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GENERALIZED BESSEL FUNCTIONS IN TERMS OF GENERALIZED HERMITE POLYNOMIALS

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Abstract: The Hermite polynomials represent a powerful tool to investigate the properties of many families of Special Functions. We present some relevant results where the generalized Hermite polynomials of Kampé de Fériet type, simplify the definitions and the operational properties of the two-variable, generalized Bessel functions and their modified. We also discuss a special class of polynomials, recognized as Hermite polynomials, which present a flexible form to describe the two-index, one-variable Bessel functions. By using the generating function method, we will obtain some relations involving these classes of Hermite polynomials and we can also compare them with the Humbert polynomials and functions.

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1. Introduction

In a previous paper [1], we have discussed the representation of the the Humbert

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functions and the related Humbert polynomials [2] through the generalized Hermite polynomials. In particular, we described the two-index Bessel functions [3,4,5] and the related Humbert functions by using the 2-dimensional generalized Hermite polynomials with two indexes and two variables [1,5]. It is possible to consider a further class of two-index, two-variable Hermite polynomials that can be used to derive relevant operational properties for functions recognized as Bessel functions. We define the following class of Hermite polynomials, by putting directly the related generating function:

$$\exp\left[x(u+v)+uv\right]=\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\frac{u^n}{n!}\frac{v^m}{m!}Q_{n,m}(x) \tag{1}$$

which can be exploited in the form:

$$\begin{aligned}&\exp\left[x(u+v)+uv\right]=\\&=\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty}\frac{x^{n-s}u^{n-s}x^{m-s}v^{m-s}}{(n-s)!(m-s)!}\frac{u^s v^s}{s!}\end{aligned} \tag{2}$$

to give:

$$\begin{aligned}&\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}u^n v^m\sum_{s=-\infty}^{+\infty}\frac{x^{n+m-2s}}{(n-s)!(m-s)!s!}=\\&=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{u^n}{n!}\frac{v^m}{m!}Q_{n,m}(x)\end{aligned} \tag{3}$$

We note that the second series in l.h.s. of the above relation does not make sense when:

$$\begin{aligned}&s<0\\&n-s<0\\&m-s<0\end{aligned}$$

and then, by equating the terms of the same power of the indexes n and m, we can state the explicit form of the two-index, one-variable generalized Hermite polynomial [6]:

$$Q_{n,m}(x)=\sum_{s=0}^{\min(n,m)}s!\binom{n}{s}\binom{m}{s}x^{n+m-2s} \tag{4}$$

Some basic properties for this class of Hermite polynomials can be derived in a straightforward way. We start by deriving with respect to x both sides of equation (1):

$$\begin{aligned}&(u+v)\exp\left[x(u+v)+uv\right]=\\&=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{u^n}{n!}\frac{v^m}{m!}\frac{d}{dx}Q_{n,m}(x)\end{aligned} \tag{5}$$

which, once exploited, gives:

$$\begin{aligned}
&\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\tfrac{u^n v^m}{(n-1)!m!}Q_{n-1,m}(x)+\\
&+\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\tfrac{u^n v^m}{n!(m-1)!}Q_{n,m-1}(x)=\\
&=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\tfrac{u^n}{n!}\tfrac{v^m}{m!}\tfrac{d}{dx}Q_{n,m}(x)
\end{aligned} \tag{6}$$

and by equating the same power of the indexes, we find the following recurrence relation:

$$\frac{d}{dx}Q_{n,m}(x) = nQ_{n-1,m}(x) + mQ_{n,m-1}(x) \tag{7}$$

Following the same procedure, by deriving in equation (1) separately with respect to u and with respect to v, we obtain the further recurrence relations:

$$xQ_{n,m}(x) + mQ_{n,m-1}(x) = Q_{n+1,m}(x) \tag{8}$$

and:

$$xQ_{n,m}(x) + nQ_{n-1,m}(x) = Q_{n,m+1}(x) \tag{9}$$

It is interesting to note that Humbert [1,2] polynomials have the following generating function:

$$\exp(u + v + xuv) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\tfrac{u^m}{m!}\tfrac{v^n}{n!}g_{m,n}(x) \tag{10}$$

and they satisfy similar recurrence relations, indeed:

$$\begin{aligned}
&\tfrac{d}{dx}g_{m,n}(x) = mng_{m-1,n-1}(x),\\
&g_{m+1,n}(x) = g_{m,n}(x) + mxg_{m-1,n}(x),\\
&g_{m,n+1}(x) = g_{m,n}(x) + nxg_{m,n-1}(x).
\end{aligned} \tag{11}$$

