ON WEAKLY gw -CLOSED SETS IN w -SPACES

Won Keun Min

Department of Mathematics
Kangwon National University
Chuncheon, 24341, KOREA

Abstract: The purpose of this note is to introduce the notions of weakly gw -closed set and weakly gw -open set in w -spaces and to study some basic properties of such the notions.

AMS Subject Classification: 54A05, 54B10, 54C10, 54D30

Key Words: w -spaces, gw -open, gw -closed, weakly gw -open, weakly gw -closed

1. Introduction

Siwiec [19] introduced the notions of weak neighborhoods and weak base in a topological space. We introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in [13]. The weak neighborhood system induces a weak neighborhood space which is independent of neighborhood spaces [4] and general topological spaces [2]. The notions of weak structure and w -space were investigated in [14]. In fact, the set of all g -closed subsets [5] in a topological space is a kind of weak structure.

The one purpose of this research is to extend the notion of generalize w -open sets (generalize w -closed sets) in w -spaces. So, we introduce the new notions of weakly generalized w -closed sets and weakly generalized w -open sets in weak spaces, and investigate some basic properties of such notions.

Received: October 29, 2016

Revised: January 6, 2017

Published: February 1, 2017

© 2017 Academic Publications, Ltd.

url: www.acadpubl.eu

2. Preliminaries

Definition 2.1 ([14]). Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* on X if it satisfies the following:

- (1) $\emptyset \in w_X$ and $X \in w_X$.
- (2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a *w-space* on X . Then $V \in w_X$ is called a *w-open* set and the complement of a *w-open* set is a *w-closed* set.

Let S be a subset of a topological space X . The closure (resp., interior) of S will be denoted by clS (resp., $intS$). A subset S of X is called a *preopen* set [11] (resp., α -open set [18], *semi-open* [6]) if $S \subset int(cl(S))$ (resp., $S \subset int(cl(int(S)))$, $S \subset cl(int(S))$). The complement of a preopen set (resp., α -open set, *semi-open*) is called a *preclosed* set (resp., α -closed set, *semi-closed*). The family of all preopen sets (resp., α -open sets, semi-open sets) in X will be denoted by $PO(X)$ (resp., $\alpha(X)$, $SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on X .

A subset A of a topological space (X, τ) is said to be:

- (a) *g-closed* [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X ;
- (b) *gp-closed* [7] if $pCl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X ;
- (c) *gs-closed* [1, 3] if $sCl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X ;
- (d) $g\alpha$ -closed [9] if $\tau^\alpha Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α -open in X where $\tau^\alpha = \alpha(X)$;
- (e) $g\alpha^*$ -closed [8] if $\tau^\alpha Cl(A) \subseteq int(U)$ whenever $A \subseteq U$ and U is α -open in X ;
- (f) $g\alpha^{**}$ -closed [8] if $\tau^\alpha Cl(A) \subseteq int(cl(U))$ whenever $A \subseteq U$ and U is α -open in X ;
- (g) αg -closed [9] if $\tau^\alpha Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X ;
- (h) $\alpha^{**}g$ -closed [9] if $\tau^\alpha Cl(A) \subseteq int(cl(U))$ whenever $A \subseteq U$ and U is open in X .

Then the family τ , $GO(X)$, $g\alpha O(X)$, $g\alpha^* O(X)$, $g\alpha^{**} O(X)$, $\alpha g O(X)$ and $\alpha^{**} g O(X)$ are all weak structures on X . But $PO(X)$, $GPO(X)$ and $SO(X)$ are not weak structures on X . A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* on X [10] if $\emptyset \in m_X$ and $X \in m_X$. Thus clearly every weak structure is a minimal structure.

Definition 2.2 ([14]). Let (X, w_X) be a *w-space*. For a subset A of X , the *w-closure* of A and the *w-interior* of A are defined as follows:

- (1) $wC(A) = \cap \{F : A \subseteq F, X - F \in w_X\}$.
- (2) $wI(A) = \cup \{U : U \subseteq A, U \in w_X\}$.

Theorem 2.3 ([14]). Let (X, w_X) be a w -space and $A \subseteq X$.

