International Journal of Pure and Applied Mathematics
Volume 112 No. 3 2017, 631-643
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v112i3.14

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RELATIONSHIPS BETWEEN L-NEIGHBORHOOD SYSTEMS AND L-FUZZY TOPOLOGIES

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Abstract: In this paper, we investigate two L-neighborhood systems induced by an L-fuzzy topology in a complete residuated lattice L. We study the relationships among L-fuzzy topology, L-topologies and L-neighborhood systems. Finally, we give their examples.

AMS Subject Classification: 03E72, 06A15, 06F07, 54A05
Key Words: complete residuated lattice, L-neighborhood system, L-(fuzzy) topological space

1. Introduction

Hájek [6] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structures [1,7,12].

Many researcher introduced the notion of L-fuzzy topological structures in unit interval [0,1]([2,8]), complete distributive lattices ([4,5,9]), commutative unital quantales and complete quasi-monoidal lattices ([3,7,12]). Ramadan et al.[10,11] investigated the relationships between L-fuzzy quasi-uniform structures and L-fuzzy topological structures in a complete residuated lattice.

Received: May 18, 2016
Revised: February 8, 2017
Published: February 9, 2017

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In this paper, we investigate two L-neighborhood systems induced by an L-fuzzy topology in a complete residuated lattice L. We study the relationships among L-fuzzy topology, L-topologies and L-neighborhood systems. Finally, we give their examples.

2. Preliminaries

Definition 2.1. [1,6,7] An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $(L, \leq, \vee, \wedge, 0, 1)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume that $(L, \vee, \wedge, \odot, \rightarrow, ^*, 1, 0)$ be a complete residuated lattice with a strong negation *. For $\alpha \in L, \lambda \in L^X$, we denote $(\alpha \rightarrow \lambda), (\alpha \odot \lambda), \alpha_X \in L^X$ as $(\alpha \rightarrow \lambda)(x) = \alpha \rightarrow \lambda(x)$, $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$, $\alpha_X(x) = \alpha$.

Lemma 2.2. [1,6,7] Let $(L, \vee, \wedge, \odot, \rightarrow, ^*, 1, 0)$ be a complete residuated lattice with a strong negation *. For each $x, y, z, x_i, y_i \in L$, the following properties hold.

(1) If $y \leq z$, then $x \odot y \leq x \odot z$.

(2) If $y \leq z$, then $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.

(3) $x \rightarrow y = 1$ iff $x \leq y$.

(4) $x \rightarrow 1 = 1$ and $1 \rightarrow x = x$.

(5) $x \odot y \leq x \wedge y$.

(6) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$.

(7) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$.

(8) $\bigvee_{i\in\Gamma} x_i \rightarrow \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \rightarrow y_i)$ and $\bigwedge_{i\in\Gamma} x_i \rightarrow \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \rightarrow y_i)$.

(9) $(x \rightarrow y) \odot x \leq y$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq (x \rightarrow z)$.

(10) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$ and $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$.

(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(12) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$ and $(x \odot y)^* = x \rightarrow y^*$.

(13) $x^* \rightarrow y^* = y \rightarrow x$ and $(x \rightarrow y)^* = x \odot y^*$.

(14) $y \rightarrow z \leq x \odot y \rightarrow x \odot z$.

(15) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$,

Definition 2.3.[1,4,9] Let X be a set. A mapping $R : X \times X \to L$ is called an *L-partial order* if it satisfies the following conditions:
(E1) reflexive if $R(x,x) = 1$ for all $x \in X$,
(E2) transitive if $R(x,y) \odot R(y,z) \leq R(x,z)$, for all $x,y,z \in X$,
(E3) antisymmetric if $R(x,y) = R(y,x) = 1$, then $x = y$.

Lemma 2.4. [1,4,9] For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by

$$S(\lambda,\mu) = \bigwedge_{x \in X} (\lambda(x) \to \mu(x)).$$

Then, for each $\lambda,\mu,\rho,\nu \in L^X$, and $\alpha \in L$, the following properties hold.
(1) S is an L-partial order on L^X.
(2) $\lambda \leq \mu$ iff $S(\lambda,\mu) \geq 1$,
(3) If $\lambda \leq \mu$, then $S(\rho,\lambda) \leq S(\rho,\mu)$ and $S(\lambda,\rho) \geq S(\mu,\rho)$,
(4) $S(\lambda,\mu) \odot S(\nu,\rho) \leq S(\lambda \odot \nu, \mu \odot \rho)$ and $S(\lambda,\mu) \odot S(\nu,\rho) \leq S(\lambda \oplus \nu, \mu \oplus \rho)$,
(5) $S(\mu,\rho) \odot S(\lambda,\mu) \leq S(\lambda,\rho)$ and $S(\mu,\rho) \leq S(\rho,\lambda) \to S(\mu,\lambda)$,
(6) $\bigvee_{\mu \in L^X} (S(\mu,\rho) \odot S(\lambda,\mu)) = S(\lambda,\rho)$.
(7) If $\phi : X \to Y$ is a map, then for $\lambda,\mu \in L^X$ and $\rho,\nu \in L^Y$,

$$S(\lambda,\mu) \leq S(\phi^{\to}(\lambda), \phi^{\to}(\mu)),$$

$$S(\rho,\nu) \leq S(\phi^{\leftarrow}(\rho), \phi^{\leftarrow}(\nu)),$$

and the equalities hold if ϕ is bijective.

