A NOTE ON AF-GROUPS AND FNS-GROUPS

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Abstract: The quantities $c(G)$, $q(G)$ and $p(G)$ for finite groups were defined by H. Behravesh. In this paper, we will calculate these quantities for $L_{p^n}$, where $p$ is an odd prime, $n \in \mathbb{N}$, $n \geq 3$.

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1. Introduction

Let $G$ be a finite $p$-group. $G$ is an FNS group, if for any normal subgroup $H$ of $G$, either $G' \leq H$ or $H \leq Z(G)$. For instance if $p$ be a prime and $G$ be a nonabelian group of order $p^3$ or $p^4$, then $G$ is an FNS-group, [8, Proposition 2.1].

A nonabelian $p$-group $G$ is called AF-group(Abelian Factor) if every proper quotient of $G$ is abelian. For example the groups $D_8$ and $Q_8$ are AF-groups.

By [8, Proposition 2.2], every AF-group is FNS-group. It is known that a group $H$ is called capable if there exists a group $G$ such that $H \cong G/Z(G)$.

By [8, Theorem 2.2] the only capable AF-$p$-groups are:

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a) $D_8$ if $p=2$.

b) $L_{p^3}$ if $p \neq 2$. That is:

$$L_{p^n} = \langle a, b, c : a^{p^n-2} = b^p = c^p = [a, b] = [a, c] = 1, [b, c] = a^{p^n-3} >, \quad n \in \mathbb{N}, \quad n \geq 3.$$

If $G$ is a finite linear group of degree $n$, that is, a finite group of automorphisms of an $n$-dimensional complex vector space, we shall say that $G$ is quasi-permutation group if the trace of every element of $G$ has a non-negative rational integer. The trace of the automorphism corresponding to an element $x$ of $G$ equal to the number of letters left fixed by $x$, and so is non-negative integer. Thus, a permutation group of degree $n$ has a representation as a quasi-permutation group of degree $n$. See [10].

By a quasi-permutation matrix we mean a square matrix over the complex field $C$ with non-negative integral trace. Thus every permutation matrix over $C$ is a quasi-permutation matrix. For a given finite group $G$, let $p(G)$ denote the minimal degree of a faithful permutation representation of $G$, let $q(G)$ denote the minimal degree of a faithful representation of $G$ by quasi-permutation matrices over the rational field $Q$, and let $c(G)$ be the minimal degree of a faithful representation of $G$ by complex quasi-permutation matrices. See [3].

It is easy to see $c(G) \leq q(G) \leq p(G)$, where $G$ is a finite group.

Now we want to calculating $c(G), q(G)$ and $p(G)$, for capable AF-$p$-groups and in particular for $L_{p^n}, n \geq 4$ and the central product of two groups $L_{p^n}$, where $p$ is an odd prime, $n \in \mathbb{N}, n \geq 3$.

### 2. Quasi-Permutation Representation of $L_{p^n}$

In this section we want to calculating quasi-permutation representation of $L_{p^n}$ and the central product of two groups $L_{p^n}$, where $p$ is an odd prime, $n \in \mathbb{N}, n \geq 3$.

**Theorem 1.** Let $G$ be a finite $p$-group and $|G'|=p$. Then $G$ is FNS-$p$-group.

**Proof.** Let $H$ be a normal subgroup in $G$. Since $G$ has class 2, then $G' \leq Z(G)$ and $1 \leq [G, H] \leq G'$. Therefore we have two possible case.

If $[G, H]=1$, then $H \leq Z(G)$.

If $[G, H] = G'$ and since $H$ be a normal subgroup in $G$, then $[G, H] \leq H$ and therefore $G' \leq H$. \qed
Theorem 2. Let $|G'|=p$ a prime. Assume that $G' \leq Z(G)$. Then $\chi(1)^2 = |G : Z(G)|$ for every non-linear $\chi \in \text{Irr}(G)$.

Proof. Let $\chi$ be a non-linear, irreducible character of $G$. From [7, Corollary 2.28], we have $Z(G) = \bigcap\{Z(\chi) : \chi \in \text{Irr}(G)\}$. So $G' \leq Z(p) \leq Z(\chi), G/Z(p)$ is abelian. From [7, Theorem 2.31], we have $\chi(1)^2 = |G : Z(\chi)|$. Since $|G'| = p$, a prime, either $G' \cap \ker 1 = 1$ or $G' \leq \ker 1$. Suppose $G' \leq \ker 1$. Then $G/\ker 1$ is abelian. By [7, Lemma 2.27(e)], $\chi = \chi(1)\lambda$ for some linear character $\lambda$ of $Z(\chi)$, contracting that $\chi$ is non-linear and irreducible character of $G$. Hence $G' \cap \ker 1 = 1$. We will show that $Z(\chi) \leq Z(G)$.

