s).x = r.(s.x)$
4. $1_R.x = x.$

The operation of the ring on M is called scalar multiplication, and is usually written by juxtaposition, i.e. as rx for r in R and x in M , though here it is denoted as $r.x$ to distinguish it from the ring multiplication operation, denoted here by position. The notation ${}_R M$ indicates a left R -module M . A right R -module M or M_R is defined similarly, except that the ring acts on the right; i.e., scalar multiplication takes the form $. : M \times R \rightarrow M$, and the above axioms are written with scalars r and s on the right of x and y .

3. Normed Z -Modules

Definition 5. Let $(G, *)$ be a commutative group. Let a non-negative, real-valued mapping, namely a norm, $\|\cdot\|_Z : G \rightarrow R$ is defined on and for all $g_1, g_2 \in G$ it satisfies the following conditions:

1. $\|g\|_Z = 0 \Leftrightarrow g = 0$,
2. $\|kg\|_Z = |k|\|g\|_Z$
3. $\|g_1 * g_2\|_Z \leq \|g_1\|_Z + \|g_2\|_Z$.

Then, G is called a generalized normed Z -module (or Z -norm). A Z -module with a norm $\|\cdot\|_Z$ is called a Z -normed module and denoted by $(G, \|\cdot\|_Z)$.

Example 6. Let $Z_2 \times Z_4 = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{1}), (\bar{0}, \bar{2}), (\bar{0}, \bar{3}), (\bar{1}, \bar{0}), (\bar{1}, \bar{1}), (\bar{1}, \bar{2}), (\bar{1}, \bar{3})\}$. $\forall (a, b) \in Z_2 \times Z_4$ for $\|(\bar{a}, \bar{b})\|_Z = \sqrt{a^2 + b^2}$

1. $\|(\bar{0}, \bar{0})\|_Z = \sqrt{0^2 + 0^2} = 0$
2. $\|(k\bar{a}, k\bar{b})\|_Z = \sqrt{(ka)^2 + (kb)^2} = \sqrt{k(a^2 + b^2)} = |k|\sqrt{a^2 + b^2} = |k|\|(\bar{a}, \bar{b})\|_Z$
3. $\|(\bar{0}, \bar{0})\|_Z \leq \|(\bar{0}, \bar{1})\|_Z + \|(\bar{1}, \bar{0})\|_Z$ such that $\|(\bar{a}, \bar{b})\|_Z \leq \|(\bar{a}, \bar{c})\|_Z + \|(\bar{c}, \bar{b})\|_Z$.

Lemma 7. Let $(G, \|\cdot\|_Z)$ be a Z -normed module. The function $d : G \times G \rightarrow R$ which is defined by

$$d(g_1, g_2) = \|g_1 * g_2^{-1}\|_Z$$

is a metric on G .

Proof. Let $g_1, g_2, g_3 \in G$. Using the generalized norm rules,

1. $d(g_1, g_2) = \|g_1 * g_2^{-1}\|_Z > 0$

2. $d(g_1, g_3) = \|g_1 * g_3^{-1}\|_Z = \|g_1 * g_2 * g_2^{-1} * g_3^{-1}\|_Z \leq \|g_1 * g_2^{-1}\|_Z + \|g_2 * g_3^{-1}\|_Z = d(g_1, g_2) + d(g_2, g_3)$
3. $d(g_1, g_2) = \|g_1 * g_2^{-1}\|_Z = \| \| - 1 \|_Z (g_2 * g_1^{-1}) \|_Z = \|g_2 * g_1^{-1}\|_Z = d(g_2, g_1)$
4. $d(g_1, g_2) = 0 \Leftrightarrow \|g_1 * g_2^{-1}\|_Z = 0 \Leftrightarrow g_1 - g_2 = 0 \Leftrightarrow g_1 = g_2.$

Therefore, d satisfies the metric space axioms. As a result, if X is a normed vector space, then X is also a normed Z -module. □

Example 8. Let $Z^{n \times m}$ denote the set of matrices whose entries are integers. For the Z -module $G = (g_{ij}) \in Z^{n \times m}$, the function defined by

$$\|G\|_Z = \max_j \left(\sum_{i=1}^n |g_{ij}| \right)$$

is a generalized norm.