- (1) $x \in wI(A)$ if and only if there exists an element $U \in W(x)$ such that $U \subseteq A$.
- (2) $x \in wC(A)$ if and only if $A \cap V \neq \emptyset$ for all $V \in W(x)$.
- (3) If $A \subseteq B$, then $wI(A) \subseteq wI(B)$; $wC(A) \subseteq wC(B)$.
- (4) $wC(X - A) = X - wI(A)$; $wI(X - A) = X - wC(A)$.
- (5) If A is w -closed (resp., w -open), then $wC(A) = A$ (resp., $wI(A) = A$).

3. Main Results

Let (X, w_X) be a w -space and $A \subseteq X$. Then A is called a *generalized w -closed set* (simply, *gw -closed set*) [15] if $wC(A) \subseteq U$, whenever $A \subseteq U$ and U is w -open. If the w_X -structure is a topology, the generalized w -closed set is exactly a generalized closed set in sense of Levine in [5]. Obviously, every w -closed set is generalized w -closed, but in general, the converse is not true.

Now, we introduce an extended notion of generalized w -closed set in w -spaces as the following:

Definition 3.1. Let (X, w_X) be a w -space and $A \subseteq X$. Then A is called a *weakly generalized w -closed set* (simply, *weakly gw -closed set*) if $wCwI(A) \subseteq U$, whenever $A \subseteq U$ and U is w -open.

We obviously obtain the next theorem:

Theorem 3.2. Every generalized w -closed set is weakly generalized w -closed in a given w -space.

In general, the converse of the above theorem is not true as the next example:

Example 3.3. Let $X = \{a, b, c, d\}$ and $w_X = \{\emptyset, \{d\}, \{b, d\}, \{a, d\}, X\}$ be a w -structure in X . Now, consider $A = \{b\}$. Then $wI(A) = \emptyset$ and $wC(A) = \{b, c\}$. So A is weakly gw -closed but not gw -closed.

Let (X, w_X) be a w -space and $P \subseteq X$. Then P is called

- (1) a *w -preopen set* (resp., *w -preclosed set* [16] if $P \subseteq wI(wC(P))$) (resp., $wC(wI(P)) \subseteq P$).
- (2) a *w -regular open set* (resp., *w -regular closed set* [17] if $A = wI(wC(A))$) (resp., $A = wC(wI(A))$).
- (3) a *$w\sigma$ -open set* (resp., *$w\sigma$ -closed set* [17] if $A = wI(A)$) (resp., $A = wC(A)$).

Theorem 3.4. Let (X, w_X) be a w -space. Then the following things hold:

- (1) Every w -preclosed set is weakly gw -closed.
- (2) Every w -regular closed set is weakly gw -closed.
- (3) Every $w\sigma$ -closed set is weakly gw -closed.

Proof. (1) and (2) Obvious.

(3) Let A be a $w\sigma$ -closed set. Then $wCwI(A) \subseteq wC(A) = A$. So obviously A is weakly gw -closed. □

In general, the intersection (union) of two weakly gw -closed sets is not weakly gw -closed:

Example 3.5. For $X = \{a, b, c, d\}$, let a w -structure $w = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, d\}, \{a, b, c\}, X\}$ in X .

(1) Consider $A = \{a, b, c\}$ and $B = \{a, c, d\}$. Then obviously A and B are weakly gw -closed. But since $wCwI(A \cap B) = wCwI(\{a, c\}) = wC(\{a, c\}) = \{a, c, d\}$ and $\{a, c\}$ is w -open, $A \cap B$ is not weakly gw -closed.

(2) Consider $A = \{a, b\}$ and $B = \{b, c\}$. Then obviously A and B are weakly gw -closed. Note that $A \cup B = \{a, b, c\}$ is w -open and $wCwI(A \cup B) = wCwI(\{a, b, c\}) = wC(\{a, b, c\}) = X$. Consequently, $A \cup B$ is not weakly gw -open.

Lemma 3.6. Let (X, w_X) be a w -space and $A, B \subseteq X$. If A and B are $w\sigma$ -open, then $wI(A) \cup wI(B) = wI(A \cup B)$.

