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ON WG-CONTINUOUS FUNCTIONS IN ASSOCIATED WEAK SPACES

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Abstract: The purpose of this paper is to introduce the notions of wg-continuity and wg^*-continuity by defined wg_τ-open sets in associated w-spaces, and to study some properties and the relationships among such notions and the other continuity.

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1. Introduction

Siwiec [18] introduced the notions of weak neighborhoods and weak base in a topological space. We introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in [8]. The weak neighborhood system induces a weak neighborhood space which is independent of neighborhood spaces [2] and general topological spaces [1]. The notions of weak structure, w-space, W-continuity and W^*-continuity were investigated in [9]. In fact, the set of all g-closed subsets [3] in a topological space is a kind of weak structure.

In [11], we studied the notion of generalized w-closed sets (simply, gw-closed sets) in a w-space.

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In [10], we introduced the notion of an associated weak space (simply, associated w_τ-space) containing a given topology τ, and moreover, we studied the notion of generalized w_τ-closed sets (simply, gw_τ-closed sets) [14] in an associated w_τ-space with a topology τ.

In [12], we introduced the notions of gw_τ-continuous and gw^*_τ-continuous functions in associated w-spaces, and studied their characterizations and the relationships among them.

In [15], we studied w_τ-generalize closed sets (simply, wg-closed) in an associated weak space w_τ in the similar way introduced by Levine [3] in topological spaces.

In this paper, we are going to introduce the notions of wg-continuity and wg^*-continuity in associated w-spaces by defined wg-open sets, and to study some properties and the relationships among such notions and the other continuity in associated w-spaces.

2. Preliminaries

Definition 2.1 ([9]). Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* on X if it satisfies the following:

(1) $\emptyset \in w_X$ and $X \in w_X$.

(2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a *w-space* on X. Then $V \in w_X$ is called a *w-open* set and the complement of a w-open set is a *w-closed* set.

The collection of all w-open sets (resp., w-closed sets) in a w-space X will be denoted by $W(X)$ (resp., $WC(X)$). We set $W(x) = \{U \in W(X) : x \in U\}$.

Let S be a subset of a topological space X. The closure (resp., interior) of S will be denoted by clS (resp., $intS$). A subset S of X is called a *preopen* set [6] (resp., α-open set [17], *semi-open* [4]) if $S \subset int(cl(S))$ (resp., $S \subset int(cl(int(S))), S \subset cl(int(S))$). The complement of a preopen set (resp., α-open set, *semi-open*) is called a *preclosed* set (resp., α-closed set, *semi-closed*). The family of all preopen sets (resp., α-open sets, semi-open sets) in X will be denoted by $PO(X)$ (resp., $\alpha(X), SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on X.

Moreover, a subset S of X is said to be g-closed [3] if $cl(A) \subset U$ whenever $A \subset U$ and U is open in X.

Then the family $GO(X) = \{U \subseteq X : U \text{ is } g\text{-open }\}$, $O(X) = \{U \subseteq X : U \text{ is open }\}$ and $CL(X) = \{F \subseteq X : F \text{ is closed }\}$ are all weak structures on X. But $PO(X)$, $GPO(X)$ and $SO(X)$ are not weak structures on X. A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* on X [5] if $\emptyset \in w_X$ and $X \in w_X$. Thus clearly every weak structure is a minimal structure.

Let (X, w_X) be a w-space. For a subset A of X, the *w-closure* of A and the *w-interior* of A are defined as follows:

(1) $wC(A) = \cap\{F : A \subseteq F, X - F \in w_X\}$.

(2) $wI(A) = \cup\{U : U \subseteq A, U \in w_X\}$.

Theorem 2.2 ([9]). *Let (X, w_X) be a w-space and $A \subseteq X$. Then the following things hold:*

(1) If $A \subseteq B$, then $wI(A) \subseteq wI(B)$; $wC(A) \subseteq wC(B)$.

