BIPOTENTIAL AND BIHARMONIC POTENTIALS ON INFINITE NETWORKS

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Abstract: On an infinite network X with transition functions that need not be symmetric, biharmonic and bisuperharmonic functions are defined and their properties derived. A representation for positive bisuperharmonic functions that are superharmonic on X is obtained by using the Choquet integral representation theorem.

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1. Introduction

On an infinite network X with transition function that need not be symmetric, the discrete Laplace operator is denoted by Δ . A real-valued function u on X is said to be bisuperharmonic if $-\Delta u$ is superharmonic; biharmonic and bisubharmonic functions are defined analogously. If X is restricted to be an infinite tree T without terminal vertices (in particular if T is homogeneous tree), then it is known that every superharmonic function v on T generates a bisuperharmonic function u , $-\Delta u = v$; consequently when the tree is hyperbolic or parabolic, there are many bisuperharmonic functions on T . But in a general network X , this result may not be valid, hence a bisuperharmonic classification of infinite networks.

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Adhering to the classical definition of biharmonic functions on $\mathbf{R}^n$, we introduce bisuperharmonic functions, biharmonic potentials and bipotentials on the infinite network X ; extensions to general polyharmonic functions on X can be easily visualized.

This note builds up some auxiliary results leading to a representation of any function in the convex cone S of non-negative bisuperharmonic functions that are superharmonic on X , by using the Choquet integral representation theorem.

It is relevant here to remark that biharmonic functions found a way from $\mathbf{R}^n$ to locally compact spaces. In fact, based on Brelot axiomatic potential theory on a locally compact space Ω [8], Smyrnalis [12], [13] has defined a biharmonic space on Ω and develops a theory of positive superharmonic pairs which form a convex cone whose discrete version is the cone S of non-negative bisuperharmonic functions that are superharmonic. A probabilistic interpretation of biharmonic spaces has been given in Bouleau [6]; Boboc and Mustata [5] is another related paper which deals with an axiomatic study of poly-superharmonic functions.

2. Preliminaries

The infinite network $\{X, t(x, y)\}$ has the following characteristics:

- i) There are countably infinite vertices and countably infinite edges in X ; two vertices x, y are said to be neighbours, written $x \sim y$ if there is an edge connecting x and y .
- ii) Connected, that is, any two vertices x, y can be joined by a path $\{x = a_0, a_2, \dots, a_n = y\}$ where $a_i \sim a_{i+1}$.
- iii) Locally finite, that is, every vertex has only a finite number of neighbours.
- iv) There is no self-loop, that is $[x, x]$ is not an edge for any x .
- v) The function $t : X \times X \rightarrow \mathbf{R}^+$ is a transition function such that $t(x, y) > 0$ if and only if $x \sim y$; $t(x, y)$ may be different from $t(y, x)$. Write $t(x) = \sum_{y \in X} t(x, y) = \sum_{y \sim x} t(x, y)$.

Notation. i) For any subset E of X , the set $\overset{o}{E}$ is the set of all x such that x and all its neighbours in X are in E ; $\partial E = E \setminus \overset{o}{E}$; $V(E)$ is the union of E

and all vertices z in X such that z has a neighbour in E ; note that $V(E)$ is connected if E is connected.

ii) The discrete Laplace operator Δ is defined as follows:

for a real-valued function $u(x)$ on X , $\Delta u(x) = \sum_{y \sim x} t(x, y)[u(y) - u(x)]$.

A real-valued function u on a subset E of X is said to be *superharmonic* on E if $\Delta u(x) \leq 0$ for every $x \in \overset{\circ}{E}$; u is harmonic on E if $\Delta u(x) = 0$ for every $x \in \overset{\circ}{E}$ and u is subharmonic on E if $\Delta u(x) \geq 0$ for $x \in \overset{\circ}{E}$. Some of the properties of superharmonic functions are given in [1], [2], [3]. A potential $p(x)$ on X is a non-negative superharmonic function such that if $u(x)$ is subharmonic and $u(x) \leq p(x)$ then $u(x) \leq 0$. The network X is said to be parabolic if the only potential on X is 0. If there is a potential $p > 0$ on X , then X is called hyperbolic network; in that case, for any $z \in X$ there exists a potential $G_z(x) > 0$ which is harmonic at every vertex $x \neq z$. Remark that if $a(z) \geq 0$ is a sequence of numbers such that $p(x) = \sum_{z \in X} a(z)G_z(x)$ is finite at one vertex, then $p(x)$ is a potential on X . Conversely, if $p(x)$ is any potential on X then $p(x) = \sum_z [-\Delta p(z)]G_z(x)$.