Proof. If A and B are $w\sigma$ -open, then $wI(A) \cup wI(B) \subseteq wI(A \cup B) \subseteq A \cup B = wI(A) \cup wI(B)$. Hence, $wI(A) \cup wI(B) = wI(A \cup B)$. □

Lemma 3.7 ([15]). Let (X, w_X) be a w -space and $A, B \subseteq X$. Then the following things hold:

- (1) $wI(A) \cap wI(B) = wI(A \cap B)$.
- (2) $wC(A) \cup wC(B) = wC(A \cup B)$.

Theorem 3.8. Let (X, w_X) be a w -space. If A and B are $w\sigma$ -open and weakly gw -closed sets, then $A \cup B$ is weakly gw -closed.

Proof. Let A and B be any two $w\sigma$ -open and weakly gw -closed sets. Let G be any w -open set such that $A \cup B \subseteq G$. Then $A \subseteq G$ and $B \subseteq G$. Since A and B are weakly gw -closed sets, $wCwI(A) \subseteq G$ and $wCwI(B) \subseteq G$. Now, by Lemma 3.6 and Lemma 3.7, $wCwI(A \cup B) = wC(wI(A) \cup wI(B)) = wCwI(A) \cup wCwI(B) \subseteq G$. Hence, $A \cup B$ is weakly gw -closed. □

Theorem 3.9. *Let (X, w_X) be a w -space. Then if A is a weakly gw -closed set, then $wCwI(A) - A$ contains no any non-empty w -closed set.*

Proof. Suppose that there is a w -closed set F such that $F \subseteq wCwI(A) - A$. Then $A \subseteq X - F$, and since $X - F$ is w -open and A is weakly gw -closed, we have $wCwI(A) \subseteq X - F$. It implies that $F \subseteq X - wCwI(A)$, and so $F \subseteq wCwI(A) \cap (X - wCwI(A)) = \emptyset$. Hence, $F = \emptyset$. $\square$

In general, the converse in Theorem 3.9 is not true as shown in the next example.

Example 3.10. Let $X = \{a, b, c, d\}$ and $w_X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, d\}, X\}$ be a w -structure in X . Consider $A = \{a\}$. Note $wCwI(A) = \{a, d\}$ and $wCwI(A) - A = \{a, d\} - \{a\} = \{d\}$. Since $\{d\}$ is not w -closed, $wCwI(A) - A$ contains no any non-empty w -closed set, but A is not weakly gw -closed.

Theorem 3.11. *Let (X, w_X) be a w -space. Then if A is a weakly gw -closed set and $A \subseteq B \subseteq wCwI(A)$, then B is weakly gw -closed.*

Proof. Let U be any w -open set such that $B \subseteq U$. Since $A \subseteq B \subseteq wCwI(A)$, obviously $wCwI(A) = wCwI(B)$. Since A is weakly gw -closed and $A \subseteq U$, $wCwI(B) \subseteq U$. So B is weakly gw -closed. $\square$

Definition 3.12. Let (X, w_X) be a w -space and $A \subseteq X$. Then A is called a *weakly generalized w -open set* (simply, weakly gw -open set) if $X - A$ is weakly gw -closed.

Theorem 3.13. *Let (X, w_X) be a w -space and $A \subseteq X$. Then A is weakly gw -open if and only if $F \subseteq wIwC(A)$ whenever $F \subseteq A$ and F is w -closed.*

Proof. Obvious. $\square$

Theorem 3.14. *Let (X, w_X) be a w -space and $A \subseteq X$. Then if A is weakly gw -open, then $U = X$, whenever $wIwC(A) \cup (X - A) \subseteq U$ and U is w -open.*

Proof. Let U be any w -open set and $wIwC(A) \cup (X - A) \subseteq U$. Then $X - U \subseteq (X - wIwC(A)) \cap A = wCwI(X - A) \cap A = wCwI(X - A) - (X - A)$. Since $X - A$ is weakly gw -closed, by Theorem 3.9, the w -closed set $X - U$ must be empty. Hence, $U = X$. $\square$

Theorem 3.15. *Let (X, w_X) be a w -space. Then if A is a weakly gw -open set and $wIwC(A) \subseteq B \subseteq A$, then B is weakly gw -open.*

Proof. It is similar to the proof of Theorem 3.11. $\square$

Theorem 3.16. *Let (X, w_X) be a w -space. Then if A is a weakly gw -closed set, then $wIwC(A) - A$ is weakly gw -open.*

