A NOTE ON AF-GROUPS AND FNS-GROUPS

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Abstract: The quantities $c(G)$, $q(G)$ and $p(G)$ for finite groups were defined by H. Behraves. In this paper, we will calculate these quantities for L_{p^n} , where p is an odd prime, $n \in \mathbb{N}$, $n \geq 3$.

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1. Introduction

Let G be a finite p -group. G is an FNS group, if for any normal subgroup H of G , either $G' \leq H$ or $H \leq Z(G)$. For instance if p be a prime and G be a nonabelian group of order p^3 or p^4 , then G is an FNS-group, [8, Proposition 2.1].

A nonabelian p -group G is called AF-group (Abelian Factor) if every proper quotient of G is abelian. For example the groups D_8 and Q_8 are AF-groups.

By [8, Proposition 2.2], every AF-group is FNS-group. It is known that a group H is called capable if there exists a group G such that $H \cong G/Z(G)$.

By [8, Theorem 2.2] the only capable AF- p -groups are:

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a) D_8 if $p=2$.

b) L_{p^3} if $p \neq 2$. That is:

$$L_{p^n} = \langle a, b, c : a^{p^{n-2}} = b^p = c^p = [a, b] = [a, c] = 1, \\ [b, c] = a^{p^{n-3}} \rangle, \quad n \in N, n \geq 3.$$

If G is a finite linear group of degree n , that is, a finite group of automorphisms of an n -dimensional complex vector space, we shall say that G is quasi-permutation group if the trace of every element of G has a non-negative rational integer. The trace of the automorphism corresponding to an element x of G equal to the number of letters left fixed by x , and so is non-negative integer. Thus, a permutation group of degree n has a representation as a quasi-permutation group of degree n . See [10].

By a quasi-permutation matrix we mean a square matrix over the complex field C with non-negative integral trace. Thus every permutation matrix over C is a quasi-permutation matrix. For a given finite group G , let $p(G)$ denote the minimal degree of a faithful permutation representation of G , let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field Q , and let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices. See [3].

It is easy to see $c(G) \leq q(G) \leq p(G)$, where G is a finite group.

Now we want to calculating $c(G)$, $q(G)$ and $p(G)$, for capable AF- p -groups and in particular for L_{p^n} , $n \geq 4$ and the central product of two groups L_{p^n} , where p is an odd prime, $n \in N$, $n \geq 3$.

2. Quasi-Permutation Representation of L_{p^n}

In this section we want to calculating quasi-permutation representation of L_{p^n} and the central product of two groups L_{p^n} , where p is an odd prime, $n \in N$, $n \geq 3$.

Theorem 1. *Let G be a finite p -group and $|G'|=p$. Then G is FNS- p -group.*

Proof. Let H be a normal subgroup in G . Since G has class 2, then $G' \leq Z(G)$ and $1 \leq [G, H] \leq G'$. Therefore we have two possible case.

If $[G, H]=1$, then $H \leq Z(G)$.

If $[G, H] = G'$ and since H be a normal subgroup in G , then $[G, H] \leq H$ and therefore $G' \leq H$. □

Theorem 2. *Let $|G'|=p$ a prime. Assume that $G' \leq Z(G)$. Then $\chi(1)^2 = |G : Z(G)|$ for every non linear $\chi \in \text{Irr}(G)$.*

Proof. Let χ be a non-linear, irreducible character of G . From [7, Corollary 2.28], we have $Z(G) = \bigcap \{Z(\chi) : \chi \in \text{Irr}(G)\}$. So $G' \leq Z(G) \leq Z(\chi)$, $G/Z(\chi)$ is abelian. From [7, Theorem 2.31], we have $\chi(1)^2 = |G : Z(\chi)|$. Since $|G'| = p$, a prime, either $G' \cap \ker \chi = 1$ or $G' \leq \ker \chi$. Suppose $G' \leq \ker \chi$. Then $G/\ker \chi$ is abelian. By [7, Lemma 2.27(e)], $\chi = \chi(1)\lambda$ for some linear character λ of $Z(\chi)$, contracting that χ is non-linear and irreducible character of G . Hence $G' \cap \ker \chi = 1$. We will show that $Z(\chi) \leq Z(G)$.

