A NOTE ON SPECTRUM PRESERVING ADDITIVE MAPS BETWEEN C^* -ALGEBRAS

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Abstract: Let A be a unital C^* -algebra of real rank zero and B a unital semi-simple Banach algebra. Let ϕ be a surjective spectrum preserving additive map from A onto B . Then ϕ is continuous and Jordan isomorphism on self-adjoint element of A .

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1. Introduction

All algebras we deal with in this paper are complex and unital. The identity element of an algebra A will be denoted by e , or e_A , for distinction. For a given algebra A and $a \in A$, $\sigma(a)$ will denote the *spectrum* of a . Given two algebras A, B , a map $\phi : A \rightarrow B$ is called *spectrum preserving* if $\sigma(\phi(a)) = \sigma(a)$ for every $a \in A$. Furthermore, ϕ is said to be *unital* if $\phi(e_A) = e_B$ and it is called *invertibility preserving* if $\phi(a)$ is invertible in B whenever a is invertible in A . Spectrum preserving linear mappings were studied for the first time by G. Frobenius [7]. He proved that a surjective linear mapping $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ which preserves the spectrum has one of the forms $\phi(T) = ATA^{-1}$ or $\phi(T) = A^tTA^{-1}$ for some invertible A .

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Kaplansky [11] asked the below problem:

Let A and B be two semi-simple complex Banach algebras and let ϕ be a spectrum preserving linear mapping from A onto B . Is it true that ϕ is Jordan isomorphism between A and B ?

This problem was motivated by the well-known Gleason-Kahane-Zelazko theorem [12], [10], [17], which constitutes a positive answer to the preceding problem in the commutative case. (The Gleason-Kahane-Zelazko Theorem says: Every unital invertibility preserving linear functional on a unital complex Banach algebra is necessarily multiplicative). Positive answers in the non-commutative case are due to B. Aupetit [1] for von Neumann algebras and to A.R. Sourour [16] for the algebra $B(X)$ of bounded linear operators on a Banach space X . This problem is still unsolved. The most important partial results obtained in this direction are [3], [4], [9], [16].

In [9] Jafarian and Sourour proved that a surjective linear map preserving spectrum from $B(X)$ onto $B(Y)$ is either an isomorphism or an anti-isomorphism where X and Y are complex Banach spaces and $B(X)$ is the Banach algebra of all bounded linear operators acting on X . In [14] Omladic and Semrl proved that the conclusions Jafarian and Sourour [9] is valid under the less restrictive condition that map ϕ is only additive.

Recently Aupetit [1] showed that a spectrum preserving surjective linear map from a Von Neumann algebra onto another is a Jordan isomorphism. In [8] Jafarian asked any spectrum-preserving additive map from a Von Neumann algebra onto another is a Jordan isomorphism.

In this paper we prove that if A be a unital C^* -algebra of real rank zero and B a unital semi-simple Banach algebra. Let ϕ be a surjective spectrum preserving additive map from A onto B . Then ϕ is continuous and Jordan isomorphism on self-adjoint element of A . Also, we obtain some interesting results in direction.

The following results will be often used in the sequel.

Lemma 1.1. [6, Lemma 5.1.5] *Let $(A, \|\cdot\|)$ be a unital Banach algebra, let $a \in A$, and let $\epsilon > 0$. Then there is a norm $\|\cdot\|$ on A such that $\|\cdot\|$ is equivalent to $\|\cdot\|$, and $\|e\| = 1$, and $\|a\| \leq r_A(a) + \epsilon$.*

Lemma 1.2. [15] *Let A be a Banach algebra, let $p \in A[\mathbb{C}]$, where $A[\mathbb{C}]$ denote the algebra of all polynomials with coefficients in A . Then*

$$r_A(p(1))^2 \leq \sup_{|z|=R} r_A(p(z)) \cdot \sup_{|z|=\frac{1}{R}} r_A(p(z)),$$

for every $R > 1$.

Theorem 1.3. [1, Theorem 1.1] *An element p of a semi-simple complex Banach algebra A is idempotent if and only if $\sigma_A(p) \subseteq \{0, 1\}$ and there exist real numbers $r > 0$ and $C > 0$ such that*

$$\sigma_A(a) \subseteq \sigma_A(p) + c\|a - p\|$$

for all $a \in A$ with $\|a - p\| \leq r$.

Definition 1.4. Let A and B be two unital Banach algebras. Let $\phi : A \rightarrow B$ be a additive map. Then the separating space of ϕ is $C(\phi) = \{b \in B : \text{there exists } a_n \rightarrow 0 \text{ in } A \text{ with } \phi(a_n) \rightarrow b\}$.