(2) $wI(wI(A)) = wI(A)$; $wC(wC(A)) = wC(A)$.

(3) $wC(X - A) = X - wI(A)$; $wI(X - A) = X - wC(A)$.

(4) If A is w-closed (resp., w-open), then $wC(A) = A$ (resp., $wI(A) = A$).

Let X be a nonempty set and let (X, τ) be a topological space. A subfamily w of the power set $P(X)$ is called *an associated weak structure* (simply, w_τ) [10] on X if $\tau \subseteq w$ and w is a weak structure. Then the pair (X, w_τ) is called *an associated w-space* with τ.

Let (X, w_τ) be an associated w-space with a topology τ and $A \subseteq X$. Then A is called *a w_τ-generalized closed set* (simply, wg-closed set) [15] if $wC(A) \subseteq U$, whenever $A \subseteq U$ and U is open. Then A is called a *w_τ-generalized open set* (simply, wg-open set) if $X - A$ is wg-closed.

We recall that: A is called *a generalized closed set* (simply, g-closed set) [3] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is open. Then if $w_\tau = \tau$, then a w_τ-generalized closed set is exactly a g-closed set. In general, the intersection of two wg-closed sets is not wg-closed and the union of two wg-open sets is not wg-open.

Let (X, w_τ) be an associated w-space with a topology τ. For a subset A of X, *$w_\tau g$-closure* of A and *$w_\tau g$-interior* of A are defined as the following:

(1) $w_\tau gC(A) = \cap\{F : A \subseteq F, F$ is $w_\tau g$-closed$\}$.

(2) $w_\tau gI(A) = \cup\{U : U \subseteq A, U$ is $w_\tau g$-open$\}$.

Theorem 2.3 ([15]). *Let (X, w_τ) be an associated w-space with a topology τ and $A \subseteq X$.*

(1) If A is wg-open, then $wgI(A) = A$.

(2) If A is wg-closed, then $wgC(A) = A$.

(3) If $A \subseteq B$, then $wgI(A) \subseteq wgI(B)$ and $wgC(A) \subseteq wgC(B)$.

(4) $wgC(X - A) = X - wgI(A)$; $wgI(X - A) = X - wgC(A)$.

(5) $x \in wgI(A)$ if and only if there exists a $w_\tau g$-open set U containing x such that $U \subseteq A$. (6) $x \in wgC(A)$ if and only if $A \cap V \neq \emptyset$ for all wg-open set V containing x.

Theorem 2.4 ([15]). *Let (X, w_τ) be an associated w-space with a topology τ and $A, B \subset X$.*

(1) $\emptyset = wgC(\emptyset)$; $X = wgI(X)$.

(2) $A \subseteq wgC(A)$; $wgI(A) \subseteq A$.

(3) $wgC(A \cup B) = wgC(A) \cup wgC(B)$; $wgI(A \cap B) = gI(A) \cap wgI(B)$.

(4) $wgC(wgC(A)) = wgC(A)$; $wgI(wgI(A)) = wgI(A)$.

3. wg-continuity; wg^*-continuity

Definition 3.1. Let $f : X \to Y)$ be a function in two associated w-spaces. Then f is said to be

(1) *wg-continuous* if for $x \in X$ and for each open set V containing $f(x)$, there is a wg-open set U containing x such that $f(U) \subseteq V$:

(2) *wg^*-continuous* if for every open set V in Y, $f^{-1}(V)$ is a $w_\tau g$-open set in X.

Obviously we obtain the following theorem:

Theorem 3.2. *Every wg^*-continuous function is wg-continuous.*

Example 3.3. Let $X = \{a, b, c, d\}$, a topology $\tau = \{\emptyset, \{a, c\}, X\}$ and an associated w-structure $w_X = \{\emptyset, \{a, c\}, \{a\}, \{b\}, \{c\}, \{a, d\}, X\}$.