Suppose $z \in Z(\chi)$. By [7, Lemma 2.27(f)], we have $Z(\chi)/\ker 1 = Z(G/\ker 1)$. So, for any $g \in G, [z, g] \in \ker 1$, i.e $[z, g] \in \ker 1 \cap G' = 1$. Hence $z \in Z(G)$ and so $Z(\chi) \leq Z(G)$ and $\chi(1)^2 = |G : Z(G)|$.

Theorem 3. [5, Theorem B] Let $G$ be a finite $p$-group such that $|G : Z(G)|$ be a square and we denote character degree of $G$ by cd$(G)$ . Then the following statements are equivalent:

a) cd$(G) = \{1, |G : Z(G)|^{1/2}\}$.

b) For any normal subgroup $H$ of $G$, either $G' \leq H$ or $H \leq Z(G)$.

Corollary 4. Let $G$ be a finite p-group such that $|G : Z(G)|$ be a square. Then $G$ is an FNS-p-group iff cd$(G) = \{1, |G : Z(G)|^{1/2}\}$.

Proof. The result is immediate by Theorem 3.

Corollary 5. Let $G$ be a finite $p$-group such that $G/Z(G) \cong Z_p \times Z_p$. Then $G$ is an FNS-p-group.

Proof. Since $G$ is a finite $p$-group and $G/Z(G) \cong Z_p \times Z_p$, therefore $G/Z(G)$ is abelian and $|G : Z(G)| = p^2$. By Corollary 4, that is enough to prove cd$(G) = \{1, p\}$.

Since $\chi(1)^2 ||G : Z(G)||$ and $|G : Z(G)| = p^2$ therefore $\chi(1)^2|p^2$. Then $|\chi(1)| = 1$ or $|\chi(1)| = p$ and cd$(G) = \{1, p\}$.

Theorem 6. [1, Theorem 4.12] Let $G$ be a finite $p$-group of class 2 and let $Z(G)$ be cyclic. Then $c(G) = |Z(G)| \cdot |G : Z(G)|^{1/2}$. Moreover when $p \neq 2$ then $c(G) = q(G) = p(G)$.

Now we want to calculating $c(G), q(G)$ and $p(G)$, for capable AF-$p$-groups.

Lemma 1. Let $G$ be capable AF-$p$-group. Then By [8, Theorem 2.2], $G$ is of the form:

a) $D_8$ if $p=2$,
b) $L_{p^3}$ if $p \neq 2$,
in this case we have $c(D_8) = 4, c(L_{p^3}) = p^2$.

Proof. Let $G = L_{p^3}$. Then

$$G = \langle a, b, c : a^p = b^p = c^p = [a, b] = [a, c] = 1, [b, c] = a >, Z(G) = \langle a > = G'$$,

and $|G : Z(G)| = p^2$. So $G/Z(G)$ is abelian and $G' \leq Z(G)$. Hence $G$ has class 2, by Theorem 6,

$$c(G) = q(G) = p(G) = |Z(G)| \cdot |G : Z(G)|^{1/2} = p^2$$.

If $G = D_8$, then by [2, Remark (3), part (b)], $c(G) = q(G) = p(G) = 4$. □

**Theorem 7.** [4, Theorem 2] Let $G$ be a nilpotent group and $G = H \times K$. Then $p(G) = p(H) + p(K)$.

**Theorem 8.** Let $G$ be a finite $p$-group of the form $L_{p^n}$. Then

a) $G/G' \cong C_{p^n-2} \times C_p \times C_p, |Z(G)| = p^{n-2}$,
b) The number of nonlinear irreducible character of $L_{p^n}$ equal to $\phi(|Z(G)|),

c) c(G) = q(G) = p(G) = p^{n-1}$.

Proof. a) Let

$$G = L_{p^n} = \langle a, b, c : a^{p^{n-2}} = b^p = c^p = [a, b] = [a, c] = 1, [b, c] = a^{p^{n-3}} >$$.

Then we have:

$$G' = \langle a^{p^{n-3}} >, |G'| = p, |G : G'| = p^{n-1}, Z(G) = \langle a > \cong C_{p^{n-2}}, |Z(G)| = p^{n-2}$$.