It is clear that ϕ is continuous if and only if $C(\phi) = \{0\}$.

2. Main Results

Let A be a $*$ -algebra, we say elements a, b of A are unitary equivalent if there exists a unitary element $u \in A$ such that $b = uau^*$. Note that $\sigma_A(a) = \sigma_A(b)$ if a and b are unitary equivalent, [13, p. 36].

Theorem 2.1. *Let A be a Banach $*$ -algebra and B be semi-simple Banach algebra and let $\phi : A \rightarrow B$ be a surjective spectrum preserving additive map and $b \in C(\phi)$, where $C(\phi)$ is separating space of ϕ . Then we have:*

- (i) ϕ is injective and $\phi(e_A) = e_B$.
- (ii) $r_B(\phi(a)) \leq r_B(\phi(a) - b)$, for every $a \in A$.
- (iii) $r_B(b) = 0$ and ϕ is continuous.

Proof. (i) see [1, Proposition 2.1 (i) and (ii)].

- (ii) Let (a_n) be in A with $a_n \rightarrow 0$ and $\phi(a_n) \rightarrow b$ as $n \rightarrow \infty$, and take $a \in A$.

For each $\epsilon > 0$, we may, by Lemma 1.1, choose norms on A and B which are equivalent to the given norms and which are such that

$$\|a\| \leq r_A(a) + \epsilon, \|\phi(a) - b\| \leq r_B(\phi(a) - b) + \epsilon \quad (a \in A, b \in B).$$

We apply Lemma 1.2 in the case where $p = (\phi(a) - \phi(a_n)) + (\phi(a_n))\mathbb{C}$ be long to $B[\mathbb{C}]$, where $n \in N$. We have $p(1) = \phi(a)$, and so, for each $R > 1$,

$$r_B(\phi(a))^2 \leq \sup_{|z|=R} r_B(p(z)) \cdot \sup_{|z|=\frac{1}{R}} r_B(p(z)) \quad (*).$$

Now $r_B(p(z)) \leq \|\phi(a) - \phi(a_n)\| + |z|\|\phi(a_n)\|$. Since ϕ is onto, for every $n \in N$, there exists $x_n \in A$ such that $\phi(x_n) = z\phi(a_n)$. We have $p(z) = \phi(a) - \phi(a_n) + z\phi(a_n) = \phi(a - a_n + x_n)$, and so, by hypothesis,

$$r_B(p(z)) = r_B(\phi(a - a_n + x_n)) = r_A(a - a_n + x_n) \leq \|a - a_n\| + \|x_n\|.$$

Thus, from (*), we have

$$r_B(\phi(a))^2 \leq (\|a - a_n\| + \|x_n\|)(\|\phi(a) - \phi(a_n)\| + \frac{\|\phi(a_n)\|}{R}).$$

This holds for each $n \in N$, and so, letting $n \rightarrow \infty$, we see that

$$r_B(\phi(a))^2 \leq (\|a\| + \|x_n\|)(\|\phi(a) - b\| + \frac{b}{R}).$$

Note that $x_n \rightarrow 0$, since

$$\sigma_A(x_n) = \sigma_B(\phi(x_n)) = \sigma_B(z\phi(a_n)) = z\sigma_A(a_n) = \sigma_A(za_n).$$

Hence x_n and za_n are unitary equivalent, so that for every $n \in N$, there exists a unitary element $u_n \in A$ such that $x_n = u_n a_n u_n^*$. It follows that $\|x_n\| \leq \|u_n\| \|a_n\| \|u_n^*\| \rightarrow 0$, since $a_n \rightarrow 0$. Therefore $x_n \rightarrow 0$. So

$$r_B(\phi(a))^2 \leq \|a\|(\|\phi(a) - b\| + \frac{b}{R}).$$

But This holds for each $R > 1$, and so, letting $R \rightarrow \infty$, we obtain,

$$r_B(\phi(a))^2 \leq \|a\| \|\phi(a) - b\| \leq (r_A(a) + \epsilon)(r(\phi(a) - b) + \epsilon).$$

Finally this holds for each $\epsilon > 0$, and so $r_B(\phi(a))^2 \leq r_A(a)r_B(\phi(a) - b)$, for every $a \in A$.

Since $r_A(a) = r_B(\phi(a))$, for every $a \in A$, we have

$$r_A(a)r_B(\phi(a)) = r_B(\phi(a))^2 \leq r_A(a)r_B(\phi(a) - b),$$

for every $a \in A$, and so $r_B(\phi(a)) \leq r_B(\phi(a) - b)$.