A superharmonic function $s(x)$ on X is said to be admissible if $s(x)$ has a subharmonic minorant outside a finite set. If X is hyperbolic, then $s(x)$ is the sum of a potential and a harmonic function on X .

3. Bisuperharmonic Functions

Definition 1. A real-valued function $u(x)$ on a network X is said to be bisuperharmonic if $-\Delta u = v$ is a superharmonic function on X ; analogous definitions for bisubharmonic and biharmonic functions.

Discrete Poisson equation: Suppose T is an infinite tree in which every non-terminal vertices as neighbours (in particular if there is no terminal vertex in T as in a homogeneous tree). Then for any real-valued function $f(x)$ on T , there exists a real-valued function $g(x)$ such that $-\Delta g(x) = f(x)$ [3][Theorem 5.1.4]. Hence if $v(x)$ is a superharmonic function on T , it generates a bisuperharmonic function $u(x)$, $-\Delta u(x) = v(x)$. Thus there is an abundance of bisuperharmonic functions on such a tree whether it is hyperbolic or parabolic. Now for solving the discrete Poisson equation $-\Delta g = f$, it is not necessary that the base network should be a tree. For example, in the case of the Lattice network in $\mathbf{R}^n$, Duffin [11] discusses the problem of finding a solution to the discrete Poisson equation by using Fourier transforms.

However when x is any network, finding a solution to the Poisson equation $-\Delta g = f$ may not be possible without some restrictions on x or $f(x)$. We know

that for any given vertex z , there exists a unique superharmonic function $q_z(x)$ (which may be taken as the Green potential if X is hyperbolic and non-positive if X is parabolic [3, ?]) such that $-\Delta q_z(x) = \delta_z(x)$.

Proposition 2. *If $f(x) \geq 0$ is a real-valued function on a parabolic network X such that $s(x) = \sum_z f(z)q_z(x)$ is a real-valued function, then $s(x)$ is a non-positive superharmonic function on X and $-\Delta s(x) = f(x)$.*

Proof. Remark that for each vertex z , $f(z)q_z(x)$ is a non-positive superharmonic function on X . Let $\{F_n\}$ be a collection of finite subsets of X such that $F_n \subset \overset{o}{F}_{n+1}$ and $X = \cup F_n$. If $s_n(x) = \sum_{z \in F_n} f(z)q_z(x)$, then $s_n(x) \leq 0$ is a superharmonic function on X and $s(x) = \lim_n s_n(x)$. Since $s(x)$ is the limit of superharmonic functions, $s(x)$ is superharmonic on X . Also, $s(x) \leq 0$ and $-\Delta s(x) = \lim_n [-\Delta s_n(x)] = f(x)$. $\square$

Proposition 3. *In a symmetric hyperbolic network X with the Green kernel $G(x, y) = G_y(x)$, suppose $\sum_z G_e(x) < \infty$ for some vertex e . Then any bounded positive superharmonic function $s(x)$ generates a potential $q(x)$, that is $-\Delta q(x) = s(x)$.*

Proof. Since X is a symmetric network, $G_a(b) = G_b(a)$ for any pair of vertices a, b . Hence $p(x) = \sum_z G_z(x)$ is a potential on X ; for $p(x)$ is a potential if it is finite at one vertex. Now

$$p(e) = \sum_z G_z(e) = \sum_z G_e(z) < \infty.$$

Hence $p(x)$ is a potential on X . Let $s(x)$ be a bounded positive superharmonic function on X , $0 < s(x) < M$. Write $q(x) = \sum_z s(z)G_z(x)$ which is a potential since it is real-valued and each term in the sum is a potential. Remark that $-\Delta q(x) = s(x)$. $\square$

Corollary 4. *With the assumption of the above proposition, for any vertex e there exists a unique symmetric potential $Q_e(x)$ such that $-\Delta Q_e(x) = G_e(x)$.*

Proof. Since $G_e(x)$ is a potential with harmonic support e , by the Domination Principle [3, ?], $G_e(x) \leq G_e(e)$ for all x . Since $G_e(x)$ is bounded, by the above Proposition there is a potential $Q_e(x)$ such that $-\Delta Q_e(x) = G_e(x)$.