Definition 2.5. [7,12] A map $\mathcal{T} : L^X \to L$ is called an L-fuzzy topology on X if it satisfies the following conditions:
(T1) $\mathcal{T}(0_X) = \mathcal{T}(1_X) = 1$,
(T2) $\mathcal{T}(\lambda \odot \mu) \geq \mathcal{T}(\lambda) \odot \mathcal{T}(\mu), \ \forall \ \lambda,\mu \in L^X$,
(T3) $\mathcal{T}(\bigvee_i \lambda_i) \geq \bigwedge_i \mathcal{T}(\lambda_i), \ \forall \ \{\lambda_i\}_{i \in \Gamma} \subseteq L^X$.
The pair $(X,\mathcal{T})$ is called an L-fuzzy topological space. An L-fuzzy topological space is called enriched if
(R) $\mathcal{T}(\alpha \odot \lambda) \geq \mathcal{T}(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.
Let $(X,\mathcal{T}_1)$ and $(Y,\mathcal{T}_2)$ be two L-fuzzy topological spaces. A mapping $\phi : X \to Y$ is said to be L-fuzzy continuous iff for each $\lambda \in L^Y$, $\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(\phi^{\leftarrow}(\lambda))$.

Remark 2.6. [7] A set $\tau \subset L^X$ is called an L-topology on X if (t1) $0_X, 1_X \in \tau$, (t2) $(\lambda \odot \mu) \in \tau$, for each $\lambda,\mu \in \tau$, (t3) $\bigvee_i \lambda_i \in \tau$, for all $\lambda_i \in \tau$. An L-topology τ is called enriched if $\alpha \odot \lambda \in \tau$, for all $\lambda \in \tau$ and $\alpha \in L$.

Definition 2.7. [8] A map $N : X \to L^{L^X}$ is called an L-neighborhood system on X with $N(x) = N_x$ if N satisfies the following conditions

(N1) $N_x(1_X) = 1$ and $N_x(0_X) = 0$,

(N2) $N_x(\lambda) \le \lambda(x)$ for all $\lambda \in L^X$,

(N3) If $\lambda \le \mu$, then $N_x(\lambda) \le N_x(\mu)$,

(N4) $N_x(\lambda \odot \mu) \ge N_x(\lambda) \odot N_x(\mu)$ for each $\lambda, \mu \in L^X$,

An L-neighborhood system is called stratified if

(R) $N_x(\alpha \odot \lambda) \ge \alpha \odot N_x(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

The pair (X, N) is called an L-neighborhood space.

Let (X, N_X) and (Y, N_Y) be two L-neighborhood spaces. A mapping $\phi : X \to Y$ is said to be an N-map at $x \in X$ iff $(N_Y)_{\phi(x)}(\lambda) \le (N_X)_x(\phi^{\leftarrow}(\lambda))$ for each $\lambda \in L^Y$. ϕ is an N-map if it is an N-map at every $x \in X$.

3. L-Neighborhood Systems and L-Fuzzy Topologies

Theorem 3.1. Let $(X, \mathcal{T})$ be an L-fuzzy topological space. Define two maps $N^{\mathcal{T}}, M^{\mathcal{T}} : X \to L^{L^X}$ as follows:

$$N_x^{\mathcal{T}}(\lambda) = \bigvee_{\mu \in L^X} \mathcal{T}(\mu) \odot S(\mu, \lambda) \odot \mu(x)$$

$$M_x^{\mathcal{T}}(\lambda) = \bigvee_{\mu \in L^X} \{\mathcal{T}(\mu) \odot \mu(x) \mid \mu \le \lambda\}.$$

Then: (1) $(X, N^{\mathcal{T}})$ is a stratified L-neighborhood space.

(2) $(X, M^{\mathcal{T}})$ is an L-neighborhood space with $M^{\mathcal{T}} \le N^{\mathcal{T}}$. Moreover, if $\mathcal{T}$ is enriched, $M^{\mathcal{T}}$ is stratified.

(3) If $\mathcal{T}$ is enriched, then $M^{\mathcal{T}} = N^{\mathcal{T}}$.