Consider a function $f : (X, w_\tau) \to (X, w_\tau)$ defined by $f(a) = b$; $f(b) = a$; $f(c) = d$; $f(d) = c$. For the only non-trivial open set $V = \{a, c\}$ containing $f(b) = a$ and $f(d) = c$, since $A = \{b\}$ and $B = \{d\}$ are wg-open sets containing b and d, respectively, so obviously f is wg-continuous. Now, for $A = \{a, c\} = X - \{b, d\}$ and for a open set $U = \{a, c\}$ such that $A \subseteq U$, since $wC(A) = \{a, c, d\} \nsubseteq U$, A is not wg-closed. Consequently, $\{b, d\}$ is not wg-open. Finally, $f^{-1}(V) = \{b, d\}$ is not wg-open, so f is not wg^*-continuous.

Let (X, w_τ) be an associated w-space with a topology τ and $A \subseteq X$. Then A is called *a generalized w_τ-closed set* (resp., *a generalized w-closed set*) (simply, gw_τ-closed set [14]) (resp., gw-closed set) if $cl(A) \subseteq U$ (resp., $wC(A) \subseteq U$), whenever $A \subseteq U$ and U is w-open. Then since $\tau \subseteq w$, every gw_τ-closed set is gw-closed, and every gw-closed set is $w_\tau g$-closed. But, the converses are not be true in general.

We recall that: Let $f : X \to Y$ be a function in associated w-spaces. Then f is said to be:

(1) *WO-continuous* [10] if for $x \in X$ and for each open set V containing $f(x)$, there is a w-open set U containing x such that $f(U) \subseteq V$;

(2) *WK-continuous* [10] if for every open set V in Y, $f^{-1}(V)$ is a w-open set in X.

(3) *gw_τ-continuous* [12] if for $x \in X$ and for each open set V containing $f(x)$, there is a gw_τ-open set U containing x such that $f(U) \subseteq V$:

(4) *gw^*_τ-continuous* [12] if for every open set V in Y, $f^{-1}(V)$ is a gw_τ-open set in X.

(5) *gw-continuous* [16] if for $x \in X$ and for each open set V containing $f(x)$, there is a gw-open set U containing x such that $f(U) \subseteq V$:

(6) *gw^*-continuous* [16] if for every open set V in Y, $f^{-1}(V)$ is a gw-open set in X.

Obviously, the following things are obtained:

Theorem 3.4. *In associated w-spaces, (1) every WO-continuous function is wg-continuous;*

(2) every WK-continuous function is wg^*-continuous;

(3) every gw_τ-continuous function is wg-continuous;

(4) every gw^*_τ-continuous function is wg^*-continuous;

(5) every gw-continuous function is wg-continuous;

(6) every gw^*-continuous function is wg^*-continuous.

Proof. (1) and (2) Since every w-open set is wg-open, it is obtained.

(3) and (4) Since every gw_τ-open set is wg-closed, it is obtained.

(5) and (6) Since every gw-open set is wg-open, it is obtained. □

The following example supports that the converses of the above theorem are not true in general.

Example 3.5. Let $X = \{a, b, c, d\}$, a topology $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and an associated w-structure $w = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}, X\}$. Note that:

$$\begin{aligned}
WC(X) =& \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}, X\};\\
GWC(X) =& \{\emptyset, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, X\};\\
GW(X) =& \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, X\};\\
WGC(X) =& \{\emptyset, \{b\}, \{c\}, \{d\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\},\\
& \{a, c, d\}, \{b, c, d\}, X\};\\
WG(X) =& \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\},\\
& \{a, c, d\}, \{a, b, d\}, X\};\\
gw_\tau C(X) =& \{\emptyset, \{c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, X\};\\
gw_\tau(X) =& \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}.
\end{aligned}$$

Consider a function $f : (X, w) \to (X, w)$ defined by $f(a) = d; f(b) = b; f(c) = c; f(d) = a$. Then obviously f is wg-continuous and wg^*-continuous. But for a open set $V = \{a, b\}$, since $f^{-1}(V) = \{b, d\}$ is neither gw-open nor gw_τ, f is neither gw^*-continuous nor gw^*_τ-continuous. Obviously, it implies that f is neither gw-continuous nor gw_τ-continuous.