For the uniqueness, suppose $u(x)$ is a potential such that $-\Delta u(x) = G_e(x)$. Then $u(x) = Q_e(x) +$ (a harmonic function); hence by Riesz decomposition $u(x) = Q_e(x)$.

To prove the symmetry of $Q_e(x) = \sum_z G_e(z)G_z(x)$, take two vertices a and b ; note that $Q_a(b) = \sum_z G_a(z)G_z(b) = \sum_z G_z(a)G_b(z) = Q_b(a)$. $\square$

Remark. The above corollary gives some conditions for the existence of two potentials p, q on X such that $-\Delta q(x) = p(x)$. Such a pair can not be found in every hyperbolic network. In the next section this existence is discussed in more details.

Let X be a network, parabolic or hyperbolic, in which $-\Delta q - z(x) = \delta_z(x)$. Suppose $f(x)$ is a real-valued function on X , vanishing outside a finite set. Then $g(x) = \sum_z f(x)q_z(x)$ is the difference of two superharmonic functions on X and $-\Delta g(x) = f(x)$. Based on this result, we can solve $-\Delta u = v$ for any real-valued function $v(x)$ on X if the following harmonic approximation theorem is valid on X : Let F be a set in X and E be a finite connected set contained in $\overset{\circ}{F}$. Then for any harmonic function h on F and $\epsilon > 0$, there exists a harmonic function H on X such that $|h - H| < \epsilon$ on E .

Remark. In the case of the Euclidean spaces $\mathbf{R}^n$, $n \geq 2$, if the Laplace operator Δ is taken in the sense of distributions, Brelot [7] has indicated that given any locally Lebesgue integrable function $f(x)$ on $\mathbf{R}^n$, then there exists a function $g(x)$ such that $-\Delta g(x) = f(x)$. Suppose $v(x)$ is a superharmonic function on $\mathbf{R}^n$; the v is locally Lebesgue integrable so that the Poisson equation $-\Delta u = v$ has a solution $u(x)$.

Proposition 5. *If $\{u_n(x)\}$ is a sequence of bisuperharmonic functions on a network X such that $u(x) = \lim_n u_n(x)$ exists and is real valued, then $u(x)$ is bisuperharmonic function.*

Proof. Since each $-\Delta u_n(x)$ is superharmonic and since

$$-\Delta u(x) = \lim_n [-\Delta u_n(x)],$$

the function $-\Delta u(x)$ is superharmonic on X ; that is, $u(x)$ is bisuperharmonic on X . $\square$

Lemma 6. *In a hyperbolic network X , suppose $s(x) \geq 0$ is a superharmonic function, that is $-\Delta s(x) = f(x) \geq 0$. Then for any function g such that $0 \leq g \leq f$, there exists a unique potential $p(x)$ such that $-\Delta p(x) = g(x)$.*

Proof. Since $s(x)$ is a non-negative superharmonic function, then $s(x) = \sum_z [-\Delta s(z)]G_z(x) +$ (a harmonic function). Since $0 \leq g(z) \leq (-\Delta s(z))$, then $p(x) = \sum_z g(z)G_z(x)$ is real-valued, hence a potential on X such that $-\Delta p(x) = g(x)$.

For the uniqueness of $p(x)$, suppose $-\Delta p_1(x) = g(x)$ for a potential $p_1(x)$. Then $p_1(x) = p(x) +$ (a harmonic function). Consequently, $p_1 = p$. $\square$

Proposition 7. *In a hyperbolic network X , suppose $v(x) > 0$ is a bisuperharmonic function such that $-\Delta v(x) = s(x) > 0$. Then for any potential*

$p(x) > 0$ with finite harmonic support, there exists a unique potential $q(x)$, $-\Delta q(x) = p(x)$.