Proof. (N1) $N_x^{\mathcal{T}}(1_X) = \bigvee_{\mu \in L^X} \mathcal{T}(\mu) \odot S(\mu, 1_X) \odot \mu(x) \ge \mathcal{T}(1_X) \odot S(1_X, 1_X) \odot 1_X(x) = 1$.

(N2) We have to show that $N_x^{\mathcal{T}}(\lambda) \le \lambda(x)$ for each $\lambda \in L^X$ from:

$$\begin{aligned} N_x^{\mathcal{T}}(\lambda) &= \bigvee_{\mu \in L^X} (\mathcal{T}(\mu) \odot S(\mu, \lambda) \odot \mu(x)) \\ &\le \bigvee_{\mu \in L^X} (\mathcal{T}(\mu) \odot (\mu(x) \to \lambda(x)) \odot \mu(x)) \end{aligned}$$

$$\leq \bigvee_{\mu\in L^X} (\mathcal{T}(\mu)\odot\lambda(x)) \leq \lambda(x).$$

(N3)

$$\begin{aligned} N^{\mathcal{T}}_x(\lambda) &= \bigvee_{\rho\in L^X} (\mathcal{T}(\rho)\odot S(\rho,\lambda)\odot\rho(x)) \\ &\leq \bigvee_{\rho\in L^X} \Big((\mathcal{T}(\rho)\odot S(\rho,\mu)\odot\rho(x)) = N^{\mathcal{T}}_x(\mu) \end{aligned}$$

(N4)

$$\begin{aligned} & N^{\mathcal{T}}_x(\lambda)\odot N^{\mathcal{T}}_x(\mu) \\ &= \bigvee_{\rho\in L^X} (\mathcal{T}(\rho)\odot S(\rho,\lambda)\odot\rho(x))\odot \bigvee_{\nu\in L^X} (\mathcal{T}(\nu)\odot S(\nu,\mu)\odot\nu(x)) \\ &= \bigvee_{\rho,\nu\in L^X} \Big(\mathcal{T}(\rho)\odot\mathcal{T}(\nu)\odot S(\rho,\lambda)\odot S(\nu,\mu)\odot(\rho\odot\nu)(x)\Big) \\ &\leq \bigvee_{\rho,\nu\in L^X} \Big(\mathcal{T}(\rho\odot\nu)\odot S(\rho\odot\nu,\lambda\odot\mu)\odot(\rho\odot\nu)(x)\Big) \\ &\qquad \text{(by Lemma 2.4)} \\ &\leq \bigvee_{\gamma\in L^X} (\mathcal{T}(\gamma)\odot S(\gamma,\lambda\odot\mu)\odot\gamma(x)) = N^{\mathcal{T}}_x(\lambda\odot\mu). \end{aligned}$$

For all $\lambda\in L^X$ and $\alpha\in L$, since $S(\rho,\alpha\odot\lambda)\geq\alpha\odot S(\rho,\lambda)$, we have

$$\begin{aligned} N^{\mathcal{T}}_x(\alpha\odot\lambda) &= \bigvee_{\rho\in L^X} (\mathcal{T}(\rho)\odot S(\rho,\lambda)\odot\rho(x)) \\ &\geq \bigvee_{\rho\in L^X} (\mathcal{T}(\rho)\odot\alpha\odot S(\rho,\lambda)\odot\rho(x)) \\ &= \alpha\odot\bigvee_{\rho\in L^X} (\mathcal{T}(\rho)\odot S(\rho,\lambda)\odot\rho(x)) = \alpha\odot N^{\mathcal{T}}_x(\lambda). \end{aligned}$$

(2) By (1), we similarly prove that $M^{\mathcal{T}}$ is an L-neighborhood system.

$$M^{\mathcal{T}}_x(\lambda) = \bigvee_{\rho\leq\lambda} (\mathcal{T}(\rho)\odot S(\rho,\lambda)\odot\rho(x))$$

$$\leq \bigvee_{\rho\in L^X} (\mathcal{T}(\rho)\odot S(\rho,\lambda)\odot\rho(x)) = N_x^{\mathcal{T}}(\lambda)$$

If $\mathcal{T}$ is enriched, $M^{\mathcal{T}}$ is stratified from:

$$\begin{aligned}\alpha\odot M_x^{\mathcal{T}}(\lambda) &= \alpha\odot\bigvee_{\rho\leq\lambda}(\mathcal{T}(\rho)\odot\rho(x))\\ &= \bigvee_{\rho\leq\lambda}(\mathcal{T}(\rho)\odot(\alpha\odot\rho)(x))\\ &\leq \bigvee_{\alpha\odot\rho\leq\alpha\odot\lambda}(\mathcal{T}(\alpha\odot\rho)\odot(\alpha\odot\rho)(x))\\ &= N_x^{\mathcal{T}}(\alpha\odot\lambda).\end{aligned}$$

