International Journal of Pure and Applied Mathematics
Volume 112 No. 3 2017, 605-611
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v112i3.12

INVESTIGATION OF SOME PROPERTIES OF THE COHOMOLOGY OF $SO(n)$ AND ITS CLASSIFYING SPACE

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Abstract: In this paper we determine those generators of $H^*(SO(n), \mathbb{Z}_2)$ which are connected by Steenrod operations. We also study some properties of $H^*(X_G)$ using the Leray - Serre spectral sequence of the fibration $X \to X_G \to B_G$.

AMS Subject Classification: 18G40, 57T10, 55R10
Key Words: steenrod operations, spectral sequence, Borel cohomology

1. Introduction and Preliminaries

In 1878 Lucas [1] gave a method to determine the value of $\binom{n}{m}(\bmod p)$ easily: Let m_0 and n_0 be the least non-negative remainders of m and n $(\bmod p)$, respectively. Then

$$\binom{n}{m} = \binom{\left\lfloor \frac{n}{p} \right\rfloor}{\left\lfloor \frac{m}{p} \right\rfloor}\binom{n_0}{m_0}(\bmod p),$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$, and we use the convention $\binom{r}{s} = 0$ if $r < s$.

Received: January 2, 2017
Revised: January 20, 2017
Published: February 9, 2017

Expressing

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_d p^d$$

and

$$m = m_0 + m_1 p + m_2 p^2 + \cdots + m_d p^d$$

in base p (so that $0 \leq m_i, n_i \leq p-1$ for each i), this may also be expressed as

$$\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\cdots\binom{n_d}{m_d} (\bmod p).$$

2. Main Results

Theorem 1 (Cohomology Leray-Serre Spectral Sequence, see [3]). *Let R be a commutative ring with unit. Suppose $F \hookrightarrow E \to B$ is a fibration, where B is path-connected and F is connected. Then there is a first quadrant spectral sequence of algebras, $\{E_r^{*,*}, d_r\}$, converging to $H^*(E,R)$ as an algebra, with $E_2^{p,q} \cong H^p(B, H^q(F,R))$: the cohomology of the space B with local coefficients in the cohomology of the fiber F.*

Remark 2. When G is connected (its classifying space), B_G is simply connected, hence the local coefficient system is simple. Therefore (for the Leray-Serre spectral sequence for the universal G bundle $E_G \to B_G$)

$$H^p(B_G, H^q(G,R)) \cong H^p(B_G, R) \otimes_R H^q(G,R),$$

when R is a field, see [3].

From now on all cohomologies will have coefficients in $\mathbb{Z}_2$. We have

$$H^i(SO(n), \mathbb{Z}_2) = (\Delta(e_1, e_2, \ldots, e_{n-1}))^i.$$

For which, an additive basis is $\{(e_{i_1} \cdots e_{i_k}) \mid 1 \leq i_1 < \cdots < i_k,\ i_1 + i_2 + \cdots + i_k = i\}$ (with obvious commutative multiplication), see [4].

Definition 3 (Steenrod Operations, see [5]). The operations Sq^i have the following properties:

1. $Sq^i : H^j(X, A, \mathbb{Z}_2) \to H^{i+j}(X, A, \mathbb{Z}_2)$;
2. $Sq^0 = Id$ (identity homomorphism);
3. If $p < i$, $Sq^i(u) = 0$, for all $u \in H^p(X, A, \mathbb{Z}_2)$;

4. $Sq^i(u) = u^2$ (for all $u \in H^i(X, A, \mathbb{Z}_2)$);

5. $Sq^k(u \cup v) = \sum_{i+j=k} Sq^i(u) \cup Sq^j(v)$ $(u, v \in H^q(X, A, \mathbb{Z}_2))$;

6. Sq^i is natural homomorphism. If $f : (X, A) \to (Y, B)$, $H^{k+i}(f)Sq^i = Sq^i H^k(f)$;

7. $\delta^* Sq^i = Sq^i \delta^*$ where $\delta^* : H^*(A, \mathbb{Z}_2) \to H^*(X, A, \mathbb{Z}_2)$.