Furthermore, since for a open set $V = \{a, b\}$, $f^{-1}(V)$ is not w-open, f is neither WK-continuous nor WO-continuous.

Remark 3.6. For a function from an associated w-space to an associated w-space, we have the following diagram:

$$\begin{array}{ccccc}
\text{Continuity} \Rightarrow & WK\text{-continuity} & \Rightarrow & WO\text{-continuity} \\
& \Downarrow & & \Downarrow \\
& gw^*\text{-continuity} & \Rightarrow & gw\text{-continuity} \\
& \Downarrow & & \Downarrow \\
& wg^*\text{-continuity} & \Rightarrow & wg\text{-continuity} \\
& \Uparrow & & \Uparrow \\
& gw^*_\tau\text{-continuity} & \Rightarrow & gw_\tau\text{-continuity}
\end{array}$$

Theorem 3.7. *Let $f : X \to Y$ be a function in associated w-spaces. Then the following statements are equivalent:*

(1) f is wg-continuous.

(2) $f(wgC(A)) \subseteq cl(f(A))$ for $A \subseteq X$.
(3) $wgC(f^{-1}(V)) \subseteq f^{-1}(cl(V))$ for $V \subseteq Y$.
(4) $f^{-1}(int(V)) \subseteq wgI(f^{-1}(V))$ for $V \subseteq Y$

Proof. (1) $\Rightarrow$ (2) Let $x \in wgC(A)$. Suppose that $f(x)$ is not in $cl(f(A))$. Then there exists an open set V containing $f(x)$ such that $V \cap f(A) = \emptyset$. Since f is wg-continuous, there is a wg-open set U containing x such that $f(U) \subseteq V$, and so $f(U) \cap f(A) = \emptyset$. Hence $U \cap A = \emptyset$, which is a contradiction to $x \in wgC(A)$. So, $f(w_\tau gC(A)) \subseteq cl(f(A))$.

(2) $\Rightarrow$ (3) Let $A = f^{-1}(B)$ for $B \subseteq Y$. By (2), $f(w_\tau gC(A)) \subseteq cl(f(A)) = cl(f(f^{-1}(B))) \subseteq cl(B)$. So, $wgC(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (1) For each $x \in X$, let V be any open set containing $f(x)$. Then since $f(x) \in int(V)$, and by hypothesis, $x \in f^{-1}(int(V)) \subseteq wgI(f^{-1}(V))$. So, there exists a wg-open set U such that $x \in U \subseteq wgI(f^{-1}(V)) \subseteq f^{-1}(V)$. Hence, f is wg-continuous. □

Corollary 3.8. *Let $f : X \to Y$ be a function in associated w-spaces. Then the following statements are equivalent:*
(1) f is wg-continuous.
(2) $f^{-1}(V) = wgI(f^{-1}(V))$ for every open set $V \in Y$.
(3) $f^{-1}(B) = wgC(f^{-1}(B))$ for every closed set $B \subseteq Y$.

Proof. It is obvious. □

Theorem 3.9. *Let $f : X \to Y$ be a function in associated w-spaces. Then f is wg^*-continuous if and only if for every closed set F in Y, $f^{-1}(F)$ is wg-closed in X.*

Proof. It is obvious. □

Let (X, w_τ) be an associated w-space with a topology τ. Let $W_\tau g(x)$ denote the set of all $w_\tau g$-open set containing x in X, and $O(x)$ denote the set of all open set containing x in X.

A collection $\mathcal{H}$ of subsets of X is called an *m-family* [7] on X if $\cap\mathcal{H} \neq \emptyset$. Let $f : X \to Y$ be a function; then it is obvious $f(\mathcal{H}) = \{f(F) : F \in \mathcal{H}\}$ is an m-family on Y. If $\mathcal{F}$ is a filter base, we denote by $< \mathcal{F} >$ the filter generated by $\mathcal{H}$.