(3) Since $\mathcal{T}$ is enriched, then $\mathcal{T}(S(\rho,\lambda)\odot\rho)\geq\mathcal{T}(\rho)$. Thus

$$\mathcal{T}(S(\rho,\lambda)\odot\rho)\odot S(\rho,\lambda)\odot\rho\geq\mathcal{T}(\rho)\odot S(\rho,\lambda)\odot\rho.$$

Since $S(\rho,\lambda)\odot\rho\leq\lambda$, we have $M_x^{\mathcal{T}}\geq N_x^{\mathcal{T}}$.

Theorem 3.2. Let (X,N) be an L-neighborhood space. Define a map $\mathcal{T}_N : L^X\to L$ by

$$\mathcal{T}_N(\lambda) = \bigwedge_{x\in X}(\lambda(x)\to N_x(\lambda)) = S(\lambda, N_-(\lambda)).$$

Then

(1) $\mathcal{T}_N$ is an L-fuzzy topology on X,
(2) If N is stratified, then $\mathcal{T}_N$ is an enriched L-fuzzy topology.
(3) If N is stratified, $S(\lambda,\mu)\leq S(N_-(\lambda),N_-(\mu))$ for all $\lambda,\mu\in L^X$.
(4) If N is stratified, $N^{\mathcal{T}_N}\leq N$.
(5) If $N_x(\lambda)\leq\bigvee\{N_x(\mu)\mid\mu(y)\leq N_y(\lambda),\ \forall\, y\in X\}$, then $N^{\mathcal{T}_N}\geq N$.
(6) If $\mathcal{T}$ is an L-fuzzy topology on X, then $\mathcal{T}_{N^{\mathcal{T}}}\geq\mathcal{T}$.
(7) $\tau_N=\{\lambda\in L^X\mid\mathcal{T}_N(\lambda)=1\}$ is an L-topology on X,
(8) If N is stratified, then τ_N is an enriched L-topology.

Proof. (1) (T1)

$$\begin{aligned}\mathcal{T}_N(1_X) &= \textstyle\bigwedge_{x\in X}(1_X(x)\to N_x(1_X)) = 1,\\ \mathcal{T}_N(0_X) &= \textstyle\bigwedge_{x\in X}(0_X(x)\to N_x(0_X)) = 1.\end{aligned}$$

(T2)

$$\begin{aligned}
&\mathcal{T}_N(\lambda \odot \mu)\\
&= \textstyle\bigwedge_{x\in X}(\lambda(x) \odot \mu(x) \to N_x(\lambda \odot \mu))\\
&\geq \textstyle\bigwedge_{x\in X}(\lambda(x) \odot \mu(x) \to (N_x(\lambda) \odot N_x(\mu)))\\
&\quad\text{(by Lemma 2.2 (15))}\\
&\geq \textstyle\bigwedge_{x\in X}(\lambda(x) \to N_x(\lambda)) \odot \bigwedge_{x\in X}(\mu(x) \to N_x(\mu))\\
&= \mathcal{T}_N(\lambda) \odot \mathcal{T}_N(\mu).
\end{aligned}$$

(T3)

$$\begin{aligned}
&\mathcal{T}_N(\textstyle\bigvee_i \lambda_i) = \bigwedge_{x\in X}((\bigvee_i \lambda_i(x) \to N_x(\bigvee_i \lambda_i)\\
&\geq \textstyle\bigwedge_{x\in X}((\bigvee_i(\lambda_i(a)(x) \to \bigvee_i N_x(\lambda_i)\\
&\quad\text{(by Lemma 2.2 (8))}\\
&\geq \textstyle\bigwedge_i \bigwedge_{x\in X}(\lambda(x) \to N_x(\lambda_i))\\
&= \textstyle\bigwedge_i \mathcal{T}_N(\lambda_i).
\end{aligned}$$

(2) By Lemma 2.2 (14), we have

$$\begin{aligned}
&\mathcal{T}_N(\alpha \odot \lambda)\\
&= \textstyle\bigwedge_{x\in X}((\alpha \odot \lambda)(x) \to N_x(\alpha \odot \lambda))\\
&\geq \textstyle\bigwedge_{x\in X}((\alpha \odot \lambda)(x) \to (\alpha \odot N_x(\lambda)))\\
&\geq \textstyle\bigwedge_{x\in X}(\lambda(x) \to N_x(\lambda)) = \mathcal{T}_N(\lambda).
\end{aligned}$$

(3) Since N is stratified, $S(\lambda, \mu) \odot N_x(\lambda) \leq N_x(S(\lambda, \mu) \odot \lambda) \leq N_x(\mu)$. Thus, $S(\lambda, \mu) \leq N_x(\lambda) \to N_x(\mu)$.