(iii) Since ϕ is onto, there exists $t \in A$ such that $\phi(t) = b$. In (ii) taking $a = t$ we get $r_B(b) = 0$.

Now we prove that ϕ is continuous, taking $x = t + a$ for every $a \in A$. By (ii) we have $r_B(b + \phi(a)) \leq r_B(\phi(a))$ for every $a \in A$. Hence $r_B(b + q) = 0$ for every quasi-nilpotent element $q \in B$. By Zemanek's characterization of the radical [2, Theorem 5.3.1], $b \in \text{rad}(B) = \{0\}$, and so $b = 0$. Therefore ϕ is continuous. $\square$

A additive map is said to be idempotent preserving if it maps every idempotent to an idempotent.

Theorem 2.2. *Let A be a Banach $*$ -algebra and B a semi-simple Banach algebra and let $\phi : A \rightarrow B$ be a surjective spectrum preserving additive map. Then ϕ is $\mathbb{R}$ -linear and idempotent preserving.*

Proof. Evidently every additive map between two algebra is $\mathbb{Q}$ -linear. Let $a \in A$ and $r \in \mathbb{R}$. Choose sequence $\{q_n\}_{n=1}^{\infty} \subseteq \mathbb{Q}$ with $\lim q_n = r$ as $n \rightarrow \infty$. Then $\lim q_n a = ra$, we obtain that $\phi(ra) = \phi(\lim q_n a) = \lim \phi(q_n a) = \lim q_n \phi(a) = r\phi(a)$. Hence ϕ is $\mathbb{R}$ -linear. Now we prove that ϕ is idempotent preserving. Let $p = p^2$ be an idempotent element of A . If $p = 0$ then $\phi(p) = 0$, and also $p = e_A$ then $\phi(e_A) = e_B$ are idempotents in B . So suppose $p \notin \{0, e_A\}$, then $\sigma_A(p) \subseteq \{0, 1\}$. By Theorem 1.3, there exist real numbers $r > 0$ and $c > 0$ such that

$$\sigma_A(a) \subseteq \sigma_A(p) + c\|a - p\| \quad \|a - p\| < r.$$

Since ϕ is spectrum preserving by Theorem 2.1, maps ϕ, ϕ^{-1} are continuous, consequently,

$$\sigma_B(\phi(a)) \subseteq \{0, 1\} + \frac{c}{\alpha}\|\phi(a) - \phi(p)\|,$$

since the mapping ϕ is onto, we obtain

$$\sigma_B(b) \subseteq \{0, 1\} + \frac{c}{\alpha}\|b - \phi(p)\|,$$

for $b \in B$ and b near to $\phi(p)$. Hence, Theorem 1.3, implies that $\phi(p)$ is an idempotent element of B . $\square$

It is easily seen that an idempotent preserving additive map must send a set of mutually orthogonal idempotents to a set of mutually orthogonal idempotents (two idempotents p_1 and p_2 are orthogonal if $p_1 p_1 = p_2 p_1 = 0$).

We recall that a C^* -algebra A is of real rank zero if the set of all real linear combinations of orthogonal self-adjoint idempotents is dense in the set of all self-adjoint elements of A , [5]. It is clear that every von Neumann algebra is a C^* -algebra of real rank zero. In particular, $B(H)$, the algebra of all bounded linear operators on a complex Hilbert space, has real rank zero.

Theorem 2.3. *Let A be a unital C^* -algebra of real rank zero and B a unital semi-simple Banach algebra. Let $\phi : A \rightarrow B$ be a surjective spectrum preserving additive map. Then ϕ is a Jordan isomorphism on self-adjoint element.*

Proof. Let $a \in A$ such that $a^* = a$ which is a real linear combination of orthogonal self-adjoint idempotents, i.e., $a = \sum_{i=1}^n r_i p_i$ with $r_i \in \mathbb{R}$ and $p_i^2 = p_i = p_i^*$ and $p_i p_j = p_j p_i$ if $i \neq j$. ϕ is continuous and maps mutually orthogonal

Hermitian idempotents to mutually orthogonal idempotents by Theorem 2.1 and Theorem 2.2. So $\phi(a^2) = \phi(a)^2$. Now, since A is a C^* -algebra of real rank zero, the set of self-adjoint elements, which are finite real linear combinations of orthogonal self-adjoint idempotents, is dense in the set of all self-adjoint elements in A , we see that $\phi(a^2) = \phi(a)^2$ hold for every self-adjoint elements a . $\square$

Remark 2.4. Since every von Neumann algebra is a C^* -algebra of real rank zero, so Theorem 2.3, is valid if A be a von Neumann algebra

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