This aspect suggests to derive the related differential equations involving the generalized Hermite polynomials of type $Q_{n,m}(x)$. From equations (7), (8) and (9), we deduce in fact:

$$\left(x + \frac{d}{dx}\right)Q_{n,m}(x) = nQ_{n-1,m}(x) + Q_{n+1,m}(x) \tag{12}$$

and

$$\left(x + \frac{d}{dx}\right)Q_{n,m}(x) = mQ_{n,m-1}(x) + Q_{n,m+1}(x) \tag{13}$$

Further relations can be obtained by manipulating the above identities and the recurrence relations stated before.

2. Two-Index, Three-Variable Modified Hermite Polynomials

In the previous section we have introduced a class of two-index generalized Hermite polynomials that present analogies with the two-index Humbert polynomials. In some previous paper [1,7], we have described the structure of the Humbert functions and polynomials and, in particular, we have seen that the so-called modified Humbert functions can be expressed in terms of Hermite polynomials. Moreover, Hermite polynomials play an important role in many branches of special functions since they can be used to generalize the two-index Bessel functions and also to describe the properties of the pseudo-Bessel functions [7]. From this point of view, we can follow the same way explored for the modified Humbert functions (see [8]) to introduce a more general class of the Hermite polynomials described until here. We introduce the two-index, three-variable generalized Hermite polynomials by using the following generating function:

$$\exp\left(xu + yv + zuv\right) = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{u^n}{n!} \frac{v^m}{m!} Q^{\square}_{n,m}(x, y, z) \tag{14}$$

By manipulating the l.h.s. of the above relation, we get:

$$\exp\left(xu + yv + zuv\right) = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} \frac{u^n v^m x^{n-s} y^{m-s} z^s}{(n-s)!(m-s)!s!} \tag{15}$$

As in the case of the polynomials $Q_{n,m}(x)$, the series in the above equation has meaningful when:

$$\begin{aligned} & s \geq 0 \\ & n - s \geq 0 \\ & m - s \geq 0 \end{aligned}$$

and then, by equating the same power of the indexes in the identities (14) and (15), we obtain the explicit form of the two-index, three-variable, modified, generalized Hermite polynomials:

$$Q^{\square}_{n,m}(x, y, z) = \sum_{s=0}^{\min(n,m)} s! \binom{n}{s} \binom{m}{s} x^{n-s} y^{m-s} z^s \tag{16}$$

It is immediate to note that generalized Hermite polynomials of type $Q_{n,m}(x)$ can be seen as a particular case of the polynomials introduced in this section. Indeed, by setting $x = y$ and $z = 1$ in equation (16), we have:

$$Q^{\square}_{n,m}(x, x, 1) = \sum_{s=0}^{\min(n,m)} s! \tbinom{n}{s} \tbinom{m}{s} x^{n-s} x^{m-s} = Q_{n,m}(x) \tag{17}$$

We can find similar recurrence relations for this class of multi-index Hermite polynomials by following the same procedure showed above. In this case, the presence of three variables establish different relations. We have, in fact, by deriving with respect to x:

$$\frac{\partial}{\partial x} Q_{n,m}^{\square}(x,y,z) = nQ_{n-1,m}^{\square}(x,y,z) \tag{18}$$

While, by deriving separately with respect to y and z:

$$\begin{aligned} &\tfrac{\partial}{\partial y} Q_{n,m}^{\square}(x,y,z) = mQ_{n,m-1}^{\square}(x,y,z) \\ &\tfrac{\partial}{\partial z} Q_{n,m}^{\square}(x,y,z) = nmQ_{n-1,m-1}^{\square}(x,y,z) \end{aligned} \tag{19}$$

The above recurrence relations can be used to state a relevant identity for this class of generalized Hermite polynomials. We start by observing that, by combining equation (18) with the first equation in (19), we can write:

$$\begin{aligned} &\tfrac{\partial}{\partial x}\left[\tfrac{\partial}{\partial y} Q_{n,m}^{\square}(x,y,z)\right] = \tfrac{\partial}{\partial x}\left[mQ_{n,m-1}^{\square}(x,y,z)\right] = \\ &= nmQ_{n-1,m-1}^{\square}(x,y,z) \end{aligned} \tag{20}$$

and that, by using the second relation in equation (19), we obtain:

$$\frac{\partial^2}{\partial x \partial y} Q_{n,m}^{\square}(x,y,z) = \frac{\partial}{\partial z} Q_{n,m}^{\square}(x,y,z) \tag{21}$$