Theorem 4 (see [4])**.** *Let k be a field of characteristic 2.*

1. For $e_i \in H^i(SO(n), k)$, one has

$$H^*(SO(n), k) = \Delta(e_1, e_2, \dots, e_{n-1}).$$

2. ${e_i}^2 = e_{2i}, ({e_i}^2 = 0$ if $2i \geq n)$, hence

$$H^*(SO(n), \mathbb{Z}_2) = \frac{\mathbb{Z}_2[x_1, x_3, \dots, x_{2m-1}]}{(e_1^{a_1}, e_3^{a_2}, \dots, e_{2m-1}^{a_m})},$$

for $m = \left\lfloor \frac{n}{2} \right\rfloor$, a_i is the smallest power of 2 for which $a_i(2i-1) \geq n$.

3. If $k = \mathbb{Z}_2$, then $Sq^j(e_i) = \binom{i}{j}_2 e_{i+j}$, for $j \leq i$, $i+j < n$, $(0$, for $i+j \geq n)$.

Let the generators of $H^*(SO(n), \mathbb{Z}_2)$ (e_i $(1 \leq i < n)$) be vertices and let us connect e_i and e_j if $i < j$ and $Sq^{j-i} e_i = e_j$. We will call this the graph of $H^*(SO(n); \mathbb{Z}_2)$

Theorem 5. *Graph of $H^*(SO(n); \mathbb{Z}_2)$ is connected for $n \neq 2^k$. For $n = 2^k$ graph is disconnected, with one isolated vertex corresponding to e_{2^k-1}.*

Proof. By induction. For, $n = 2, 3, 4$ the claim is true.

1. Suppose that for $n = 2^k$ claim is true. We prove the claim for $n+1$:

$$\begin{aligned} H^*(SO(2^k), \mathbb{Z}_2) &= \triangle\,(e_1, e_2, \dots, e_{2^k-1}), \\ H^*(SO(2^k+1), \mathbb{Z}_2) &= \triangle\,(e_1, e_2, \dots, e_{2^k}), \\ Sq^1 e_{2^k-1} &= \binom{2^k-1}{1}_2 e_{2^k} = e_{2^k}, \\ Sq^2 e_{2^k-2} &= \binom{2^k-2}{2}_2 e_{2^k} \\ &= \frac{(2^k-2)!}{(2^k-4)!2!} e_{2^k} = \frac{(2^k-3)(2^k-2)}{2} e_{2^k} \end{aligned}$$

$$=(2^k-3)(2^{k-1}-1)e_{2^k}=e_{2^k}\ (\bmod 2).$$

2. For $n\neq 2^k$ and $n+1\neq 2^k$, we prove that the claim is true:

$$H^*(SO(n),\mathbb{Z}_2)=\triangle\,(e_1,e_2,\ldots,e_{n-1}),$$
$$H^*(SO(n+1),\mathbb{Z}_2)=\triangle\,(e_1,e_2,\ldots,e_n).$$

Let $1\leq 2^k-1\leq n-1$. Since

$$Sq^{1-2^k+n}e_{2^k-1}=\binom{2^k-1}{1-2^k+n}_2 e_n=e_n$$

(by Lucas's Theorem) for $n\neq 2^k$ and $n+1\neq 2^k$. So, our hypothesis is true.

3. Let $n\neq 2^k$ and $n+1=2^k$. Then

$$H^*(SO(n+1),\mathbb{Z}_2)=\triangle(e_1,e_2,\ldots,e_n),$$

$\forall i\leq 2^k-1$. We will show that there is no path from e_i to e_n.

For $j\neq 0,\ j\leq i,\ i+j=n=2^k-1$, $Sq^je_i=\binom{i}{j}_2e_{i+j}=0e_{2^k-1}=0$, by Lucas' Theorem. □

Claim 6. *If $f:SO(n)\to X$ be a continuous map such that $e_{2^k-1}\in \operatorname{Im} f^*$, $0<k<\frac{\ln n}{\ln 2}$ (where $f^*:H^*(X,\mathbb{Z}_2)\to H^*(SO(n),\mathbb{Z}_2)$ is the induced homomorphism in cohomology). Then f^* is onto.*

Proof. By Lucas' theorem, for every $i\leq 2^{k-1}-1$, we have $\binom{2^{k-1}-1}{i}_2=1$.

