INVESTIGATION OF SOME PROPERTIES OF THE COHOMOLOGY OF \(SO(n)\) AND ITS CLASSIFYING SPACE

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Abstract: In this paper we determine those generators of \(H^*(SO(n),\mathbb{Z}_2)\) which are connected by Steenrod operations. We also study some properties of \(H^*(X_G)\) using the Leray - Serre spectral sequence of the fibration \(X \to X_G \to BG\).

AMS Subject Classification: 18G40, 57T10, 55R10
Key Words: steenrod operations, spectral sequence, Borel cohomology

1. Introduction and Preliminaries

In 1878 Lucas [1] gave a method to determine the value of \(\binom{n}{m} \mod p\) easily: Let \(m_0\) and \(n_0\) be the least non-negative remainders of \(m\) and \(n \mod p\), respectively. Then

\[
\binom{n}{m} = \left\lfloor \frac{n}{p} \right\rfloor \binom{m_0}{n_0} \mod p,
\]

where \(\lfloor x \rfloor\) denotes the greatest integer \(\leq x\), and we use the convention \(\binom{r}{s} = 0\) if \(r < s\).
Expressing

\[ n = n_0 + n_1 p + n_2 p^2 + \cdots + n_d p^d \]

and

\[ m = m_0 + m_1 p + m_2 p^2 + \cdots + m_d p^d \]

in base \( p \) (so that \( 0 \leq m_i, n_i \leq p - 1 \) for each \( i \)), this may also be expressed as

\[
\binom{n}{m} \equiv \binom{n_0}{m_0} \binom{n_1}{m_1} \cdots \binom{n_d}{m_d} \pmod{p}.
\]

## 2. Main Results

**Theorem 1** (Cohomology Leray-Serre Spectral Sequence, see [3]). Let \( R \) be a commutative ring with unit. Suppose \( F \hookrightarrow E \to B \) is a fibration, where \( B \) is path-connected and \( F \) is connected. Then there is a first quadrant spectral sequence of algebras, \( \{E_r, d_r\} \), converging to \( H^*(E, R) \) as an algebra, with \( E_2^{p,q} \cong H^p(B, H^q(F, R)) \): the cohomology of the space \( B \) with local coefficients in the cohomology of the fiber \( F \).

**Remark 2.** When \( G \) is connected (its classifying space), \( B_G \) is simply connected, hence the local coefficient system is simple. Therefore (for the Leray-Serre spectral sequence for the universal \( G \) bundle \( E_G \to B_G \))

\[
H^p(B_G, H^q(G, R)) \cong H^p(B_G, R) \otimes_R H^q(G, R),
\]

when \( R \) is a field, see [3].

From now on all cohomologies will have coefficients in \( \mathbb{Z}_2 \). We have

\[
H^i(SO(n), \mathbb{Z}_2) = (\Delta(e_1, e_2, \ldots, e_{n-1}))^i.
\]

For which, an additive basis is \( \{(e_{i_1} \cdots e_{i_k}) \mid 1 \leq i_1 < \cdots < i_k, \ i_1 + i_2 + \cdots + i_k = i\} \) (with obvious commutative multiplication), see [4].

**Definition 3** (Steenrod Operations, see [5]). The operations \( Sq^i \) have the following properties:

1. \( Sq^i : H^j(X, A, \mathbb{Z}_2) \to H^{i+j}(X, A, \mathbb{Z}_2) \);
2. \( Sq^0 = \text{Id} \) (identity homomorphism);
3. If \( p < i \), \( Sq^i(u) = 0 \), for all \( u \in H^p(X, A, \mathbb{Z}_2) \);
4. \( Sq^i(u) = u^2 \) (for all \( u \in H^i(X, A, \mathbb{Z}_2) \));

5. \( Sq^k(u \cup v) = \sum_{i+j=k} Sq^i(u) \cup Sq^j(v) \) (\( u, v \in H^q(X, A, \mathbb{Z}_2) \));

6. \( Sq^i \) is natural homomorphism. If \( f : (X, A) \to (Y, B) \), \( H^{k+i}(f) Sq^i = Sq^i H^k(f) \);

7. \( \delta^* Sq^i = Sq^i \delta^* \) where \( \delta^* : H^*(A, \mathbb{Z}_2) \to H^*(X, A, \mathbb{Z}_2) \).