(4)

$$\begin{aligned}
N_x^{\mathcal{T}_N}(\lambda) &= \bigvee_{\mu\in L^X} (\mathcal{T}_N(\mu) \odot S(\mu, \lambda) \odot \mu(x))\\
&= \bigvee_{\mu\in L^X} (S(\mu, N_-(\mu)) \odot S(\mu, \lambda) \odot \mu(x))\\
&\quad\text{(by the definition of } \mathcal{T}_N)\\
&\leq \bigvee_{\mu\in L^X} S(\mu, N_-(\mu)) \odot S(N_-(\mu), N_-(\lambda)) \odot \mu(x)\\
&\leq \bigvee_{\mu\in L^X} S(\mu, N_-(\lambda)) \odot \mu(x) \text{ (by Lemma 2.4(5))}\\
&\leq N_x(\lambda).
\end{aligned}$$

(5) Since $N_x(\lambda) \leq \bigvee\{N_x(\mu) \mid \mu(y) \leq N_y(\lambda),\ \forall\, y \in X\}$, we have $N_x(\lambda) \leq N_x(N_-(\lambda))$. By (N2), $N_x(N_-(\lambda)) \leq N_x(\lambda)$. Hence $N_x(N_-(\lambda)) = N_x(\lambda)$. Thus,

$$N_x^{\mathcal{T}_N}(\lambda) = \bigvee_{\mu \in L^X} (\mathcal{T}_N(\mu) \odot S(\mu, \lambda) \odot \mu(x))$$
$$\geq S(N_-(\lambda), N_-(N_-(\lambda))) \odot S(N_-(\lambda), \lambda) \odot N_x(\lambda) \text{ (put } \mu = N_-(\lambda))$$
$$\geq N_x(\lambda).$$

(6)

$$\begin{aligned}
\mathcal{T}_{N_\mathcal{T}}(\lambda) &= S\Big(\lambda, N_-^{\mathcal{T}}(\lambda)\Big) \\
&= \bigwedge_{x \in X} \Big(\lambda(x) \to N_x^{\mathcal{T}}(\lambda)\Big) \\
&= \bigwedge_{x \in X} \Big(\lambda(x) \to \bigvee_{\mu \in L^X} \mathcal{T}(\mu) \odot S(\mu, \lambda) \odot \mu(x)\Big) \\
&\geq \bigwedge_{x \in X} \Big(\lambda(x) \to \mathcal{T}(\lambda) \odot S(\lambda, \lambda) \odot \lambda(x)\Big) \\
&\geq \mathcal{T}(\lambda) \odot S(\lambda, \lambda) = \mathcal{T}(\lambda).
\end{aligned}$$

(7)and (8) are easily proved from (1) and (2).

Theorem 3.3. Let (X, τ) be an L-topological space. Define two maps $M^\tau, N^\tau : X \to L^{L^X}$ by

$$M_x^\tau(\lambda) = \bigvee\{\rho(x) \mid \rho \leq \lambda,\ \rho \in \tau\},$$

$$N_x^\tau(\lambda) = \bigvee_{\rho \in \tau} (\rho(x) \odot S(\rho, \lambda)).$$

Then the following properties hold.

(1) (X, M^τ) is an L-neighborhood space satisfying

$$M_x(\lambda) \leq \bigvee\{M_x(\mu) \mid \mu(y) \leq M_y(\lambda),\ \forall\, y \in X\}.$$

(2) (X, N^τ) is a stratified L-neighborhood space.

(3) $M^\tau = M^{\mathcal{T}_\tau}$ and $N^\tau = N^{\mathcal{T}_\tau}$ with an L-fuzzy topology as

$$\mathcal{T}_\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \tau, \\ 0, & \text{otherwise.} \end{cases}$$

(4) If τ is enriched, then M^τ is stratified and $M^\tau = N^\tau$.

(5) $\mathcal{T}_{M^\tau}$ and $\mathcal{T}_{N^\tau}$ are L-fuzzy topologies on X,

(6) If τ is enriched, then $\mathcal{T}_{M^\tau} = \mathcal{T}_{N^\tau}$ is an enriched L-fuzzy topology.

(7) $N^{\mathcal{T}_{N^\tau}} \leq N^\tau$ and $S(\lambda,\mu) \leq S(N^\tau_-(\lambda), N^\tau_-(\mu))$ for all $\lambda,\mu \in L^X$.

(8) If τ is enriched, $N^{\mathcal{T}_{M^\tau}} \leq M^\tau$ and $S(\lambda,\mu) \leq S(M^\tau_-(\lambda), M^\tau_-(\mu))$ for all $\lambda,\mu \in L^X$.