Proof. Since $s(x)$ is a positive superharmonic function and $p(x)$ is a potential with finite harmonic support, then by the Domination Principle, $\alpha p(x) \leq s(x)$ for some $\alpha > 0$. Then by the above lemma there exists a unique potential $\varphi(x)$ such that $-\Delta \varphi(x) = \alpha p(x)$. Take $q(x) = \frac{1}{\alpha} \varphi(x)$ to conclude $-\Delta q(x) = p(x)$. $\square$

4. Bipotentials and Biharmonic Potentials

Definition 8. A potential $q(x)$ in a hyperbolic network X is said to be bipotential if and only if there exists a potential $p(x)$ such that $-\Delta q(x) = p(x)$. If there exists a positive bipotential on X , then X is said to be bipotential network.

Definition 9. A potential $b(x)$ in a hyperbolic network X is said to be biharmonic potential if and only if there exists a nonnegative harmonic function $h(x)$ on X such that $-\Delta b(x) = h(x)$. If there exists a positive biharmonic potential on X , then X is said to be biharmonic potential network.

Remark. i) A real-valued function $q(x)$ is a bipotential if and only if it is of the form $q(x) = \sum_z p(z)G_z(x)$ where $p(x)$ is a potential on X .

ii) A real-valued function $b(x)$ is a biharmonic potential if and only if it is of the form $b(x) = \sum_z h(z)G_z(x)$ where $h(x)$ is a nonnegative harmonic function on X .

Continuous Case: In the Euclidean space $\mathbf{R}^n$, $n \geq 2$, using the above terminology and defining the Laplace operator Δ in the sense of distributions, we note the following: $|x|^{4-n}$ is a bipotential in $\mathbf{R}^n$, $n \geq 5$; in $\mathbf{R}^3$, $\mathbf{R}^4$ even though positive potentials exist, there is no positive bipotential; in $\mathbf{R}^2$ there is no positive potential, hence there can not be any positive bipotential. There is no positive biharmonic potential in any of the Euclidean spaces $\mathbf{R}^n$, $n \geq 2$.

Proposition 10. In a biharmonic potential network X , there are bipotentials.

Proof. Let $b(x)$ be biharmonic potential on X , $-\Delta b(x) = h(x)$ is a positive harmonic function. Let A be a finite non-empty set in X . Then the reduced function (balayage) $R_h^A = p$ is a positive potential, $p \leq h$. Then by Lemma 7, there is a unique potential $q(x)$ such that $-\Delta q(x) = p(x)$. $\square$

Proposition 11. *In a hyperbolic network X with the Green kernel $G(x, y) = G_y(x)$ the following are equivalent:*

- i) *there exist two positive superharmonic functions u, v such that $-\Delta u = v$.*
- ii) *X is a bipotential network.*
- iii) *For any vertex z , there exists a potential Q_z on X such that $-\Delta Q_z(x) = G_z(x)$.*

Proof. i) $\Rightarrow$ ii) Let u, v be positive superharmonic functions on X such that $-\Delta u = v$. Now $v = (\text{a potential } p) + (\text{a non-negative harmonic function } h)$. Since $p \leq v$, by Lemma 7, there is a potential $q(x)$ such that $-\Delta q(x) = p(x)$.

ii) $\Rightarrow$ iii) Let $-\Delta q(x) = p(x)$. Let $G_z(x)$ be the Green potential with harmonic support z . Then by the Domination Principle, $G_z(x) \leq \alpha p(x)$ for some α . An application of Lemma 7 gives the unique potential $Q_z(x)$ such that $-\Delta Q_z(x) = G_z(x)$.

iii) $\Rightarrow$ i) Obvious. $\square$

Definition 12. The Kernel $Q(x, z) = Q_z(x)$ introduced in the above proposition is referred to as the biharmonic Green kernel in the potential network X .

It follows that a hyperbolic network X with the Green kernel $G(x, y)$ is a bipotential network if and only if $Q(x) = \sum_y G_z(y)G_y(x)$ is a potential on X for any fixed z . Hence there are no bipotentials on X if and only if $\sum_y G_z(y)G_y(x) = \infty$ for one (and hence only) vertex y in X . Some other characteristics for the network not to be a bipotential network can be given (Proposition 13) as a reformulation of Proposition 11. Recall that a superharmonic function on X is said to be admissible if it has a harmonic minorant outside a finite set.