Definition 3.10. Let $\mathcal{H}$ be an m-family on X. Then we say that an m-family $\mathcal{H}$ *wg-converges* to $x \in X$ if $\mathcal{H}$ is finer than $W_\tau g(x)$ i.e., $W_\tau g(x) \subseteq \mathcal{H}$.

Theorem 3.11. *Let $f : X \to Y$ be a function in associated w-spaces. Then if f is wg-continuous, then for an m-family $\mathcal{H}$ wg-converging to $x \in X$, a filter $< f(\mathcal{H}) >$ converges to $f(x)$.*

Proof. Suppose that f is wg-continuous and $\mathcal{H}$ is an m-family wg-converging to $x \in X$. Since f is wg-continuous, for an open set V containing $f(x)$, there exists a wg-open set U containing x such that $f(U) \subseteq V$. Since $f(W_\tau g(x)) \subseteq f(\mathcal{H})$, $V \in < f(\mathcal{H}) >$ i.e., $O(f(x)) \subseteq < f(\mathcal{H}) >$. Hence the filter $< f(\mathcal{H}) >$ converges to $f(x)$. □

Theorem 3.12. *Let $f : X \to Y$ be a bijective function in associated w-spaces. Then f is wg^*-continuous iff for an m-family $\mathcal{H}$ wg-converging to $x \in X$, the filter $< f(\mathcal{H}) >$ converges to $f(x)$.*

Proof. Suppose that f is wg^*-continuous and $\mathcal{H}$ is an m-family wg-converging to $x \in X$. Since f is wg^*-continuous and surjective, it is satisfied that $O(f(x)) \subseteq f(W_\tau g(x)) \subseteq f(\mathcal{H})$. So, the filter $< f(\mathcal{H}) >$ converges to $f(x)$.

Conversely, let $U \in O(f(x))$ for $U \subseteq Y$. Since the family $W_\tau g(x)$ clearly wg-converges to x, by hypothesis, it is obtained that $O(f(x)) \subseteq < f(W_\tau g(x)) >$ for $x \in X$. Since f is injective, $f^{-1}(U) \in W_\tau g(x)$. So, f is wg^*-continuous. □

We recall that: Let $f : X \to Y$ be a function on w-spaces. Then f is said to be *W-continuous* [9] if for $x \in X$ and for each w-open set V containing $f(x)$, there is a w-open set U containing x such that $f(U) \subseteq V$.

Theorem 3.13 ([9]). *Let $f : (X, w_X) \to (Y, w_Y)$ be a function in two w-spaces. Then f is W-continuous if and only if $f^{-1}(wI(B)) \subseteq wI(f^{-1}(B))$ for $B \subseteq Y$.*

Theorem 3.14 ([15]). *Let (X, w_τ) be an associated w-space with a topology τ and $A \subseteq X$. Then A is wg-open if and only if $F \subseteq wI(A)$ whenever $F \subseteq A$ and F is closed.*

Theorem 3.15. *Let $f : X \to Y$ be a function in associated w-spaces. Then if f is W-continuous and closed, then f is wg^*-continuous, i.e., for every open subset B in Y, $f^{-1}(B)$ is wg-open.*

Proof. Let B be any open subset in Y and F be a closed set in X such that $F \subseteq f^{-1}(B)$. Now, we show that $F \subseteq wI(f^{-1}(B))$. Since B is open and $f(F)$ is closed, $f(F) \subseteq B = int(B) \subseteq wI(B)$. From (4) of Theorem 3.13, it follows that $F \subseteq f^{-1}(wI(B)) \subseteq wI(f^{-1}(B))$. Hence, $f^{-1}(B)$ is wg-open. Hence, f is wg^*-continuous. □

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