(9) $M^{\mathcal{T}_{M^\tau}} \geq M^\tau$.

Proof. (1) We easily prove that (X, M^τ) is an L-neighborhood space. Put $M^\tau_-(\lambda) = \bigvee\{\rho \mid \rho \leq \lambda,\ \rho \in \tau\}$ with $M^\tau_-(\lambda)(x) = M^\tau_x$. Then $M^\tau_-(\lambda) \in \tau$. By (N3) and the definition of M^τ,

$$M^\tau_x(M^\tau_-(\lambda)) = M^\tau_x(\lambda).$$

$$M^\tau_x(\lambda) = M^\tau_x(M^\tau_-(\lambda)) \leq \bigvee\{M^\tau_x(\rho) \mid \rho(y) \leq M^\tau_y(\lambda)\}.$$

(2) It similarly proved as Theorem 3.1(1).

(3)

$$\begin{aligned} M^\tau_x(\lambda) &= \bigvee\{\rho(x) \mid \rho \leq \lambda,\ \rho \in \tau\} = \bigvee\{\mathcal{T}_\tau(\rho) \odot \rho(x) \mid \rho \leq \lambda\}, \\ N^\tau_x(\lambda) &= \bigvee\nolimits_{\rho\in\tau}(\rho(x) \odot S(\rho,\lambda)) = \bigvee(\mathcal{T}_\tau(\rho) \odot \rho(x) \odot S(\rho,\lambda)). \end{aligned}$$

(4)

$$\begin{aligned} &\alpha \odot M^\tau_x(\lambda) = \alpha \odot \bigvee\{\rho \mid \rho \leq \lambda,\ \rho \in \tau\} \\ &\leq \bigvee\{\alpha \odot \rho \mid \alpha \odot \rho \leq \alpha \odot \lambda,\ \alpha \odot \rho \in \tau\} \leq M^\tau_x(\alpha \odot \lambda). \end{aligned}$$

Let ρ with $\rho \leq \lambda$ and $\rho \in \tau$. Then $\rho(x) \odot S(\rho,\lambda) = \rho(x) \odot 1 = \rho(x)$. Thus $\rho(x) \leq N^\tau_x(\lambda)$. Therefore $M^\tau_x(\lambda) \leq N^\tau_x(\lambda)$.

Let $\rho(x) \odot S(\rho,\lambda)$ with $\rho \in \tau$. Since τ is enriched, $\rho \odot S(\rho,\lambda) \in \tau$ and $\rho(x) \odot S(\rho,\lambda) \leq \rho(x) \odot (\rho(x) \to \lambda(x)) \leq \lambda(x)$. Then $N^\tau_x(\lambda) \leq M^\tau_x(\lambda)$.

Other cases are easily proved from Theorem 3.2.

Theorem 3.4. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L-fuzzy topological spaces. If a mapping $\phi : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is L-fuzzy continuous, then $\phi : (X, N^{\mathcal{T}_X}) \to (Y, N^{\mathcal{T}_Y})$ is an N- map and $\phi : (X, M^{\mathcal{T}_X}) \to (Y, M^{\mathcal{T}_Y})$ is an N- map.

Proof.

$$N^{\mathcal{T}_Y}_{\phi(x)}(\lambda) = \bigvee_{\mu \in L^Y} \mathcal{T}_Y(\mu) \odot S(\mu, \lambda) \odot \mu(\phi(x))$$
$$\leq \bigvee_{\mu \in L^Y} (\mathcal{T}_X(\phi^{\leftarrow}(\mu)) \odot S(\phi^{\leftarrow}(\mu), \phi^{\leftarrow}(\lambda)) \odot \phi^{\leftarrow}(\mu)(x)) \leq N^{\mathcal{T}_X}_x(\phi^{\leftarrow}(\lambda)).$$

$$M^{\mathcal{T}_Y}_{\phi(x)}(\lambda) = \bigvee_{\mu \leq \lambda} \mathcal{T}_Y(\mu) \odot \mu(\phi(x))$$
$$\leq \bigvee_{\phi^{\leftarrow}(\mu) \leq \phi^{\leftarrow}(\lambda)} (\mathcal{T}_X(\phi^{\leftarrow}(\mu)) \odot \phi^{\leftarrow}(\mu)(x)) \leq M^{\mathcal{T}_X}_x(\phi^{\leftarrow}(\lambda)).$$

Theorem 3.5. Let (X, N_X) and (Y, N_Y) be L-neighborhood spaces. If a mapping $\phi : (X, N_X) \to (Y, N_Y)$ is an N-map, then $\phi : (X, \mathcal{T}_{N_X}) \to (Y, \mathcal{T}_{N_Y})$ is L-fuzzy continuous.