Solution:

$$1. \ G = 0 = \begin{pmatrix} 0 & . & 0 \\ . & . & . \\ . & . & . \\ 0 & . & 0 \end{pmatrix} \Rightarrow \|G\|_Z = |0| + |0| + \dots + |0| = 0,$$

$$\begin{aligned}
 2. \ \|k.G\|_Z &= \left\| \begin{pmatrix} k.g_{11} & k.g_{12} & \dots & k.g_{1n} \\ k.g_{21} & k.g_{22} & \dots & k.g_{2n} \\ . & . & . & . \\ . & . & . & . \\ k.g_{m1} & k.g_{m2} & \dots & k.g_{mn} \end{pmatrix} \right\| \\
 &= \max_j \left(\sum_{i=1}^n |k.g_{ij}| \right) \\
 &= |k.g_{m1}| + |k.g_{m2}| + \dots + |k.g_{mn}| \\
 &= |k|. |g_{m1}| + |k|. |g_{m2}| + \dots + |k|. |g_{mn}| \\
 &= |k|. (|g_{m1}| + |g_{m2}| + \dots + |g_{mn}|) \\
 &= |k|. (\max_j (\sum_{i=1}^n |g_{ij}|)) \\
 &= |k|. \|G\|_Z
 \end{aligned}$$

$$3. \ \|G * H\|_Z \leq \|G\|_Z + \|H\|_Z.$$

Lemma 9. Let $(G, \|\cdot\|_Z)$ be a generalized normed Z -module. Then, for all $g, h \in G$ and $g_1, g_2, \dots, g_n \in G$

1. $\|g\|_Z - \|h\|_Z \leq \|g * h\|_Z$
2. $\|g_1 * g_2 * \dots * g_n\|_Z \leq \|g_1\|_Z + \|g_2\|_Z + \dots + \|g_n\|_Z$.

Proof. 1. Firstly,

$$\|g\|_Z = \|g * h^{-1} * h\|_Z \leq \|g * h^{-1}\|_Z + \|h\|_Z \Rightarrow \|g\|_Z - \|h\|_Z \leq \|g * h^{-1}\|_Z \dots (i)$$

On the other hand,

$$\|h\|_Z = \|h * g^{-1} * g\|_Z \leq \|h * g^{-1}\|_Z + \|g\|_Z \Rightarrow \|h\|_Z - \|g\|_Z \leq \|h * g^{-1}\|_Z \dots (ii)$$

Using (i) and (ii), we have $\|g\|_Z - \|h\|_Z \leq \|g * h\|_Z$ as desired.

2. This can be proved using axiom (N3) and induction. □

Definition 10. Let $\{g_n\}$ be a sequence on the generalized normed Z -module $(G, \|\cdot\|_Z)$.

1. Let $g \in G$. If for all $\varepsilon > 0$ and $n \geq N$, there exists such that

$$\|g * g_n^{-1}\|_Z < \varepsilon,$$

then we say $\{g_n\}$ converges to $g \in G$ (or $\{g_n\}$ is convergent). We denote it by

$$\lim_{n \rightarrow \infty} g_n = g \text{ or } g_n \rightarrow g.$$

2. If for $\varepsilon > 0$ and $m, n \in N$, there exists $N \in N$ such that for all $m, n \geq N$,

$$\|g_m * g_n^{-1}\|_Z < \varepsilon,$$

then $\{g_n\}$ is called a Cauchy sequence.

Example 11. Let G, H be Z -modules on a ring Z and let us show the cartesian product of G and H as $G \times H = \{(g, h) : g \in G, h \in H\}$. Let $\|\cdot\|_{Z_1}$ be the generalized norm on G and let $\|\cdot\|_{Z_2}$ be the generalized norm on H . Then $\{(g_n, h_n)\}$ is a Cauchy sequence on $G \times H$ if and only if $\{g_n\}$ is a Cauchy sequence on G and $\{h_n\}$ is a Cauchy sequence on H .