Proposition 13. *In a hyperbolic network X , the following are equivalent:*

- i) *any bisuperharmonic function that is positive superharmonic on X is constant.*
- ii) *There are no positive bipotentials on X .*
- iii) *For any vertex z , if $-\Delta u(x) = G_z(x)$ then $u(x)$ is a superharmonic function that is not admissible.*
- iv) *If $s > 0$ is any superharmonic function on X and if $-\Delta v(x) = s(x)$ then $v(x)$ is a superharmonic function that is not admissible.*

We introduce in an example of a hyperbolic network on which there is no positive bipotential (based on Yamasaki [14, ?]).

Example 14. Let $\{0, 1, 2, \dots\}$ be an infinite graph with transition functions $t(n-1, n) = (n+1)\sqrt{n} + n\sqrt{n+1}$ if $n \geq 1$. Suppose $h(x)$ is a harmonic function on X . Then $\Delta h(0) = 0$ implies that $h(1) = h(0)$ and $0 = \Delta h(1) = t(1, 0)[h(0) - h(1)] + t(1, 2)[h(2) - h(1)]$ so that $h(2) = h(1)$. This procedure leads to the conclusion that any harmonic function on X is a constant.

Let $u(n) = \frac{1}{\sqrt{n+1}}$. Then we have

$\Delta u(n) = t(n, n-1)[u(n-1) - u(n)] + t(n, n+1)[u(n+1) - u(n)] = 0$ where $n \geq 1$ and $\Delta u(0) = t(0, 1)[u(1) - u(0)] < 0$.

Hence $u(n) > 0$ is a superharmonic function with harmonic support at the vertex 0. Since $u(n)$ tends to 0 when n tends to ∞ and since constants are the only harmonic functions on X , then $u(n)$ is a potential on X . Consequently $G_0(n) = u(n)$ is the Green potential on X with harmonic support at the vertex 0.

Now if X is a bipotential network, then there exists a potential $q(n)$ such that $-\Delta q(n) = G_0(n)$ so that for any k in X

$$\begin{aligned} q(n) &= \sum_k [-\Delta q(k)] G_k(n) \\ &= \sum_k [-\Delta q(k)] G_n(k), \end{aligned}$$

by the symmetry of transition function $G_k(n) = G_n(k)$. In particular $q(0) = \sum_k G_0(k) G_0(k) = \sum_k \frac{1}{k+1} = \infty$, a contradiction. We conclude therefore that there is no bipotential on X .

Now we discuss an example of a network with biharmonic potentials.

Example 15. Let T be a homogeneous tree in which each vertex has $(q+1)$, $q \geq 2$, neighbourhoods. Fix a vertex e in T . Then if $d(x, e)$ denotes the distance of the vertex x from e , we have (Cartier [9]) the Green potential $G(x, e) = G_e(x) = \frac{q}{q-1} \cdot \frac{1}{q^{d(x, e)}}$. Consequently

$$\sum_{x \in T} G_e(x) = \frac{q}{q-1} \sum_n q(q-1)^{n-1} \cdot \frac{1}{q^n} < \infty,$$

Since there are $q(q-1)^{n-1}$ vertices at a distance n from e . Hence if $h(x) > 0$ is any bounded harmonic function on T , then by Proposition 3 there exists a potential $u(x)$ on T such that $-\Delta u(x) = h(x)$. That is, $u(x)$ is a biharmonic potential. In fact there are at least $(q+1)$ bounded non-proportional positive

harmonic functions on T [3, ?]. Hence there are at least $(q+1)$ non-proportional biharmonic potentials on T . In particular the homogeneous tree T is bipotential tree (Proposition 10).

Theorem 16. *Let $u(x)$ be a non-negative bisuperharmonic function that is superharmonic on X . Then $u(x)$ is the unique sum of a bipotential q , a biharmonic potential b and a non-negative harmonic function H on X , $u = q + b + H$.*

Proof. The assumption on $u(x)$ show that $u \geq 0$, $-\Delta u = v \geq 0$ and v is superharmonic on X . Hence $u(x) = \sum_z v(z)G_z(x) + H(x)$ where $H(x)$ is a non-negative harmonic function on X . Since $v(x)$ is a non-negative superharmonic function, $v(x) = (\text{a potential } p(x)) + (\text{a non-negative harmonic function } h(x))$.