Proof. Since $(N_Y)_{\phi(x)}(\lambda) \leq (N_X)_x(\phi^{\leftarrow}(\lambda))$, we have

$$\mathcal{T}_{N_Y}(\lambda) = \bigwedge_{y \in Y} \Big(\lambda(y) \to (N_Y)_y(\lambda)\Big)$$
$$\leq \bigwedge_{x \in X} \Big(\phi^{\leftarrow}(\lambda)(x) \to (N_Y)_{\phi(x)}(\lambda)\Big)$$
$$\leq \bigwedge_{x \in X} (\phi^{\leftarrow}(\lambda)(x) \to (N_X)_x(\phi^{\leftarrow}(\lambda)))\Big) = \mathcal{T}_{N_X}(\phi^{\leftarrow}(\lambda))$$

Example 3.6. Let $(L = [0,1], \odot, \to)$ be a complete residuated lattice defined by

$$x \odot y = (x + y - 1) \vee 0, \;\; x \to y = (1 - x + y) \wedge 1.$$

Let $X = \{x, y, z\}$ be a set and $\rho, \rho \odot \rho \in L^X$ such that

$$\rho(x) = 0.5, \rho(y) = 0.6, \rho(z) = 0.6,$$

$$\rho \odot \rho(x) = 0, \rho \odot \rho(y) = 0.2, \rho \odot \rho(z) = 0.2.$$

(1) We define an L-topology $\tau = \{0_X, 1_X, \rho, \rho \odot \rho\}$. We obtain two $[0,1]$-neighborhood systems $N^\tau, M^\tau : X \to [0,1]^{[0,1]^X}$ as follows:

$$M_x^\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = 1_X, \\ 0.5, & \text{if } \lambda \geq \rho, \\ 0, & \text{otherwise.} \end{cases} \qquad M_y^\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = 1_X, \\ 0.6, & \text{if } \lambda \geq \rho, \\ 0.2, & \text{if } \lambda \geq \rho \odot \rho, \\ & \lambda \not\geq \rho, \\ 0, & \text{otherwise.} \end{cases}$$

$$M_z^\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = 1_X, \\ 0.6, & \text{if } \lambda \geq \rho, \\ 0.2, & \text{if } \lambda \geq \rho \odot \rho, \lambda \not\geq \rho, \\ 0, & \text{otherwise.} \end{cases}$$

$$N_x^\tau(\lambda) = S(1, \lambda) \vee (\rho(x) \odot S(\rho, \lambda)) \vee ((\rho \odot \rho)(x) \odot S(\rho \odot \rho, \lambda)).$$

We obtain L-fuzzy topologies $\mathcal{T}_{M^\tau}, \mathcal{T}_{N^\tau} : L^X \to L$ as follows

$$\mathcal{T}_{M^\tau}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 1_X,\ \lambda = 0_X, \\ \wedge_{x \in X}(\lambda(x) \to \rho(x)), & \text{if } \lambda \geq \rho, \\ \wedge_{x \in X}(\lambda(x) \to (\rho \odot \rho)(x)), & \text{if } \lambda \geq \rho \odot \rho, \lambda \not\geq \rho, \\ \wedge_{x \in X}(\lambda(x) \to 0_X), & \text{otherwise.} \end{cases}$$

$$\mathcal{T}_{N^\tau}(\lambda) = \bigwedge_{x \in X} (\lambda(x) \to N_x^\tau(\lambda)).$$

Then $\lambda \in \tau$ iff $\mathcal{T}_{M^\tau}(\lambda) = 1$. $\mathcal{T}_{M^\tau}$ is not enriched from:

$$\mathcal{T}_{M^\tau}((0.1, 0.1, 0.1)) = \mathcal{T}_{M^\tau}((0.1) \odot 1_X) = \wedge_{x \in X}((0.1, 0.1, 0.1) \to 0_X)$$
$$= 0.9 \not\geq \mathcal{T}_{M^\tau}(1_X) = 1.$$

We have $\mathcal{T}_{N^\tau}((0.1, 0.1, 0.1)) = 1$ but $(0.1, 0.1, 0.1) \notin \tau$.