Solution:($\Rightarrow$) Let $\{(g_n, h_n)\}$ be a Cauchy sequence on $G \times H$. So, for all $m, n \geq N$, there exists some $N \in N$ such that

$$\|g_n * g_m^{-1}, h_n * h_m^{-1}\|_Z = \|(g_n, h_n) * (h_m, g_m)^{-1}\|_Z \leq \varepsilon.$$

Thus, for all $m, n \geq N$,

$$\begin{aligned}\|g_n * g_m^{-1}\|_{Z_1} &\leq \|g_n * g_m^{-1}\|_{Z_1} + \|h_n * h_m^{-1}\|_{Z_2} \\ &= \|(g_n * g_m^{-1}) * (h_n * h_m^{-1})\|_Z \\ &\leq \varepsilon\end{aligned}$$

and

$$\begin{aligned}\|h_n * h_m^{-1}\|_{Z_2} &\leq \|g_n * g_m^{-1}\|_{Z_1} + \|h_n * h_m^{-1}\|_{Z_2} \\ &= \|(g_n * g_m^{-1}) * (h_n * h_m^{-1})\|_Z \\ &\leq \varepsilon\end{aligned}$$

As a result, $\{g_n\}$ is a Cauchy sequence on G and $\{h_n\}$ is a Cauchy sequence on H .

($\Leftarrow$) Let $\{g_n\}$ be a Cauchy sequence on G and $\{h_n\}$ be a Cauchy sequence on H . For $m, n \geq N_1$, there exists $N_1 \in N$

$$\|g_m * g_n^{-1}\|_{Z_1} \leq \frac{\varepsilon}{2}$$

and for $m, n \geq N_2$, there exists $N_2 \in N$ such that

$$\|h_m * h_n^{-1}\|_{Z_2} \leq \frac{\varepsilon}{2}.$$

Let $N_0 = \max\{N_1, N_2\}$. Then for all $m, n \geq N_0$, we have

$$\begin{aligned}\|(g_n, h_n) * (g_m, h_m)^{-1}\|_Z &= \|(g_n * g_m^{-1}), (h_n * h_m^{-1})\|_Z \\ &\leq \|(g_n * g_m^{-1})\|_{Z_1} * \|(h_n * h_m^{-1})\|_{Z_2} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

This completes the proof that $\{(g_n, h_n)\}$ is a Cauchy sequence on G .

Definition 12. Let $(G, \|\cdot\|_Z)$ be a normed Z -module and A be a subset of $(G, \|\cdot\|_Z)$.

$$d(A) = \sup\{\|(g_1 * g_2^{-1})\|_Z : g_1 \in A, g_2 \in A\} \geq 0$$

is called the diameter of A . If the diameter of a subset $A \subset G$ is finite, then A is called a bounded set.

Definition 13. Let $(G, \|\cdot\|_Z)$ be a generalized normed Z -module, $g_0 \in G$ and $r > 0$. Then

$$S_r(g_0) = \{g \in G : \|(g * g_0^{-1})\|_Z < r\}$$

is called the open ball with center g_0 and radius r .

$$\bar{S}_r(g_0) = \{g \in G : \|(g * g_0^{-1})\|_Z \leq r\}$$

is called the closed ball with center g_0 and radius r . And

$$\bar{\sigma}_r(g_0) = \{g \in G : \|(g * g_0^{-1})\|_Z = r\}$$

is called the surface of the ball with center g_0 and radius r .

Theorem 14. *Let (g_n) be a sequence on the normed space $(G, \|\cdot\|_Z)$ and let $g_0 \in G$. If*

$$\lim_{n \rightarrow \infty} g_n = g_0,$$

then the following are true:

1. *The limit g_0 is unique.*
2. *(g_n) is bounded.*
3. *The limits of all (g_{n_k}) subsequences of (g_n) are also g_0 .*
4. *For all $a, a_n \in K$, if $(a_n) \rightarrow a$ then $a_n \cdot g_n \rightarrow a \cdot g_0$.*
5. *If, in addition, for $h, h_0 \in G$, then $g_n + h_n \rightarrow g_0 + h_0$.*

Proof. For all $g, h \in G$, let $d(g, h) = \|g - h\|_Z$. So (G, d) is a metric Z -module.

1. Let

$$\lim_{n \rightarrow \infty} \|(g_n * g_0^{-1})\|_Z \Rightarrow \lim_{n \rightarrow \infty} d(g_n, g_0) = 0$$

$$\lim_{n \rightarrow \infty} \|(g_n * g_1^{-1})\|_Z \Rightarrow \lim_{n \rightarrow \infty} d(g_n, g_1) = 0.$$

Then

$$0 \leq d(g_0, g_1) \leq d(g_0, g_n) + d(g_n, g_1) = 0 + 0.$$

By the M1 axiom $g_0 = g_1$.