The above partial differential equation can be seen as an ordinary linear differential equation with respect to the variable z in literature, with regard to the families of Hermite polynomials. The discussion of this kind of equations has been approached in such way very often (see for instance [9,10,11]). We further note that:

$$Q_{n,m}^{\square}(x,y,0) = x^n y^m$$

and, then, we can state the relevant identity:

$$Q_{n,m}^{\square}(x,y,z) = \exp\left[z\frac{\partial^2}{\partial x \partial y}\right] x^n y^m \tag{22}$$

3. Generalized, Two-Variable Bessel Functions and Related Modified Bessel Functions in Terms of Generalized, Two-Variable Hermite Polynomials

The class of Hermite polynomials of Kampé de Fériet type contains many familiel of polynomials ([5]), in particular the Hermite polynomials defined by the following generating function:

$$\exp(xt + yt^2) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x, y) \tag{23}$$

can be used to describe the two-variable, generalized Bessel functions. By noting that the generating function of the generic Bessel function has the form [4]:

$$\exp\left[\tfrac{x}{2}\left(t - \tfrac{1}{t}\right) + \tfrac{y}{2}\left(t^2 - \tfrac{1}{t^2}\right)\right] = \sum_{n=-\infty}^{+\infty} t^n J_n(x, y) \tag{24}$$

we have:

$$\exp\left[\tfrac{1}{2}\left(xt + yt^2\right) - \tfrac{1}{2}\left(\tfrac{x}{t} - \tfrac{y}{t^2}\right)\right] = \sum_{n=-\infty}^{+\infty} t^n J_n(x, y)$$

and, from equation (23), we get:

$$\sum_{n=-\infty}^{+\infty} t^n J_n(x, y) = \sum_{m=0}^{+\infty} \tfrac{t^m}{m!} H_m\left(\tfrac{x}{2}, \tfrac{y}{2}\right) \sum_{r=0}^{+\infty} \tfrac{t^{-r}}{r!} H_r\left(-\tfrac{x}{2}, -\tfrac{y}{2}\right)$$

By setting $n = m - r$ and by rearranging the indexes, we obtain:

$$\sum_{n=-\infty}^{+\infty} t^n J_n(x, y) = \sum_{n=0}^{+\infty} t^n \sum_{r=0}^{+\infty} \tfrac{1}{(n+r)!r!} H_{n+r}\left(\tfrac{x}{2}, \tfrac{y}{2}\right) H_r\left(-\tfrac{x}{2}, -\tfrac{y}{2}\right)$$

and, by equating the same powers of n, we can finally conclude with:

$$J_n(x, y) = \sum_{r=0}^{+\infty} \frac{1}{(n+r)!r!} H_{n+r}\left(\frac{x}{2}, \frac{y}{2}\right) H_r\left(-\frac{x}{2}, -\frac{y}{2}\right) \tag{25}$$

which represents the generic, two-variable, generalized Bessel function in terms of generalized Hermite polynomials of type $H_n(x, y)$. It is important to note that the above relation gives a representation of the Bessel function $J_n(x, y)$ on the positive, real axis, that is on $(0, +\infty)$, due to the nature of the Hermite polynomials. That said, by noting that the Bessel function satisfies the reflection property [4]:

$$J_{-n}(x, y) = J_n(-x, -y)$$

it is immediate to conclude that equation (25) can be written also for $-n$, giving:

$$J_{-n}(x,y)=\sum_{r=0}^{+\infty}\frac{1}{(n+r)!r!}H_{n+r}\left(\frac{x}{2},\frac{y}{2}\right)H_r\left(-\frac{x}{2},-\frac{y}{2}\right),\ n>0 \tag{26}$$

Equations (25) and (26) give the complete representation of the Bessel function $J_n(x,y)$ in terms of two-variable, generalized Hermite polynomials on the real axis; in fact equation (26) gives, again, a representation on the positive axis $(0,+\infty)$ for the Bessel function $J_{-n}(x,y)$, that is a representation of the Bessel function $J_n(x,y)$ on the axis $(-\infty,0)$. In conclusion, equation (25) shows the complete representation of the Bessel function of two variables in terms of the Hermite polynomials $H_n(x,y)$. The generalized, two-variable modified Bessel function is defined by the following generating function:

$$\exp\left[\frac{x}{2}\left(t+\frac{1}{t}\right)+\frac{y}{2}\left(t^2+\frac{1}{t^2}\right)\right]=\sum_{n=-\infty}^{+\infty}t^n I_n(x,y) \tag{27}$$

with the explicit form given by:

$$I_n(x,y)=\sum_{r=-\infty}^{+\infty}I_{n-2r}(x)I_r(y)$$

It is evident that the generating functions, of the Bessel functions $J_n(x,y)$ and $I_n(x,y)$ have a difference only in the sign; then, we can write the general generating function in the following form:

$$\exp\left[\frac{x}{2}\left(t\pm\frac{1}{t}\right)+\frac{y}{2}\left(t^2\pm\frac{1}{t^2}\right)\right] \tag{28}$$

to indicate both the generating functions of $J_n(x,y)$ and $I_n(x,y)$. It is possible to rearrange the argument (indicated with A) of the above exponential in the form:

$$A=\frac{x}{2}\left(t\mp\frac{1}{t}\right)+\frac{y}{2}\left(t^2\mp\frac{1}{t^2}\right)\pm\left(\frac{x}{t}+\frac{y}{t^2}\right)$$

and then, we can write the exponential in equation (28) as follows:

$$\exp\left[\frac{x}{2}\left(t\mp\frac{1}{t}\right)+\frac{y}{2}\left(t^2\mp\frac{1}{t^2}\right)\right]\exp\left[\pm\left(\frac{x}{t}+\frac{y}{t^2}\right)\right] \tag{29}$$

The above expression allows us to state two relevant relations relating the generalized Bessel functions and their relative modified functions with generalized Hermite polynomials. By considering the expression (29) with the above signs, we have:

$$\sum_{n=-\infty}^{+\infty}t^n I_n(x,y)=\sum_{m=-\infty}^{+\infty}t^m J_m(x,y)\sum_{r=0}^{+\infty}\frac{t^r}{r!}H_r(x,y)$$

by setting $n = m - r$ and by rearranging the indexes, we have:

$$\sum_{n=-\infty}^{+\infty} t^n I_n(x,y) = \sum_{m=-\infty}^{+\infty} t^n \sum_{r=0}^{+\infty} \tfrac{1}{r!} J_{n+r}(x,y) H_r(x,y)$$

and finally, we state the following relation involving the Bessel function, its modified form, and the Hermite polynomials:

$$I_n(x,y) = \sum_{r=0}^{+\infty} \frac{1}{r!} J_{n+r}(x,y) H_r(x,y) \tag{30}$$

By following the same procedure, by considering the down sign in expression (29), we have:

$$\sum_{n=-\infty}^{+\infty} t^n J_n(x,y) = \sum_{m=-\infty}^{+\infty} t^m I_m(x,y) \sum_{r=0}^{+\infty} \tfrac{1}{r!} H_r(x,y)$$

and, then we obtain the second relation linking the Bessel functions, the classical and modified ones, and Hermite polynomials:

$$J_n(x,y) = \sum_{r=0}^{+\infty} \tfrac{1}{r!} I_{n-r}(x,y) H_r(-x,-y) \tag{31}$$

It is well known that the two-variable, generalized Bessel functions of the form $J_n(x,y)$ satisfy the following multiplication theorem [4]:

$$J_n(\lambda x, \mu y) = \lambda^n \sum_{p=0}^{+\infty} J_{n+p}\left(x, y; \tfrac{\mu}{\lambda^2}\right) F_p\left(x, y; \tfrac{\lambda^2}{\mu}\right) \tag{32}$$

where $\lambda, \mu \in \mathbb{R}^-\{-\infty, +\infty\}$ and the one-parameter, two-variable, generalized Bessel function has the form:

$$\sum_{n=-\infty}^{+\infty} t^n J_n(x,y;\tau) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right) + \frac{y}{2}\left(t^2\tau - \frac{1}{t^2\tau}\right)\right] \tag{33}$$

and:

$$F_p\left(x, y; \frac{\lambda^2}{\mu}\right) = \sum_{v=0}^{[p/2]} \tfrac{(1-\lambda^2)^{p-2v}(1-\mu^2)}{(p-2v)!v!} \left(\tfrac{x}{2}\right)^{p-2v} \left(\tfrac{y}{2}\right)^v \left(\tfrac{\lambda^2}{\mu}\right)^v$$