From Theorem 3.1, we obtain two $[0,1]$-neighborhood systems $M_x^{\mathcal{T}_{M^\tau}}, N_x^{\mathcal{T}_{N^\tau}} : X \to [0,1]^{[0,1]^X}$ as follows:

$$M_x^{\mathcal{T}_{M^\tau}}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 1_X, \\ \bigvee_{\mu \leq \lambda}(\wedge_{x \in X}(\mu(x) \to \rho(x)) \odot \mu(x)), & \text{if } \lambda \geq \rho, \\ \bigvee_{\mu \leq \lambda}(\wedge_{x \in X}(\mu(x) \to (\rho \odot \rho)(x)) \odot \mu(x)), & \text{if } \lambda \geq \rho \odot \rho, \\ & \lambda \not\geq \rho, \\ \bigvee_{\mu \leq \lambda}(\wedge_{x \in X}(\mu(x) \to 0) \odot \mu(x)) = 0, & \text{otherwise.} \end{cases}$$

$$N_x^{\mathcal{T}_{N^\tau}}(\lambda) = \bigvee_{\mu \in L^X} \mathcal{T}_{N^\tau}(\mu) \odot S(\mu, \lambda) \odot \mu(x)$$

Then, if $\lambda \geq \rho$, since $\bigvee_{\mu\leq\lambda}(\wedge_{x\in X}(\mu(x) \to \rho(x)) \odot \mu(x)) \geq (\wedge_{x\in X}(\rho(x) \to \rho(x)) \odot \rho(x)) = \rho(x)$, $M_x^{\mathcal{T}_{M^\tau}}(\lambda) \geq M_x^\tau(\lambda)$. Other case, we similarly prove $M_x^{\mathcal{T}_{M^\tau}}(\lambda) \geq M_x^\tau(\lambda)$.

If $\lambda \geq \rho$, $N_x^{\mathcal{T}_{N^\tau}}(\lambda) \geq \mathcal{T}_{N^\tau}(\rho) \odot S(\rho,\lambda) \odot \rho(x) = \rho = N_x^\tau(\lambda)$.

If $\lambda \geq \rho\odot\rho$, $N_x^{\mathcal{T}_{N^\tau}}(\lambda) \geq \mathcal{T}_{N^\tau}(\rho\odot\rho) \odot S(\rho\odot\rho,\lambda) \odot (\rho\odot\rho)(x) = (\rho\odot\rho)(x) = N_x^\tau(\lambda)$. Hence $N_x^{\mathcal{T}_{N^\tau}}(\lambda) \geq N_x^\tau(\lambda)$.

(2) Let $\omega, \omega\odot\omega \in [0,1]^X$ such that

$$\omega(x) = 0.1, \omega(y) = 0.8, \omega(z) = 0.7,$$

$$\omega\odot\omega(x) = 0, \omega\odot\omega(y) = 0.6, \omega\odot\omega(z) = 0.4.$$

We define an $[0,1]$-fuzzy topology $\mathcal{T} : [0,1]^X \to [0,1]$ as follows

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 1_X,\ \lambda = 0_X, \\ 0.6, & \text{if } \lambda = \omega, \\ 0.3, & \text{if } \lambda = \omega\odot\omega, \\ 0, & \text{otherwise.} \end{cases}$$

We obtain two $[0,1]$-neighborhood systems $M_-^{\mathcal{T}}, N_-^{\mathcal{T}} : X \to [0,1]^{[0,1]^X}$ as follows

$$M_-^{\mathcal{T}}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 1_X, \\ (0.6\odot\omega(-)) \vee (0.3\odot(\omega\odot\omega)(-)), & \text{if } 1_X \neq \lambda \geq \omega, \\ 0.3\odot(\omega\odot\omega)(-), & \text{if } \lambda \geq \omega\odot\omega, \\ & \lambda \ngeq \omega, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} N_-^{\mathcal{T}}(\lambda) &= \bigvee_{\mu\in L^X}(\mathcal{T}(\mu)\odot S(\mu,\lambda)\odot\mu(-)) \\ &= S(1_X,\lambda) \vee (0.6\odot S(\omega,\lambda)\odot\omega(-)) \\ &\quad \vee(0.3\odot S(\omega\odot\omega,\lambda)\odot(\omega\odot\omega)(-)). \end{aligned}$$

For $\lambda_1 = (0.9, 0.4, 0.2)$ and $\lambda_2 = (0,1,1)$,

$$\begin{aligned} M_-^{\mathcal{T}}(\lambda_1) &= (0,0,0), \\ M_-^{\mathcal{T}}(\lambda_2) &= 0.3\odot(\omega\odot\omega)(-) = (0,0,0), \end{aligned}$$

$$\begin{aligned} N_-^{\mathcal{T}}(\lambda_1) &= (0.2\odot 1_X) \vee (0.6\odot 0.5\odot\omega(-)) \\ &\quad \vee(0.3\odot 0.8\odot(\omega\odot\omega)(-)) = (0.2, 0.2, 0.2), \\ N_-^{\mathcal{T}}(\lambda_2) &= (0\odot 1_X) \vee (0.6\odot 0.7\odot\omega(-)) \\ &\quad \vee(0.3\odot 1\odot(\omega\odot\omega)(-)) = (0, 0.3, 0.2). \end{aligned}$$

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