**Theorem 4** (see [4]). Let \( k \) be a field of characteristic 2.

1. For \( e_i \in H^i(SO(n), k) \), one has
   \[
   H^*(SO(n), k) = \Delta(e_1, e_2, \ldots, e_{n-1}).
   \]

2. \( e_i^2 = e_{2i}, (e_i^2 = 0 \text{ if } 2i \geq n) \), hence
   \[
   H^*(SO(n), \mathbb{Z}_2) = \frac{\mathbb{Z}_2[x_1, x_3, \ldots, x_{2m-1}]}{(e_1^a_1, e_3^a_3, \ldots, e_{2m}^a_{2m-1})},
   \]
   for \( m = \left\lfloor \frac{n}{2} \right\rfloor \), \( a_i \) is the smallest power of 2 for which \( a_i(2i - 1) \geq n \).

3. If \( k = \mathbb{Z}_2 \), then \( Sq^i(e_i) = (\sum_{j}^i) e_{i+j} \), for \( j \leq i \), \( i+j < n \), \( (0, \text{ for } i+j \geq n) \).

Let the generators of \( H^*(SO(n), \mathbb{Z}_2) \) (\( e_i \ (1 \leq i < n) \)) be vertices and let us connect \( e_i \) and \( e_j \) if \( i < j \) and \( Sq^i e_i = e_j \). We will call this the graph of \( H^*(SO(n); \mathbb{Z}_2) \).

**Theorem 5.** Graph of \( H^*(SO(n); \mathbb{Z}_2) \) is connected for \( n \neq 2^k \). For \( n = 2^k \) graph is disconnected, with one isolated vertex corresponding to \( e_{2^{k-1}} \).

**Proof.** By induction. For \( n = 2, 3, 4 \) the claim is true.

1. Suppose that for \( n = 2^k \) claim is true. We prove the claim for \( n + 1 \):
   \[
   H^*(SO(2^k), \mathbb{Z}_2) = \Delta(e_1, e_2, \ldots, e_{2^{k-1}}),
   \]
   \[
   H^*(SO(2^k + 1), \mathbb{Z}_2) = \Delta(e_1, e_2, \ldots, e_{2^k}),
   \]
   \[
   Sq^1 e_{2^{k-1}} = \binom{2^k - 1}{1} e_{2^k} = e_{2^k},
   \]
   \[
   Sq^2 e_{2^{k-2}} = \binom{2^k - 2}{2} e_{2^k} = \frac{(2^k - 2)!}{(2^k - 4)!2!} e_{2^k} = \frac{(2^k - 3)(2^k - 2)}{2} e_{2^k}
   \]

\[(2^k - 3)(2^{k-1} - 1)e_{2^k} = e_{2^k} \pmod{2}.\]

2. For \(n \neq 2^k\) and \(n + 1 \neq 2^k\), we prove that the claim is true:

\[
H^*(SO(n), \mathbb{Z}_2) = \Delta(e_1, e_2, \ldots, e_{n-1}),
\]
\[
H^*(SO(n+1), \mathbb{Z}_2) = \Delta(e_1, e_2, \ldots, e_n).
\]

Let \(1 \leq 2^k - 1 \leq n - 1\). Since

\[
Sq^{1-2^k+n}e_{2^k-1} = \left(\frac{2^k - 1}{1 - 2^k + n}\right)_{e_n}^2 = e_n
\]

(by Lucas's Theorem) for \(n \neq 2^k\) and \(n + 1 \neq 2^k\). So, our hypothesis is true.

3. Let \(n \neq 2^k\) and \(n + 1 = 2^k\). Then

\[
H^*(SO(n + 1), \mathbb{Z}_2) = \Delta(e_1, e_2, \ldots, e_n),
\]

\(\forall i \leq 2^k - 1\). We will show that there is no path from \(e_i\) to \(e_n\).

For \(j \neq 0, j \leq i, i + j = n = 2^k - 1\), \(Sq^j e_i = (i)_{j}^2 e_{i+j} = 0 e_{2^k-1} = 0\), by Lucas' Theorem.

\[\Box\]

**Claim 6.** If \(f : SO(n) \rightarrow X\) be a continuous map such that \(e_{2^k-1} \in \text{Im } f^*\), \(0 < k < \frac{\ln n}{\ln 2}\) (where \(f^* : H^*(X, \mathbb{Z}_2) \rightarrow H^*(SO(n), \mathbb{Z}_2)\) is the induced homomorphism in cohomology ). Then \(f^*\) is onto.

\[\text{Proof.}\] By Lucas' theorem, for every \(i \leq 2^{k-1} - 1\), we have \((2^{k-1}-1)_{i}^2 = 1\).