It is possible to derive a similar result for the related modified Bessel function $I_n(x,y)$. By choosing in equation (27) parameters $\lambda, \mu \in \mathbb{R}^-\{-\infty, +\infty\}$, we can formally write:

$$\sum_{n=-\infty}^{+\infty} t^n I_n(\lambda x, \mu y) = \exp\left[\tfrac{\lambda x}{2}\left(t + \tfrac{1}{t}\right) + \tfrac{\mu y}{2}\left(t^2 + \tfrac{1}{t^2}\right)\right] \tag{34}$$

Setting A the argument in the exponential, we can recast it in the following form:

$$A = \tfrac{x}{2}\left(\lambda t + \tfrac{1}{\lambda t}\right) + \tfrac{y}{2}\left(\mu t^2 + \tfrac{1}{\mu t^2}\right) + \tfrac{x}{2}\left(\tfrac{\lambda^2 - 1}{\lambda t}\right) + \tfrac{y}{2}\left(\tfrac{\mu^2 - 1}{\mu t^2}\right)$$

so that equation (34) can be written as:

$$\sum_{n=-\infty}^{+\infty} t^n I_n(\lambda x, \mu y) = \sum_{r=-\infty}^{+\infty} (\lambda t)^r I_r(x) \sum_{s=-\infty}^{+\infty} \mu^s t^{2s} I_s(y) \cdot \\ \cdot \exp\left[\tfrac{x}{2}\left(\tfrac{\lambda^2-1}{\lambda t}\right) + \tfrac{y}{2}\left(\tfrac{\mu^2-1}{\mu t^2}\right)\right]$$

and by expanding the exponential in the r.h.s., we have:

$$\sum_{n=-\infty}^{+\infty} t^n I_n(\lambda x, \mu y) = \sum_{r=-\infty}^{+\infty} (\lambda t)^r I_r(x) \sum_{s=-\infty}^{+\infty} \mu^s t^{2s} I_s(y) \cdot \\ \cdot \sum_{k=0}^{+\infty} \tfrac{1}{k!}\left(\tfrac{\lambda^2-1}{\lambda t}\right)^k \left(\tfrac{x}{2}\right)^k \sum_{v=0}^{+\infty} \tfrac{1}{v!}\left(\tfrac{\mu^2-1}{\mu t^2}\right)^v \left(\tfrac{y}{2}\right)^v$$

By setting $r + 2s = m$ in the first two series, $k + 2v = p$ in the second two series and by rearranging the indexes, we get:

$$\sum_{n=-\infty}^{+\infty} t^n I_n(\lambda x, \mu y) = \sum_{m=-\infty}^{+\infty} (\lambda t)^m \sum_{s=-\infty}^{+\infty} \left(\tfrac{\mu}{\lambda^2}\right)^s I_{m-2s}(x) I_s(y) \cdot \\ \cdot \sum_{p=0}^{+\infty} \left(\tfrac{1}{\lambda t}\right)^p \sum_{v=0}^{+\infty} \tfrac{(\lambda^2-1)^{p-2v}}{(p-2v)!} \tfrac{(\mu^2-1)^v}{v!} \left(\tfrac{x}{2}\right)^{p-2v} \left(\tfrac{y}{2}\right)^v$$

By using the definition of the one-parameter, two-variable generalized Bessel function (see equation (32)), we can easily deduce the expression of the related modified function for the parameter $\frac{\mu}{\lambda^2}$, that is:

$$\sum_{s=-\infty}^{+\infty} \left(\tfrac{\mu}{\lambda^2}\right)^s I_{m-2s}(x) I_s(y) = I_m\left(x, y; \tfrac{\mu}{\lambda^2}\right)$$

and, by setting:

$$F_p\left(x, y; \tfrac{\lambda^2}{\mu}\right) = \sum_{v=0}^{[p/2]} \tfrac{(1-\lambda^2)^{p-2v}(1-\mu^2)}{(p-2v)!v!} \left(\tfrac{x}{2}\right)^{p-2v} \left(\tfrac{y}{2}\right)^v \left(\tfrac{\lambda^2}{\mu}\right)^v$$

where we notice that summation is meaningful up to $v = p/2$, we have:

$$\sum_{n=-\infty}^{+\infty} t^n I_n(\lambda x, \mu y) = \sum_{m=-\infty}^{+\infty} (\lambda t)^m I_m\left(x, y; \tfrac{\mu}{\lambda^2}\right) \cdot \\ \cdot \sum_{p=0}^{+\infty} \left(\tfrac{1}{\lambda t}\right)^p F_p\left(x, y; \tfrac{\lambda^2}{\mu}\right)$$

and, finally, by setting $n = m - p$, rearranging the indexes and equating the same powers of n, we find with the following multiplication theorem related to the modified Bessel function of type $I_n(x, y)$:

$$I_n(\lambda x, \mu y) = \lambda^n \sum_{p=0}^{+\infty} I_{n+p}\left(x, y; \frac{\mu}{\lambda^2}\right) F_p\left(x, y; \frac{\lambda^2}{\mu}\right) \tag{35}$$