Furthermore since

$$G/G' = \langle aG', bG', cG' >, (bG')^p = G', (cG')^p = G', (aG')^{p^{n-3}} = a^{p^{n-3}} G' = G'$$

so

$$G/G' = \langle a, b, c : a^{p^{n-3}} = b^p = c^p = [a, b] = [a, c] = 1, [b, c] = a^{p^{n-3}} >$$,

$$G/G' \cong C_{p^{n-2}} \times C_p \times C_p.$$
b) Since 

$$|G| = |G : G'| + \sum_{\chi \in \text{Irr}(G), \chi(1) > 1} \chi(1)^2$$

so

$$p^n = p^{n-1} + \alpha p^2 \Rightarrow \alpha = p^{n-2} - p^{n-3} = p^{n-2}(1 - 1/p) = \phi(p^{n-2}) = \phi(|Z(G)|)$$

therefore the number of nonlinear irreducible character of $L_{p^n}$ equal to $\phi(|Z(G)|)$.

c) Since $|G/Z(G)| = p^2$, so $G/Z(G)$ is abelian and $G' \leq Z(G)$. Hence $G$ has class 2, by Theorem 6,

$$c(G) = q(G) = p(G) = |Z(G)| \cdot |G : Z(G)|^{1/2} = p^{n-1}. \qed$$

**Notation.** We denote the central product of groups $H, G$ by $H \ast G$.

**Lemma 2.** Let $H$ be the central product of 2 copies of the group $L_{p^n}$, with $n \geq 3$. Then:

a) For $i, j \in \{1, 2\}$ we have:

$$H = \langle a, b_1, b_2, c_1, c_2 : a^{p^{n-2}} = b_i^p = c_i^p = 1,$$

$$[a, b_i] = [a, c_i] = [b_i, b_j] = [c_i, c_j] = 1, [b_i, c_j] = a^{\delta_{ij}p^{n-3}} > .$$

b) $c(H) = q(H) = p(H) = p^n$.

**Proof.** a) By an easy calculation and by definition of group $L_{p^n}$, the first part is obviously.

b) Since $H = L_{p^n} \ast L_{p^n}$, then $H' = \langle a^{p^{n-3}} \rangle$ and $Z(H) = \langle a \rangle \cong C_{p^{n-2}}$ and so $H' \leq Z(H)$. Therefore by Theorem 6,

$$c(H) = q(H) = p(H) = |Z(H)| \cdot |H : Z(H)|^{1/2} = p^{n-2}p^2 = p^n. \qed$$

**Theorem 9.** Let $H$ be the central product of $r$ copies ($r \geq 1$) of the group $L_{p^n}$, with $n \geq 3$. Then

a) For $i, j \in \{1, 2, \ldots, r\}$ we have:

$$H = \langle a, b_1, b_2, \ldots, b_r, c_1, c_2, \ldots, c_r : a^{p^{n-2}} = b_i^p = c_i^p = 1,$$

$$[a, b_i] = [a, c_i] = [b_i, b_j] = [c_i, c_j] = 1, [b_i, c_j] = a^{\delta_{ij}p^{n-3}} > ,$$

b) $c(H) = p(H) = q(H) = p^{n+r-2}$. 

Proof. a) By definition of group $L_{p^n}$ we have:

\begin{align*}
a_1^{p_n-2} &= b_1^p = c_1^p = [a_1, b_1] = [a_1, c_1] = 1, 
[a_1, b_1, c_1] &= a_1^{p_n-3}, \\
a_2^{p_n-2} &= b_2^p = c_2^p = [a_2, b_2] = [a_2, c_2] = 1, 
[a_2, b_2, c_2] &= a_2^{p_n-3}, \\
\vdots
\end{align*}

Then

\begin{align*}
a_r^{p_n-2} &= b_r^p = c_r^p = [a_r, b_r] = [a_r, c_r] = 1, 
[a_r, b_r, c_r] &= a_r^{p_n-3}.
\end{align*}

By the second property of [4, Definition 19.3], we have $[b_i, b_j] = [c_i, c_j] = 1, [b_i, c_j] = 1$ and therefore

\begin{equation*}
H = \langle a, b_1, b_2, \ldots, b_r, c_1, c_2, \ldots, c_r : a_i^{p_n-2} = b_i^p = c_i^p = 1, 
[a, b_i] = [a, c_i] = [b_i, b_j] = [c_i, c_j] = 1, [b_i, c_j] = a_i^{\delta_{ij}p_n-3} \rangle
\end{equation*}

where $i, j \in \{1, 2, \ldots, r\}$.

b) By structure of group $H$, we have:

\begin{equation*}
Z(H) = \langle a \rangle, H' = \langle a_i^{p_n-3} \rangle.
\end{equation*}

Then by Theorem 6,

\begin{equation*}
c(H) = p(H) = q(H) = |Z(H)| \cdot |H : Z(H)|^{1/2} = p^{n-2} \cdot p^r = p^{n+r-2}.
\end{equation*}

References