Write $b(x) = \sum_z h(z)G_z(x)$ and $q(x) = \sum_z p(z)G_z(x)$ so that $u(x) = q(x) + b(x) + H(x)$ where $q(x)$ is a bpotential, $b(x)$ is a biharmonic potential and $H(x)$ is a non-negative harmonic function.

For the uniqueness, suppose $u(x) = q_1(x) + b_1(x) = H_1(x)$ be another such representation. Then $-\Delta q - \Delta b = -\Delta q_1 - \Delta b_1$. Since $-\Delta q, -\Delta q_1$ are potentials and $-\Delta b, -\Delta b_1$ are harmonic, $-\Delta q = -\Delta q_1$ and $-\Delta b = -\Delta b_1$. since q, q_1, b, b_1 are all potentials, we conclude that $q = q_1$ and $b = b_1$; Consequently $H_1 = H$. $\square$

5. Integral Representation

In a hyperbolic network, Let $\mathcal{S}$ stand for the convex cone of positive bisuperharmonic functions that are superharmonic; $\mathbf{\Gamma}$ the convex cone of bipotentials; $\mathcal{B}$ the convex cone of biharmonic potentials; and $\mathcal{H}$ the convex cone of non-negative harmonic functions. Then Theorem 16 states that $\mathcal{S} = \mathbf{\Gamma} \oplus \mathcal{B} \oplus \mathcal{H}$. In this section we give an integral representation for any function in $\mathcal{S}$ by using the Choquet theorem [10]: In a locally convex metrisable topological space Ω , Let C be a cone which is lattice for its own order. Let $\mathcal{B}$ be a compact base for C , and $\mathcal{E}$ be the set of extremal elements of $\mathcal{B}$. Then for any u in C , there exists a unique measure $\mu \geq 0$ with support in $\mathcal{E}$ such that $u = \int_{\mathcal{E}} v d\mu(v)$.

i) Compact base $\mathcal{B}$ for the cone $\mathcal{S}$

Let $\mathcal{F}$ be the set of real-valued functions on the hyperbolic network X . For a vertex z fixed, define the semi-norm for $f \in \mathcal{F}$ as $\|f\|_2 = |f(z)|$. Since there are only a countable number of vertices in X , these semi-norms define a locally convex metrisable topology on $\mathcal{F}$. Consider now the cone $\mathcal{S}$ in $\mathcal{F}$ and for a fixed vertex a , take $\mathcal{B} = \{s \in \mathcal{S} : s(a) = 1\}$. Then $\mathcal{B}$ is a base for $\mathcal{S}$ and we show now that $\mathcal{B}$ is compact.

Let $\{S_i\}$ be a collection of functions from $\mathcal{B}$. Since $S_i(a) = 1$ for every S_i , then for any $y \in X$, $\{S_i(y)\}$ is bounded by the Harnack property. Since X is countable, we can extend a convergent subsequence $\{S_n\}$ from $\{S_i\}$ such that $S(x) = \lim_n S_n(x)$. Since each $S_n(x)$ is superharmonic $S(x)$ is real-valued, the function $S(z)$ also is superharmonic. Moreover, since $-\Delta S(x) = \lim_n [-\Delta S_n(x)]$ and each $-\Delta S_n(x)$ is a non-negative superharmonic function, then $-\Delta S(x)$ is also non-negative superharmonic. Consequently $S \in \mathcal{S}$ and $S(a) = 1$. this shows that $\mathcal{B}$ is a compact base for $\mathcal{S}$.

ii) The case $\mathcal{S}$ is a lattice

Let $\mathcal{C}$ be a one of non-negative functions on Ω . Then $\mathcal{C}$ determines an order as follows: If $f, g \in \mathcal{C}$, then $f <_{\mathcal{C}} g$ if and only if there exists $\varphi \in \mathcal{C}$ such that $f + \varphi = g$. Now $f \vee_{\mathcal{C}} g$ is defined as the smallest function in $\mathcal{C}$ for the order determined by $\mathcal{C}$, such that $f \vee_{\mathcal{C}} g >_{\mathcal{C}} f$ and $f \vee_{\mathcal{C}} g >_{\mathcal{C}} g$ if it exists. Analogously, $f \wedge_{\mathcal{C}} g$ is defined. The cone $\mathcal{C}$ is said to be lattice for its own order, if for any $f, g \in \mathcal{C}$, $f \vee_{\mathcal{C}} g$ and $f \wedge_{\mathcal{C}} g$ exists in $\mathcal{C}$.