Suppose $z \in Z(\chi)$. By [7, Lemma 2.27(f)], we have $Z(\chi)/\ker \chi = Z(G/\ker \chi)$. So, for any $g \in G$, $[z, g] \in \ker \chi$, i.e $[z, g] \in \ker \chi \cap G' = 1$. Hence $z \in Z(G)$ and so $Z(\chi) \leq Z(G)$ and $\chi(1)^2 = |G : Z(G)|$. $\square$

Theorem 3. [5, Theorem B] *Let G be a finite p -group such that $|G : Z(G)|$ be a square and we denote character degree of G by $\text{cd}(G)$. Then the following statements are equivalent:*

- a) $\text{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$.
- b) For any normal subgroup H of G , either $G' \leq H$ or $H \leq Z(G)$.

Corollary 4. *Let G be a finite p -group such that $|G : Z(G)|$ be a square. Then G is an FNS- p -group iff $\text{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$.*

Proof. The result is immediate by Theorem 3. $\square$

Corollary 5. *Let G be a finite p -group such that $G/Z(G) \cong Z_p \times Z_p$. Then G is an FNS- p -group.*

Proof. Since G is a finite p -group and $G/Z(G) \cong Z_p \times Z_p$, therefore $G/Z(G)$ is abelian and $|G : Z(G)| = p^2$. By Corollary 4, that is enough to prove $\text{cd}(G) = \{1, p\}$.

Since $\chi(1)^2 |G : Z(G)|$ and $|G : Z(G)| = p^2$ therefore $\chi(1)^2 |p^2$. Then $|\chi(1)| = 1$ or $|\chi(1)| = p$ and $\text{cd}(G) = \{1, p\}$. $\square$

Theorem 6. [1, Theorem 4.12] *Let G be a finite p -group of class 2 and let $Z(G)$ be cyclic. Then $c(G) = |Z(G)| \cdot |G : Z(G)|^{1/2}$. Moreover when $p \neq 2$ then $c(G) = q(G) = p(G)$.*

Now we want to calculating $c(G)$, $q(G)$ and $p(G)$, for capable AF- p -groups.

Lemma 1. *Let G be capable AF- p -group. Then By [8, Theorem 2.2], G is of the form:*

- a) D_8 if $p=2$,

b) L_{p^3} if $p \neq 2$,

in this case we have $c(D_8) = 4, c(L_{p^3}) = p^2$.

Proof. Let $G = L_{p^3}$. Then

$$G = \langle a, b, c : a^p = b^p = c^p = [a, b] = [a, c] = 1,$$

$$[b, c] = a \rangle, Z(G) = \langle a \rangle = G',$$

and $|G : Z(G)| = p^2$. So $G/Z(G)$ is abelian and $G' \leq Z(G)$. Hence G has class 2, by Theorem 6,

$$c(G) = q(G) = p(G) = |Z(G)| \cdot |G : Z(G)|^{1/2} = p^2.$$

If $G = D_8$, then by [2, Remark (3), part (b)], $c(G) = q(G) = p(G) = 4$. $\square$

Theorem 7. [4, Theorem 2] Let G be a nilpotent group and $G = H \times K$. Then $p(G) = p(H) + p(K)$.

Theorem 8. Let G be a finite p -group of the form L_{p^n} . Then

- a) $G/G' \cong C_{p^{n-3}} \times C_p \times C_p, |Z(G)| = p^{n-2}$,
- b) The number of nonlinear irreducible character of L_{p^n} equal to $\phi(|Z(G)|)$,
- c) $c(G) = q(G) = p(G) = p^{n-1}$.

Proof. a) Let

$$G = L_{p^n} = \langle a, b, c : a^{p^{n-2}} = b^p = c^p = [a, b] = [a, c] = 1, [b, c] = a^{p^{n-3}} \rangle.$$

Then we have:

$$G' = \langle a^{p^{n-3}} \rangle, |G'| = p, |G : G'| = p^{n-1}, Z(G) = \langle a \rangle \cong C_{p^{n-2}}, |Z(G)| = p^{n-2}.$$

Furthemore since

$$G/G' = \langle aG', bG', cG' \rangle, (bG')^p = G', (cG')^p = G', (aG')^{p^{n-3}} = a^{p^{n-3}}G' = G'$$

so

$$G/G' = \langle a, b, c : a^{p^{n-3}} = b^p = c^p = [a, b] = [a, c] = 1, [b, c] = a^{p^{n-3}} \rangle,$$

$$G/G' \cong C_{p^{n-3}} \times C_p \times C_p.$$

b) Since

$$|G| = |G : G'| + \sum_{\chi \in \text{Irr}(G), \chi(1) > 1} \chi(1)^2$$

so

$$p^n = p^{n-1} + \alpha.p^2 \Rightarrow \alpha = p^{n-2} - p^{n-3} = p^{n-2}(1 - 1/p) = \phi(p^{n-2}) = \phi(|Z(G)|)$$

therefore the number of nonlinear irreducible character of L_{p^n} equal to $\phi(|Z(G)|)$.