Since \(Sq^i e_j = (i)_{2}^j e_{i+j}, i \leq j, \) for \(i \leq 2^{k-1} - 1\), we obtain \(Sq^i e_{2^k-1} = e_{2^k-1+i-1}\).

Since there are \(a_1, a_3, \ldots, a_{2^k-1} \in H^*(X)\) such that \(f^*(a_{2^k-1}) = e_{2^k-1}\) with \(0 < k < \frac{\ln n}{\ln 2}\). For \(i \leq 2^{k-1} - 1\), we obtain \(f^*(Sq^i a_{2^k-1-i}) = Sq^i (e_{2^k-1-i}) = e_{2^k-1+i-1}\), since the Steenrod squares are natural. Thus, \(f^*\) is onto. \[\Box\]

**Definition 7** (see [3]). Given a fibration \(F \hookrightarrow E \rightarrow B\), if the homomorphism \(i^* : H^*(E, k) \rightarrow H^*(F, k)\) is onto, we say \(F\) is totally nonhomologous to zero (TNHZ) (in \(F \hookrightarrow E \rightarrow B\) with respect to \(k\))

**Proposition 8** (see [3]). Given a fibration \(F \hookrightarrow E \rightarrow B\). \(F\) is totally nonhomologous to zero in \(E\) with respect to \(k\) if and only if the spectral sequence, \(E_r(B, E, F)\) (with coefficients in \(k\)), collapses at the \(E_2\)-term.
Theorem 9 (Wu's Formula, see [4]). \( H^*(BSO(n), \mathbb{Z}_2) = \mathbb{Z}_2[w_2, w_3, \ldots, w_{n-1}] \),

\[ Sq^j w_i = \sum_{k=0}^{j} \binom{i-k-1}{j-k} w_{i+j-k} w_k \quad (0 \leq j \leq i) \]

(assuming \( w_0 = 1 \) and \( w_1 = 0 \)).

Definition 10 (see [3]). Suppose \( F \hookrightarrow E \rightarrow B \) is fibration and \( x \in H^r(F) \). We say that \( x \) is transgressive if \( d_2(x) = d_3(x) = \cdots = d_r(x) = 0 \) and \( d_{r+1}(x) : H^r(F) \rightarrow H^{r+1}(B) \), \( d_{r+1}(x) \neq 0 \) The induced homomorphism from a submodule of \( H^r(F) \) to a quotient of \( H^{r+1}(B) \) is called transgression and is usually denoted by \( \tau \).

Alternative definition of the transgression map:

Definition 11 (see [2]). Given a fibration \( F \hookrightarrow E \twoheadrightarrow B \) with base \( B \) and fibre \( F \). The long exact sequence of cohomology of the pair \((E, F)\):

\[ \cdots \rightarrow H^{r-1}(F) \xrightarrow{\delta} H^r(E, F) \rightarrow H^r(E) \rightarrow \cdots \]

Let \( \tau \) denote the homomorphism, \( \tau : \delta^{-1}(\text{Im } p^*) \rightarrow \frac{H^r(B)}{\text{Ker } p^*} \), given by \( \tau(x) = y + \text{Ker } p^* \) where \( \delta(x) = p^*(y) \). The homomorphism \( \tau \) is called the transgression.

Proposition 12 (see [2]). If \( x \in H^*(F) \) is transgressive then \( Sq^j(x) \) is also transgressive and \( \tau(Sq^j(x)) = Sq^j(\tau(x)) \) (coefficients in \( \mathbb{Z}_2 \)).

Theorem 13 (see [3]). Given a fibration \( F \hookrightarrow E \rightarrow B \) with \( B \) path connected and \( F \) connected, the following hold for its associated cohomology Leray-Serre spectral sequence

1. \( E_{n,0}^{0,0} \cong \frac{H^n(B)}{\text{Ker } p^*} \);
2. \( E_{n,0}^{0,n-1} \cong \delta^{-1}(\text{Im } p^*_0) \subseteq H^{n-1}(F) \);
3. \( d_n \) : \( E_{n,0}^{0,n-1} \rightarrow E_{n,n}^{0,0} \) is the transgression.