Proof. If A is a weakly gw -closed set, then by Theorem 3.11, $\emptyset$ is the only one w -closed subset of $wCwI(A) - A$. So, for the only w -closed subset $\emptyset$ of $wCwI(A) - A$, $\emptyset \subseteq wCwI(A) - A$ and $\emptyset \subseteq wIwC(wCwI(A) - A)$. Hence, $wCwI(A) - A$ is weakly gw -open. $\square$

Theorem 3.17. *Let (X, w_X) be a w -space. Then if A is a weakly gw -open set, then $wIwC(A) \cup (X - A)$ is weakly gw -closed.*

Proof. If A is a weakly gw -open set, then by Theorem 3.14, X is the only one w -open set containing $wIwC(A) \cup (X - A)$. It implies that $wIwC(A) \cup (X - A) \subseteq X$ and $wCwI(wIwC(A) \cup (X - A)) \subseteq X$. So, $wIwC(A) \cup (X - A)$ is weakly gw -closed. $\square$

References

- [1] P. Bhattacharyya and B. K. Lahiri; Semi-generalized closed sets in topology, Indian J. Math., 29(3) (1987), 375-382.
- [2] Á. Császár; Generalized Topology, Generalized Continuity, Acta Math. Hungar., 96 (2002), 351-357.
- [3] J. Dontchev and H. Maki; On sg-closed sets and semi- λ -closed sets, Q & A in General Topology, 15(1997), 259-266.
- [4] D. C. Kent and W. K. Min; Neighborhood Spaces, International Journal of Mathematics and Mathematical Sciences, 32(7) (2002), 387-399.
- [5] N. Levine; Generalized closed sets in topology, Rend. Cir. Mat. Palermo, 19(1970), 89-96.
- [6] N. Levine; Semi-open sets and semi-continuity in topological spaces, Ams. Math. Monthly, 70(1963), 36-41.
- [7] H. Maki, J. Umehara and T. Noiri; Every topological space is pre- $T_{\frac{1}{2}}$, Mem. Fac. Sci. Kochi Univ. Ser. A, 17(1996), 33-42.
- [8] H. Maki, R. Devi and K. Balachandran; Associated topologies of generalized α -closed sets and α -generalized closed sets, Mem. Fac. Sci. Kochi Univ. Ser. A, 15(1994), 51-63.
- [9] H. Maki, R. Devi and K. Balachandran;; Generalized α -closed maps and α -generalized closed maps, Indian J. pure appl. Math., 29(1)(1998), 37-49.
- [10] H. Maki; On generalizing semi-open and preopen sets, Report for Meeting on Topological Spaces Theory and its Applications, August 1996, Yatsushiro College of Technology, 13-18.
- [11] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb; On precontinuous and weak precontinuous mappings, Proc. Math. and Phys. Soc. Egypt, 53 (1982), 47-53.

- [12] W. K. Min; Some Results on Generalized Topological Spaces and Generalized Systems, *Acta Math. Hungar.*, 108 (1-2) (2005), 171-181.
- [13] W. K. Min; On Weak Neighborhood Systems and Spaces, *Acta Math. Hungar.*, 121(3) (2008), 283-292.
- [14] W. K. Min and Y. K. Kim; On Weak Structures and w -spaces, *Far East Journal of Mathematical Sciences*, 97(5) (2015), 549-561.
- [15] W. K. Min and Y. K. Kim; On generalized w -closed sets in w -spaces, *International Journal of Pure and Applied Mathematics*, 110(2) (2016), 327-335.
- [16] W. K. Min and Y. K. Kim, w -preopen sets and W -precontinuity in weak spaces, *International Journal of Mathematical Analysis* 10(21) (2016), 1009-1017.
- [17] W. K. Min and Y. K. Kim; On $w\sigma$ -open sets and w -regular open sets in weak spaces, *Far East Journal of Mathematical Sciences*, 100(12) (2016), 1997-2006.
- [18] O. Njastad; On some classes of nearly open sets, *Pacific Journal of Mathematics*, 15(3)(1964), 961-970.
- [19] F. Siwiec; On Defining a Space by a Weak Base, *Pacific Journal of Mathematics*, 52(1) (1974), 351-357.