As an example, the non-negative harmonic cone $\mathcal{H}$ in X is a lattice. for if $h_1, h_2 \in \mathcal{H}$, then $h_1 \vee_{\mathcal{H}} h_2$ is the least harmonic majorant of the subharmonic function $\sup(h_1, h_2)$; and $h_1 \wedge_{\mathcal{H}} h_2$ is the greatest harmonic minorant of $\inf(h_1, h_2)$.

We show now that the cone $\mathcal{S}$ is a lattice for the order it determines. Let $s_1, s_2 \in \mathcal{S}$. Then by the unique representation (Theorem 16), $s_i = q_i + b_i + H_i$, $i = 1, 2$. Hence $q_i(x) = \sum_z \alpha_i(z) Q_z(x)$ and $b_i(x) = \sum_z u_i(z) G_z(x)$ where $\alpha_i(x)$ are non-negative constants and u_i are non-negative harmonic functions. Let $\beta(z) = \max[\alpha_1(z), \alpha_2(z)]$, $u = u_1 \vee_{\mathcal{H}} u_2$ and $H = H_1 \vee_{\mathcal{H}} H_2$. Take $q(x) = \sum_z \beta(z) Q_z(x)$ and $b(x) = \sum_z u(x) G_z(x)$. Then $s(x) = q(x) + b(x) + H(x) \in \mathcal{S}$ and $s = s_1 \vee_{\mathcal{S}} s_2$.

Analogously, defining $s_1 \wedge_{\mathcal{S}} s_2$, we conclude that $\mathcal{S}$ is a lattice for its own order.

iii) Extremal elements of the cone $\mathcal{S}$

A function $f(x)$ in a cone $\mathcal{C}$ in Ω is said to be an extremal element if $f = f_1 + f_2$ where $f_1, f_2 \in \mathcal{C}$, then for some λ , $0 \leq \lambda \leq 1$ we have $f_1 = \lambda f$ and $f_2 = (1 - \lambda)f$.

In the case of the harmonic cone $\mathcal{H}$ in a network X , $h \in \mathcal{H}$ is an extremal if and only if it is minimal in X . (that is if $u \in \mathcal{H}$ and $u \leq h$, then $u = \alpha h$ for some α , $0 \leq \alpha \leq 1$). Let us denote by $\mathcal{E}$ the set of minimal harmonic functions on X .

Now take $\mathcal{S} = \mathbf{\Gamma} \oplus \mathcal{B} \oplus \mathcal{H}$. If $s \in \mathcal{S}$ is extremal, then s is in $\mathbf{\Gamma}$ or in $\mathcal{B}$ or in $\mathcal{H}$.

1. If $s \in \mathbf{\Gamma}$, then $s(x) = \sum_z \alpha(z) Q_z(x)$. Hence if $\mathcal{S}$ is extremal then $s(x) =$

$Q_z(x)$ for some z .

2. If $s \in \mathcal{B}$, then $s(x) = \sum_z u(z)G_z(x) = r_u(x)$ where $u(x)$ is a non-negative harmonic function. Hence s is extremal if and only if $s(x) = r_h(x)$ for a minimal harmonic function $h(x)$.
3. If $s \in \mathcal{H}$, then s is extremal implies that $s(x)$ is a minimal harmonic function.

Applying the Choquet integral representation theorem, we have now:

Theorem 17. *A function $s \in \mathcal{S}$ has the unique representation*

$$\begin{aligned} s(x) &= q(x) + b(x) + H(x) \\ &= \int_X Q_z(x) d\alpha(z) + \int_{\mathcal{E}} r_h(x) d\mu(h) + \int_{\mathcal{E}} h(x) d\lambda(h), \end{aligned}$$

where $\mathcal{E}$ is the set of minimal harmonic functions in $\mathcal{S}$; $\alpha \geq 0$ is a Radon measure on X ; and μ, λ are non-negative measures supported by $\mathcal{E}$. Here the measures are uniquely determined.

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