The multiplication theorems exposed for the generalized Bessel functions $J_n(x, y)$ (equation (32)) and for their modified $I_n(x, y)$ (equation (35)) can be expressed in terms of generalized Hermite polynomials. We start to observe that the term:

$$F_p\left(x, y; \frac{\lambda^2}{\mu}\right) = \sum_{v=0}^{[p/2]} \frac{(1-\lambda^2)^{p-2v}(1-\mu^2)}{(p-2v)!v!} \left(\frac{x}{2}\right)^{p-2v} \left(\frac{y}{2}\right)^v \left(\frac{\lambda^2}{\mu}\right)^v$$

can be recast in the following form:

$$\begin{aligned} &\sum_{v=0}^{[p/2]} \frac{1}{(p-2v)!v!} \left[\frac{x}{2}(1-\lambda^2)\right]^{p-2v} \left[\frac{y}{2}(1-\mu^2)\frac{\lambda^2}{\mu}\right]^v = \\ &= \frac{1}{p!} H_p\left(\frac{x}{2}(1-\lambda^2), \frac{y}{2}(1-\mu^2)\frac{\lambda^2}{\mu}\right) \end{aligned}$$

which allows us to write the multiplication theorem related to the Bessel function $J_n(x, y)$ in the form:

$$\begin{aligned} J_n(\lambda x, \mu y) &= \lambda^n \sum_{p=0}^{+\infty} \frac{1}{p!} J_{n+p}\left(x, y; \frac{\mu}{\lambda^2}\right) \cdot \\ &\cdot H_p\left(\frac{x}{2}(1-\lambda^2), \frac{y}{2}(1-\mu^2)\frac{\lambda^2}{\mu}\right) \end{aligned} \tag{36}$$

Similarly, for the modified Bessel functions of the form $I_n(x, y)$, we obtain:

$$\begin{aligned} I_n(\lambda x, \mu y) &= \lambda^n \sum_{p=0}^{+\infty} \frac{1}{p!} I_{n+p}\left(x, y; \frac{\mu}{\lambda^2}\right) \cdot \\ &\cdot H_p\left(\frac{x}{2}(1-\lambda^2), \frac{y}{2}(1-\mu^2)\frac{\lambda^2}{\mu}\right) \end{aligned} \tag{37}$$

In this section we have showed how the generalized Hermite polynomials of Kampé de Fériet type $H_n(x, y)$ represent a useful tool to investigate the concepts and the related properties of the generalized Bessel functions and their modified functions. By using the same approach, we will show, in the next section, that the special polynomials of type $Q_{n,m}(x)$ and their generalization of the form $Q_{n,m}(x, y, z)$, can be used to deduce some relevant properties regarding the two-index Bessel functions of type $J_{n,m}(x)$.

4. Two-Index Bessel Functions in Terms of Generalized Hermite Polynomials

The multi-index Bessel functions are well known in literature [3,4,12] and they can be defined directly by using the concepts and related definitions of the ordinary cylindrical one-variable Bessel function:

$$J_n(x) = \sum_{r=0}^{+\infty} \frac{(-1)^r}{(n+r)!r!} \left(\frac{x}{2}\right)^{n+2r} \tag{38}$$

with the following generating function:

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} t^n J_n(x) \tag{39}$$

where t is a continuous real variable. So that, the two-index, one-variable Bessel function can be introduced by setting its generating function:

$$\begin{aligned} &\exp\left\{\tfrac{x}{2}\left[\left(u - \tfrac{1}{u}\right) + \left(v - \tfrac{1}{v}\right) + \left(uv - \tfrac{1}{uv}\right)\right]\right\} = \\ &= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} u^n v^m J_{n,m}(x) \end{aligned} \tag{40}$$

where u and v are continuous variable such that $0 < |u| \neq |v| < +\infty$. By expanding the exponential on l.h.s. of equation (40), we have:

$$J_{n,m}(x) = \sum_{s=-\infty}^{+\infty} J_{n-s}(x) J_{m-s}(x) J_s(x) \tag{41}$$