2. For a fixed point $a \in G$, by the M3 axiom, for all $n \in N$

$$\lim_{n \rightarrow \infty} \|(g_n * a^{-1})\|_Z \Rightarrow \lim_{n \rightarrow \infty} d(g_n, a) = 0$$

and

$$0 \leq d(g_n, a) \leq d(g_n, g_0) + d(g_0, a).$$

As

$$\lim_{n \rightarrow \infty} \|(g_n * g_0^{-1})\|_Z = 0,$$

for all $\varepsilon > 0$, there exists $n_1 \in N$ such that for all $n > n_1$

$$d(g_n, g_0) \leq 1.$$

Thus, if we take

$$M = \max\{d(g_1, g_0), d(g_2, g_0), \dots, d(g_{n_1}, g_0), 1 + d(g_0, a)\}$$

and for all $n \in N$, as $d(g_n, a) \leq M$ it is clear that $g_n \in S_M^-(a)$. This concludes that (g_n) is a bounded sequence in G .

3. Let (g_{n_k}) be subsequence of (g_n) .

As

$$\lim_{n \rightarrow \infty} g_n = g_0,$$

for all $\varepsilon > 0$, there exists $n_\varepsilon \in N$ such that for all $n > n_\varepsilon$

$$\|g_n * g_0^{-1}\|_Z < \varepsilon.$$

So, for

$$\lim_{k \rightarrow \infty} n_k = \infty,$$

there exists $k_\varepsilon \in N$ such that for all $k > k_\varepsilon$ we have

$$\|(g_n * g_0)\|_Z < \varepsilon \Rightarrow \lim_{k \rightarrow \infty} g_{n_k} = g_0.$$

4. As (g_n) is a convergent sequence in G , for all $n \in N$ there exists $M > 0$ such that

$$\|g_n\|_Z < M.$$

Let $\varepsilon > 0$ and $\lim_{n \rightarrow \infty} a_n = a$. So there exists $n'_\varepsilon \in N$ such that for all $n > n'_\varepsilon$, we have

$$|a_n - a| < \frac{\varepsilon}{2M}.$$

Let $\lim_{n \rightarrow \infty} g_n = g_0$. So there exists $n''_\varepsilon \in N$ such that for all $n > n''_\varepsilon$, we have

$$|g_n - g_0| < \frac{\varepsilon}{2(|a| + 1)}.$$

Now, let $n_\varepsilon = \max\{n', n''\}$. For all $n > n_\varepsilon$,

$$\begin{aligned} \|(a_n g_n) * (a g_0)^{-1}\|_Z &= \|(a_n * a^{-1}) g_n * a(g_n * g_0^{-1})\|_Z \\ &\leq |(a_n * a^{-1})| \|g_n\|_Z + |a| \|(g_n * g_0^{-1})\|_Z \\ &\leq \frac{\varepsilon}{2M} \cdot M + |a| \cdot \frac{\varepsilon}{2(|a|+1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This shows that

$$\lim_{n \rightarrow \infty} \|(a_n g_n) * (a g_0)^{-1}\|_Z = 0.$$

5. Let $\varepsilon > 0$ and $\lim_{n \rightarrow \infty} g_n = g_0$ So there exists $n'_\varepsilon \in N$ such that for all $n > n'_\varepsilon$, we have

$$|g_n * g_0^{-1}| < \frac{\varepsilon}{2}.$$

Let $\lim_{n \rightarrow \infty} h_n = h_0$. So there exists $n''_\varepsilon \in N$ such that for all $n > n''_\varepsilon$, we have

$$|h_n * h_0^{-1}| < \frac{\varepsilon}{2}.$$

Now, let $n_\varepsilon = \max\{n', n''\}$.

$$\begin{aligned} \|(g_n * h_n) * (g_0 * h_0)^{-1}\|_Z &= \|g_n * g_0^{-1} * h_n * h_0^{-1}\|_Z \\ &\leq \|g_n * g_0^{-1}\|_Z + \|h_n * h_0^{-1}\|_Z \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, as $n \rightarrow \infty$, $g_n * h_n \rightarrow g_0 * h_0$.

□

4. Conclusion

In this paper, We introduce for the first time the concept of normed Z -modules which is more effective and useful and studied some of its properties. Also the basic operations on normed Z -modules the cartesian product of normed Z -modules, Cauchy sequence of normed Z -modules, metric of normed Z -modules have been defined.

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