Theorem 14 (see [3]). Given a fibration \( F \hookrightarrow E \rightarrow B \) with \( B \) path connected and \( F \) connected, and for which the system of local coefficients on \( B \) is simple; then the composites

\[ H^q(B, R) = E_2^{q,0} \twoheadrightarrow E_3^{q,0} \twoheadrightarrow \cdots \twoheadrightarrow E_{q+1}^{q,0} = E_\infty^{q,0} \subseteq H^q(E, R), \]

and

\[ H^q(E, R) \twoheadrightarrow E_\infty^{0,q} = E_{q+2}^{0,q} \subseteq E_{q}^{0,q} \subseteq \cdots \subseteq E_2^{0,q} = H^q(F, R) \]

are the homomorphisms, \( p^* : H^q(B, R) \rightarrow H^q(E, R) \) and \( i^* : H^q(E, R) \rightarrow H^q(F, R) \), respectively (Edge homomorphisms).
Theorem 15. Let \( X \to E \to B_G \) be a fibration such that \( X = G = SO(n) \). Then \( e_1 \in \text{Im} \ i^* \iff e_2 \in \text{Im} \ i^* \).

Proof. The necessity is clear, by Claim 6.

Sufficiency. Let \( e_1 \notin \text{Im} \ i^* \). We’ll prove \( e_2 \notin \text{Im} \ i^* \).

Assume that \( d_2(e_1) = 0 \). Then \( e_1 \in E_3^{0,q} \) since \( E_3^{0,q} = \text{ker} \ d_2 \), \( e_1 \in E_4^{0,q} \) since \( d_3(e_1) = 0 \) and \( E_4^{0,q} = \text{ker} \ d_3 \). If we continue in this way, then \( e_1 \in E_\infty^{0,q} \). \( \square \)

Since \( E_2^{1,0} \) is 0 so is \( E_\infty^{1,0} \). Therefore \( H^1(E) \cong E_\infty^{1,0} \cong E_2^{1,0} \).

\[ \begin{array}{ccc}
H^1(E) & \xrightarrow{i^*} & H^1(X) \\
\downarrow & & \downarrow \\
E_\infty^{0,1} & & \\
\end{array} \]

The above diagram (\( i \) is edge homomorphism) is commutative. So there is \( a_1 \in H^*(E) \) such that \( i^*(a_1) = e_1 \). Contradiction. Thus if \( e_1 \notin \text{Im} \ i^* \) then \( d_2(e_1) \neq 0 \). Let \( d_2 : E_2^{0,1} \to E_2^{2,0}, \ d_2(e_1) = w_2 \neq 0, \) (so \( E_2^{2,0} = 0 \), hence \( E_\infty^{2,0} = 0 \)) Since \( d_2 \) is transgression and \( e_1 \) is transgressive, we find that \( Sq^1 e_1 = e_2 \) is transgressive and \( Sq^1 d_2 e_1 = d_3 Sq^1 e_1 \). So \( d_3 e_2 = Sq^1 w_2 \). \( Sq^1 w_2 = (1)^2 w_3 \cdot w_0 + (0)^2 w_2 \cdot w_1 = w_3 \) (since \( w_1 = 0 \) in \( H^*(B_{SO(n)}) \)) \( d_2 \colon E_2^{0,2} \to E_2^{2,1}, d_2(e_2) = 0 \) and \( d_3(e_2) \neq 0 \). \( e_2 \notin \text{Ker} \ d_3 \), since \( d_3(e_2) \neq 0 \). \( e_2 \in E_3^{0,2} \), since \( e_2 \in E_2^{0,2} \) and \( d_2(e_2) = 0 \) \( e_2 \notin \text{Im} \ E_4^{0,2}, \ldots, e_2 \notin E_\infty^{0,2} \), because \( d_3(e_2) \neq 0 \). \( E_2^{0,2} = H^2(SO(n)) = < e_2 >, E_3^{0,2} = \text{Ker} \{ E_2^{0,2} \to E_2^{2,1} \} = E_2^{0,2} = E_3^{0,2} \), since \( d_2(e_2) = 0 \). \( E_4^{0,2} = \text{Ker} \{ E_3^{0,2} \to E_3^{3,0} \}, d_3(e_2) \neq 0 \), since \( E_4^{0,2} = \{0\} \). \( E_\infty^{0,2} = \{0\} \) since \( E_\infty^{0,2} \subseteq E_4^{0,2} = 0 \).

Also, obviously \( E_\infty^{1,1} = 0 \) since \( E_2^{1,1} = 0 \). Since \( E_\infty^{0,2} = E_\infty^{1,1} = E_\infty^{2,0} = 0 \), we see that \( H^2(E) = 0 \). Therefore \( e_2 \notin \text{Im} \ i^* \).
References