that represents the explicit form of the generic two-index, one-variable Bessel function. In some previous papers [4,6,12], the link between Bessel functions and Hermite polynomials in their various classes has been shown. Regarding the modified Hermite polynomials of type $Q_{n,m}(x)$ or, more in general, the two-index, three-variable, generalized Hermite polynomials presented in second section (see eq. (16)), we can obtain some interesting relations involving two-index Bessel functions. We start by noting that the exponential in the generating function of the two-index Bessel function $J_{n,m}(x)$ can be rearranged in the following form:

$$\begin{aligned} &\exp\left\{\tfrac{x}{2}\left[\left(u - \tfrac{1}{u}\right) + \left(v - \tfrac{1}{v}\right) + \left(uv - \tfrac{1}{uv}\right)\right]\right\} = \\ &= \exp\left(\tfrac{x}{2}u + \tfrac{x}{2}v + \tfrac{x}{2}uv\right) \exp\left(-\tfrac{x}{2}\tfrac{1}{u} - \tfrac{x}{2}\tfrac{1}{v} - \tfrac{x}{2}\tfrac{1}{uv}\right) \end{aligned} \tag{42}$$

and, according to equation (14), we have:

$$
\begin{aligned}
&\exp\left\{\frac{x}{2}\left[\left(u-\frac{1}{u}\right)+\left(v-\frac{1}{v}\right)+\left(uv-\frac{1}{uv}\right)\right]\right\}= \\
&=\sum_{r=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty}\frac{u^r}{r!}\frac{v^s}{s!}Q^{\square}_{n,m}\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right)\cdot \\
&\cdot\sum_{k=-\infty}^{+\infty}\sum_{h=-\infty}^{+\infty}\frac{u^{-k}}{k!}\frac{v^{-h}}{h!}Q^{\square}_{k,h}\left(-\frac{x}{2},-\frac{x}{2},-\frac{x}{2}\right)
\end{aligned}
\tag{43}
$$

By setting $r-k=n$ and $s-h=m$, we can write the r.h.s. of the above relation in the form:

$$
\begin{aligned}
&\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}u^n v^m\sum_{k=0}^{+\infty}\sum_{h=0}^{+\infty}\frac{1}{(n+k)!(m+h)!k!h!} \\
&Q^{\square}_{n,m}\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right)Q^{\square}_{k,h}\left(-\frac{x}{2},-\frac{x}{2},-\frac{x}{2}\right)
\end{aligned}
\tag{44}
$$

and then, by comparing equation (43) with the expression of the generating function of the Bessel function $J_{n,m}(x)$, presented in equation (40), we obtain:

$$
\begin{aligned}
&J_{n,m}(x)=\sum_{k=0}^{+\infty}\sum_{h=0}^{+\infty}\frac{1}{(n+k)!(m+h)!k!h!} \\
&Q^{\square}_{n,m}\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right)Q^{\square}_{k,h}\left(-\frac{x}{2},-\frac{x}{2},-\frac{x}{2}\right)
\end{aligned}
\tag{45}
$$

that gives a relevant example of how the Hermite polynomials, in their various forms, are a powerful tool to describe many families of special functions.

5. Conclusion

We have presented some applications of generalized Hermite polynomials of type $H_n(x,y)$ in the description of the theory of generalized Bessel functions and their modified functions. We have also discussed a very special class of polynomials recognized as belonging to the family of Hermite polynomials and we have deduced some proper relations. It is showed that the polynomials of type $Q_{n,m}(x)$ and $Q_{n,m}(x,y,z)$, allow us to simplify definitions and the related properties of the two-index, one-variable Bessel functions of type $J_{n,m}(x)$. The obtained results underline that the family of Hermite polynomials, in any form, represents a powerful tool to investigate many classes of special functions and special polynomials, as for instance the orthogonal polynomials and in particular the Laguerre [13], Legendre [13-15] and Chebyshev polynomials [16-18]. In a forthcoming paper, we will present further applications of the various class of polynomials, recognized as belonging to the family of Hermite polynomials.