Since $Sq^ie_j=\binom{j}{i}_2e_{i+j},\ i\leq j,$ for $i\leq 2^{k-1}-1$, we obtain $Sq^ie_{2^{k-1}-1}=e_{2^{k-1}+i-1}$.

Since there are $a_1,a_3,\ldots,a_{2^k-1}\in H^*(X)$ such that $f^*(a_{2^k-1})=e_{2^k-1}$ with $0<k<\frac{\ln n}{\ln 2}$. For $i\leq 2^{k-1}-1$, we obtain $f^*(Sq^ia_{2^{k-1}-1})=Sq^i(e_{2^{k-1}-1})=e_{2^{k-1}-1+i}$, since the Steenrod squares are natural. Thus, f^* is onto. □

Definition 7 (see [3]). Given a fibration $F\hookrightarrow E\to B$, if the homomorphism $i^*:H^*(E,k)\to H^*(F,k)$ is onto, we say F is totally nonhomologous to zero (TNHZ) (in $F\hookrightarrow E\to B$ with respect to k)

Proposition 8 (see [3]). *Given a fibration $F\hookrightarrow E\to B$. F is totally nonhomologous to zero in E with respect to k if and only if the spectral sequence, $E_r(B,E,F)$ (with coefficients in k), collapses at the E_2-term.*

Theorem 9 (Wu's Formula, see [4]). $H^*(BSO(n),\mathbb{Z}_2) = \mathbb{Z}_2[w_2, w_3, \ldots, w_{n-1}]$,

$$Sq^j w_i = \sum_{k=0}^{j} \binom{i-k-1}{j-k}_2 w_{i+j-k} w_k \ (0 \le j \le i)$$

(assuming $w_0 = 1$ and $w_1 = 0$).

Definition 10 (see [3]). Suppose $F \hookrightarrow E \to B$ is fibration and $x \in H^r(F)$. We say that x is transgressive if $d_2(x) = d_3(x) = \cdots = d_r(x) = 0$ and $d_{r+1}(x) : H^r(F) \to H^{r+1}(B)$, $d_{r+1}(x) \neq 0$ The induced homomorphism from a submodule of $H^r(F)$ to a quotient of $H^{r+1}(B)$ is called transgression and is usually denoted by τ.

Alternative definition of the transgression map is:

Definition 11 (see [2]). Given a fibration $F \hookrightarrow E \xrightarrow{p} B$ with base B and fibre F. The long exact sequence of cohomology of the pair (E, F):

$$\cdots \to H^{r-1}(F) \xrightarrow{\delta} H^r(E,F) \to H^r(E) \to \cdots .$$

Let τ denote the homomorphism, $\tau : \delta^{-1}(\operatorname{Im} p^*) \to \frac{H^r(B)}{\operatorname{Ker} p^*}$, given by $\tau(x) = y + \operatorname{Ker} p^*$ where $\delta(x) = p^*(y)$. The homomorphism τ is called the transgression.

Proposition 12 (see [2]). *If $x \in H^*(F)$ is transgressive then $Sq^i(x)$ is also transgressive and $\tau(Sq^i(x)) = Sq^i(\tau(x))$ (coefficients in $\mathbb{Z}_2$).*

Theorem 13 (see [3]). *Given a fibration $F \hookrightarrow E \to B$ with B path connected and F connected, the following hold for its associated cohomology Leray-Serre spectral sequence*

1. $E_n^{n,0} \cong \dfrac{H^n(B)}{\operatorname{Ker} p^*}$;
2. $E_n^{0,n-1} \cong \delta^{-1}(\operatorname{Im} {p_0}^*) \subseteq H^{n-1}(F)$;
3. $d_n : {E_n}^{0,n-1} \to {E_n}^{n,0}$ *is the transgression.*

Theorem 14 (see [3]). *Given a fibration $F \hookrightarrow E \to B$ with B path connected and F connected, and for which the system of local coefficients on B is simple; then the composites*

$$H^q(B,R) = E_2^{q,0} \twoheadrightarrow E_3^{q,0} \twoheadrightarrow \cdots \twoheadrightarrow E_{q+1}^{q,0} = E_\infty^{q,0} \subset H^q(E,R),$$

and

$$H^q(E,R) \twoheadrightarrow E_\infty^{0,q} = E_{q+2}^{0,q} \subset E_q^{0,q} \subset \cdots \subset E_2^{0,q} = H^q(F,R)$$

are the homomorphisms, $p^* : H^q(B,R) \to H^q(E,R)$ and $i^* : H^q(E,R) \to H^q(F,R)$, respectively (Edge homomorphisms).