c) Since $|G/Z(G)| = p^2$, so $G/Z(G)$ is abelian and $G' \leq Z(G)$. Hence G has class 2, by Theorem 6,

$$c(G) = q(G) = p(G) = |Z(G)| \cdot |G : Z(G)|^{1/2} = p^{n-1}. \quad \square$$

Notation. We denote the central product of groups H, G by $H * G$.

Lemma 2. *Let H be the central product of 2 copies of the group L_{p^n} , with $n \geq 3$. Then:*

a) For $i, j \in \{1, 2\}$ we have:

$$H = \langle a, b_1, b_2, c_1, c_2 : a^{p^{n-2}} = b_i^p = c_i^p = 1,$$

$$[a, b_i] = [a, c_i] = [b_i, b_j] = [c_i, c_j] = 1, [b_i, c_j] = a^{\delta_{ij}.p^{n-3}} > .$$

b) $c(H) = q(H) = p(H) = p^n$.

Proof. a) By an easy calculation and by definition of group L_{p^n} , the first part is obviously.

b) Since $H = L_{p^n} * L_{p^n}$, then $H' = \langle a^{p^{n-3}} \rangle$ and $Z(H) = \langle a \rangle \cong C_{p^{n-2}}$ and so $H' \leq Z(H)$. Therefore by Theorem 6,

$$c(H) = q(H) = p(H) = |Z(H)| \cdot |H : Z(H)|^{1/2} = p^{n-2}p^2 = p^n. \quad \square$$

Theorem 9. *Let H be the central product of r copies ($r \geq 1$) of the group L_{p^n} , with $n \geq 3$. Then*

a) For $i, j \in \{1, 2, \dots, r\}$ we have:

$$H = \langle a, b_1, b_2, \dots, b_r, c_1, c_2, \dots, c_r : a^{p^{n-2}} = b_i^p = c_i^p = 1,$$

$$[a, b_i] = [a, c_i] = [b_i, b_j] = [c_i, c_j] = 1, [b_i, c_j] = a^{\delta_{ij}.p^{n-3}} > ,$$

b) $c(H) = p(H) = q(H) = p^{n+r-2}$.

Proof. a) By definition of group L_{p^n} we have :

$$\begin{aligned} a_1^{p^{n-2}} &= b_1^p = c_1^p = [a_1, b_1] = [a_1, c_1] = 1, [b_1, c_1] = a_1^{p^{n-3}}, \\ a_2^{p^{n-2}} &= b_2^p = c_2^p = [a_2, b_2] = [a_2, c_2] = 1, [b_2, c_2] = a_2^{p^{n-3}}, \\ &\vdots \\ a_r^{p^{n-2}} &= b_r^p = c_r^p = [a_r, b_r] = [a_r, c_r] = 1, [b_r, c_r] = a_r^{p^{n-3}}. \end{aligned}$$

Then

$$a^{p^{n-2}} = b_i^p = c_i^p = [a, b_i] = [a, c_i] = 1, [b_i, c_i] = a^{p^{n-3}}.$$

By the second property of [4, Definition 19.3], we have $[b_i, b_j] = [c_i, c_j] = 1, [b_i, c_j] = 1$ and therefore

$$\begin{aligned} H &= \langle a, b_1, b_2, \dots, b_r, c_1, c_2, \dots, c_r : a^{p^{n-2}} = b_i^p = c_i^p = 1, \\ &[a, b_i] = [a, c_i] = [b_i, b_j] = [c_i, c_j] = 1, [b_i, c_j] = a^{\delta_{ij} \cdot p^{n-3}} \rangle \end{aligned}$$

where $i, j \in \{1, 2, \dots, r\}$.

b) By structure of group H , we have:

$$Z(H) = \langle a \rangle, H' = \langle a^{p^{n-3}} \rangle.$$

Then by Theorem 6,

$$c(H) = p(H) = q(H) = |Z(H)| \cdot |H : Z(H)|^{1/2} = p^{n-2} \cdot p^r = p^{n+r-2}. \quad \square$$

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