As a final remark, we discuss another relevant approach to use of the Hermite polynomials in the description of the structure of the Bessel functions. It is well known that Hermite polynomials of one variable, for instance, satisfy the rules of the Monomiality principle [9,19,20], that is they can recognized as monomial under the action of two suitable operators. We have in fact that:

$$\begin{aligned} &\left(x - \tfrac{d}{dx}\right) He_n(x) = He_{n+1}(x) \\ &\left(\tfrac{1}{n}\tfrac{d}{dx}\right) He_n(x) = He_{n-1}(x) \end{aligned} \tag{46}$$

where the Kampé de Fériet Hermite polynomial reads [9]:

$$He_n(x) = n! \sum_{r=0}^{[p/2]} \tfrac{(-1)^r}{(n-2r)!r!2^r} x^{n-2r} \tag{47}$$

By setting:

$$\hat{m} = x - \frac{d}{dx} \text{ and } \hat{p} = \frac{1}{n}\frac{d}{dx}$$

we can write the relations in (46) in the more simple form:

$$\begin{aligned} &\hat{m} He_n(x) = He_{n+1}(x) \\ &\hat{p} He_n(x) = He_{n-1}(x) \end{aligned}$$

It is immediate to introduce a Bessel function based on the polynomials $He_n(x)$, instead that on the monomial x^n, by substituting in the explicit form of the cylindrical, first kind Bessel function [4] the Hermite polynomial; that is:

$${}_H J_n(x) = \sum_{s=0}^{+\infty} \frac{(-1)^s He_{n+2s}(x)}{(n-2s)!s!2^{n+2s}}$$

The operators introduced in equation (46) play the role of the multiplication and derivation operators, so that we can derive the definitions and the properties of the above so-called Hermite-Bessel [20] function, by using the correspondence:

$$\begin{aligned} &\hat{m} \to x \\ &\hat{p} \to \tfrac{d}{dx} \end{aligned}$$

we now proceed to show that it is immediate to derive the generating function for the Bessel function based on Hermite polynomials. By noting that the generating function of the cylindrical, one-variable Bessel function is expressed by the following exponential:

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right]$$

and by using the operators exposed in equation (46), we have:

$$\exp\left[\frac{\hat{m}}{2}\left(t-\frac{1}{t}\right)\right]=\exp\left[\frac{1}{2}x\left(t-\frac{1}{t}\right)-\frac{1}{2}\frac{d}{dx}\left(t-\frac{1}{t}\right)\right]$$

We note that the operators in the above r.h.s. do not commute and, since their commutator is:

$$\left[\frac{1}{2}x\left(t-\frac{1}{t}\right),\frac{1}{2}\frac{d}{dx}\left(t-\frac{1}{t}\right)\right]=\frac{1}{4}\left(t-\frac{1}{t}\right)^2$$

we obtain:

$$\exp\left[\frac{\hat{m}}{2}\left(t-\frac{1}{t}\right)\right]=\exp\left[\frac{1}{2}x\left(t-\frac{1}{t}\right)-\frac{1}{8}\left(t-\frac{1}{t}\right)^2\right]$$

where we have omitted the term contained in the derivative because its action is trivial. By expanding the above exponential it is immediate to obtain:

$$\exp\left[\frac{\hat{m}}{2}\left(t-\frac{1}{t}\right)\right]=\sum_{n=0}^{+\infty}\left(t-\frac{1}{t}\right)^n\frac{1}{2^n n!}He_n(x)$$

and by expanding the binomial in the r.h.s., after setting $n-2s=m$ we end up with:

$$\sum_{n=0}^{+\infty}\sum_{s=0}^{+\infty}\frac{t^{n-2s}(-1)^s}{2^n(n-s)!s!}He_n(x)=\sum_{m=0}^{+\infty}\sum_{s=0}^{+\infty}\frac{t^m(-1)^s}{2^{m+2s}(m+s)!s!}He_{m+2s}(x)$$

We conclude that the expression of the generating function related to the above introduced Hermite-Bessel functions:

$$\exp\left[\frac{\hat{m}}{2}\left(t-\frac{1}{t}\right)\right]=\sum_{r=0}^{+\infty}t^n\,{}_HJ_n(x)$$

The above considerations showed that the Hermite polynomials can be used also to define interesting isospectral problems regarding a large number of special functions and special polynomials. In some papers we have discussed some aspects of this approach. Still, there is room for further investigations this wide scenario offered by the nature of the Hermite polynomials. Furthermore, the generalized Hermite polynomials and the related special polynomials, cited above as Laguerre, Legendre and Chebyshev polynomials and different families of special functions, in particular the large class of functions recognized as belonging to the Bessel functions, can be efficiently employed to solve a large class of problems in field such as stochastic processes [21-23], particle physics [24,25], electromagnetisms [26-28], continuum mechanics [29,30], material sciences [31,32], transmission lines [33-35], building sciences [36-38] and applications in the field of special functions and orthogonal polynomials [39-44]. Further investigations will be carried out in the next future in other fields of interest.

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