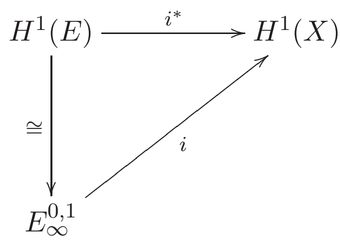
Theorem 15. *Let $X \hookrightarrow E \to B_G$ be a fibration such that $X = G = SO(n)$. Then $e_1 \in \operatorname{Im} i^* \Leftrightarrow e_2 \in \operatorname{Im} i^*$.*

Proof. The necessity is clear, by Claim 6.

Sufficiency. Let $e_1 \notin \operatorname{Im} i^*$. We'll prove $e_2 \notin \operatorname{Im} i^*$.

Assume that $d_2(e_1) = 0$. Then $e_1 \in E_3^{0,q}$ since $E_3^{0,q} = \ker d_2$, $e_1 \in E_4^{0,q}$ since $d_3(e_1) = 0$ and $E_4^{0,q} = \ker d_3$. If we continue in this way, then $e_1 \in E_\infty^{0,q}$. □

Since $E_2^{1,0}$ is 0 so is $E_\infty^{1,0}$ Therefore $H^1(E) \cong E_\infty^{1,0} \cong E_2^{1,0}$



The above diagram (i is edge homomorphism) is commutative. So there is $a_1 \in H^*(E)$ such that $i^*(a_1) = e_1$. Contradiction. Thus if $e_1 \notin \operatorname{Im} i^*$ then $d_2(e_1) \neq 0$. Let $d_2 : E_2^{0,1} \to E_2^{2,0}$, $d_2(e_1) = w_2 \neq 0$. (so $E_2^{2,0} = 0$, hence $E_\infty^{2,0} = 0$) Since d_2 is transgression and e_1 is transgressive, we find that $Sq^1 e_1 = e_2$ is transgressive and $Sq^1 d_2 e_1 = d_3 Sq^1 e_1$. So $d_3 e_2 = Sq^1 w_2$. $Sq^1 w_2 = \binom{1}{1}_2 w_3 \cdot w_0 + \binom{0}{0}_2 w_2 \cdot w_1 = w_3$(Since $w_1 = 0$ in $H^*(B_{SO(n)})$) $d_2 : E_2^{0,2} \to E_2^{2,1}$, $d_2(e_2) = 0$ and $d_3(e_2) \neq 0$. $e_2 \notin \operatorname{Ker} d_3$, since $d_3(e_2) \neq 0$. $e_2 \in E_3^{0,2}$, since $e_2 \in E_2^{0,2}$ and $d_2(e_2) = 0$ $e_2 \notin E_4^{0,2}, \ldots, e_2 \notin E_\infty^{0,2}$, because $d_3(e_2) \neq 0$. $E_2^{0,2} = H^2(SO(n)) =< e_2 >$, $E_3^{0,2} = \operatorname{Ker}\{E_2^{0,2} \to E_2^{2,1}\}$ $E_2^{0,2} = E_3^{0,2}$, since $d_2(e_2) = 0$. $E_4^{0,2} = \operatorname{Ker}\{E_3^{0,2} \to E_3^{3,0}\}$, $d_3(e_2) \neq 0$, since $E_4^{0,2} = \{0\}$ $E_\infty^{0,2} = \{0\}$ since $E_\infty^{0,2} \subseteq E_4^{0,2} = 0$.

Also, obviously $E_\infty^{1,1} = 0$ since $E_2^{1,1} = 0$. Since $E_\infty^{0,2} = E_\infty^{1,1} = E_\infty^{2,0} = 0$,we see that $H^2(E) = 0$. Therefore $e_2 \notin \operatorname{Im} i^*$.

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